MASTER OF SCIENCES MATHEMATICS

(M.Sc. Mathematics)

MMT-102

ADVANCED CALCULUS

Semester-I

Author - Dr. Manvendra N. Mishra

SURESH GYAN VIHAR UNIVERSITY

Centre for Distance and Online Education,

Mahal Road, Jagatpura, Jaipur-302017

Published by:

S. B. Prakashan Pvt. Ltd.

WZ-6, Lajwanti Garden, New Delhi: 110046

Tel.: (011) 28520627 | Ph.: 9205476295

Email: info@sbprakashan.com | Web.: www.sbprakashan.com

Designed & Graphic by : S. B. Prakashan Pvt. Ltd.

Printed at :

Suresh Gyanvihar University
Department of Mathematics
School of Science,
Rajasthan

M.Sc., Mathematics - Syllabus – I year – I Semester (Distance Mode)

COURSE TITLE : ADVANCED CALCULUS

COURSE CODE : MMT-102

COURSE CREDIT : 4

COURSE OBJECTIVES

While studying the **ADVANCED CALCULUS**, the Learner shall be able to:

CO 1: Discuss the relationship between continuous functions and differentiable functions.

CO 2: Review about the notion of Jacobian.

CO 3: Represent Taylor’s expansion of given function.

CO 4: Predict the applications of line integrals.

CO 5: Describe surface integrals.

COURSE LEARNING OUTCOMES

After completion of the **ADVANCED CALCULUS**, the Learner will be able to:

CLO 1:Interpret and able to derive basic mean value theorem which is of fundamental importance in the theory of partial differentiation

CLO 2: Describe the concept of functional dependence of two functions.

CLO 3: Enable to determine extrema of functions of two and three variables

CLO 4: Demonstrate an understanding about the knowledge about Green’s theorem which provides a formula connecting a line integral over its boundary with a double integral over a region.

CLO 5: Demonstrate an understanding to apply change of variable in evaluating multiple integrals.

BLOCKI: PARTIAL DIFFERENTIATION

Functions of several variables - Homogeneous functions - Total derivative - Higher order Derivatives, Equality of cross derivatives - Differentials - Directional Derivatives.

BLOCK II:IMPLICIT FUNCTIONS AND INVERSE FUNCTIONS

Implicit functions - Higher order derivatives - Jacobians - Dependent and independent variables - The inverse of a transformation - Inverse function theorem - Change ofvariables - Implicit function theorem - Functional dependence - Simultaneous equations.

BLOCK III: TAYLOR'S THEOREM AND APPLICATIONS

Taylor's theorem for functions of two variables - Maxima and Minima of functions of two and three variables - Lagrange Multipliers.

BLOCK IV:LINE AND SURFACE INTEGRALS

Definition of line integrals - Green's theorem - Applications - Surface integrals - Gauss theorem - Verification of Green's and Gauss theorems.

BLOCK V:TRANSFORMATION AND LINE INTEGRALS IN SPACE

Change of variables in multiple integrals - Definition of line integrals in space - Stoke'stheorem - Verification of Stoke's theorem.

REFERENCE BOOKS :

1. Widder D.V.,"Advanced Calculus", Prentice Hall of India, New Delhi, 12th Print, SecondEdition, 2002.

2. Kaplan W., "Advanced Calculus", Addison Wesley (Pearson Education, Inc.), Fifth Edition,2003.

3. Malik S.C., "Mathematical Analysis", New Age International Publishers, New Delhi, 1992.

4. Burkill J.C. and Burkill H., "A Second course in Mathematical Analysis", Cambridge University Press, New York, 2002.

5. Apostol T.M., "Mathematical Analysis", Second Edition, Narosa Publishing House, New Delhi,2013.

Block-I

Unit-1: Functions of Several Variables

Unit-2: Homogeneous functions and Differentials

Block-II

Unit-3: Jacobians

Unit-4: Inverse Functions and Implicit Functions

Block-III

Unit-5: Taylor's Theorem

Unit-6: Maximum and Minimum of functions of two and three variables

Unit-7: Lagrange's Multipliers

Block-IV

Unit-8: Line Integral

Unit-9: Green's Theorem

Unit-10: Surface Integral and Gauss's Theorem

Block-V

Unit-11: Transformation and Line Integral in Space

Contents

BLOCK I

Partial Differentiation

Unit 1

Functions of Several Variables

Learning Outcomes :

After studying this unit, students will acquire knowledge

- F To evaluate partial derivatives of functions of several variables.
- F To identify the relationship between continuous and differentiable functions.
- F To derive basic mean value theorem which is of fundamental importance in the theory of partial differentiation.
- F To understand under what condition the cross derivatives are equal.

1.1 Introduction

Constants and Variables

The quantity or parameters that does not change their value throughout a particular mathematical investigation are called constants and the quantities which take different values are called variables or arguments .

Functions

For finding the area of a triangle the base and altitude are multiplied. Here the base and the altitude can be of any value but the area of the triangle depends on these two values. So area of the triangle is called a function of the base and altitude of the triangle. The base and altitude are called the independent variables and the area is called the dependent variable.

In this unit we shall be dealing with real functions of several variables such as $u = f(x, y)$, $u = f(x, y, z)$ etc., The variables $x, y, z, \ldots$ are called the independent variables or arguments of the function, u is the dependent variable or value of the function.

Single valued and many valued functions

If the value of a function is uniquely determined by the argument we call the function single-valued or one valued function . For example $u = \dfrac{3x+5}{16x+3}$.

If the value of a function is not uniquely determined that is to each value of the argument, if there correspond more than one value of the function the function is called many-valued function or multiple-valued function.

As an example,

$$u^2 + x^2 + y^2 = a^2 \tag{1.1}$$

Multiple valued functions may be studied as combinations of single-valued functions. For example the equation (1.1) defines two single valued functions

$$u = +\sqrt{a^2 - x^2 - y^2} \tag{1.2}$$

$$u = -\sqrt{a^2 - x^2 - y^2}, \quad x^2 + y^2 \leq a^2 \tag{1.3}$$

A function of two variables clearly represents a surface in the space of the rectangular coordinates x, y, u.

Explicit and Implicit functions

If we consider a set of n independent variables $x, y, z, \ldots, t$ and one dependent variable u, the equation

$$u = f(x, y, z, \ldots, t) \tag{1.4}$$

denotes the functional relation, where u depends for its values on $x, y, z, \ldots, t$. Then, we say that the function represented by equation (1.4) is an explicit function.

But in case of several variables it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the other. Thus most of the functions of more than one variable are implicit functions, that is to say we are given a functional relation $\varphi(x, y, z, \ldots, t) = 0$ connecting n variables $x, y, z, \ldots, t$, and it is not in general possible to solve this equation to find an explicit function which expresses one of these variables say x, in terms of the other $n - 1$ variables.

For example, the equation (1.1) defines two functions (1.2) and (1.3) which are said to be defined implicitly by (1.1) or explicitly by (1.2) and (1.3).

In other cases, a function may be defined implicitly eventhough it is impossible to give its explicit form. For example, the equation

$$u + \log u = xy \tag{1.5}$$

defines one single valued function u of x and y. Given real values for the arguments, the equation could be solved by approximation methods for u. Yet u cannot be given in terms of x and y by use of a finite number of the elementary functions.

Partial derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivatives of $u = f(x, y, z)$ with respect to x is generally denoted by

$$\frac{\partial u}{\partial x} = f_1(x, y, z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y, z) = f_x,$$

while those with respect to y and z are given by

$$\frac{\partial u}{\partial y} = f_2(x, y, z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y, z) = f_y$$

and

$$\frac{\partial u}{\partial z} = f_3(x, y, z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} f(x, y, z) = f_z$$

Example 1.1.1 *If* $f(x, y, z) = 2x^2 - xy + xy^2$, *then*

$$f_1(x, y) = 4x - y + y^2$$

$$and \quad f_2(x, y) = -x + 2xy$$

Example 1.1.2 *If* $f(x, y, z) = xz^y$, *then* $f_2(x, y, z) = xz^y \log z$

Note : The partial derivatives at a particular point (x_0, y_0, z_0) are often denoted by

$$\frac{\partial u}{\partial x}\Big|_{x=x_0, y=y_0, z=z_0} = \frac{\partial f}{\partial x}\Big|_{x_0, y_0, z_0} = f_1(x_0, y_0, z_0),$$

$$\frac{\partial u}{\partial y}\Big|_{x=x_0, y=y_0, z=z_0} = \frac{\partial f}{\partial y}\Big|_{x_0, y_0, z_0} = f_2(x_0, y_0, z_0)$$

and $$\frac{\partial u}{\partial z}\Big|_{x=x_0, y=y_0, z=z_0} = \frac{\partial f}{\partial z}\Big|_{x_0, y_0, z_0} = f_3(x_0, y_0, z_0)$$

For example, $f_3(x_0, y_0, z_0) = \frac{d}{dz} f(x_0, y_0, z)\big|_{z=z_0}$

Example 1.1.3 *If* $f(x, y, z) = x\sin(yz)$, *then*

$$f_3(x, y, z) = xy\cos(yz)$$

$$\textit{and} \quad f_3(a, 1, \pi) = a\cos\pi = -a.$$

Example 1.1.4 *Consider equation (1.1)*

$$u^2 + x^2 + y^2 = a^2$$

Differentiating partially with respect to x *and* y

$$2u\frac{\partial u}{\partial x} + 2x = 0, \qquad \frac{\partial u}{\partial x} = -\frac{x}{u},$$
$$2u\frac{\partial u}{\partial y} + 2y = 0, \qquad \frac{\partial u}{\partial y} = -\frac{y}{u}.$$

Example 1.1.5 *If* $v + \log u = xy$, $u + \log v = x - y$, *find* $\frac{\partial v}{\partial x}$ *and* $\frac{\partial u}{\partial x}$.

Solution: Differentiating partially with respect to x

$$\frac{\partial v}{\partial x} + \frac{1}{u}\frac{\partial u}{\partial x} = y,$$
$$\frac{\partial u}{\partial x} + \frac{1}{v}\frac{\partial v}{\partial x} = 1,$$
$$u\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = uy,$$
$$\text{and} \quad \frac{1}{v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = 1.$$

Subtracting we get

$$\frac{\partial v}{\partial x} = \frac{v(uy - 1)}{uv - 1}.$$

Similarly, we can find

$$\frac{\partial u}{\partial x} = \frac{u(y - v)}{1 - uv}.$$

Higher order derivatives

By successive application of the differentiation, we can obtain partial derivatives of higher order. The notations used are sufficiently illustrated by the following examples. If $u = f(x, y, z)$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = f_{21}(x, y, z),$$

$$\frac{\partial^3 u}{\partial y^2 \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = f_{322}(x, y, z),$$

and
$$\frac{\partial^4 u}{\partial x \partial y \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^3 u}{\partial y \partial z^2}\right) = f_{3321}(x, y, z).$$

Remarks

1. A function of two variables has two derivatives of order one, four derivatives of order two and 2^n derivatives of order n.

2. A function of m independent variable will have m^n derivatives of order n.

3. Many of the derivatives of a given order will be equal under very general conditions. In fact,

Number of distinct derivatives of order n

= Number of terms in a homogeneous polynomial in m variables of degree n:

$$\binom{n+m-1}{n} = \frac{(n+m-1)!}{n!(m-1)!}.$$

Example 1.1.6 *Find* $\dfrac{\partial^2}{\partial r^2}\log(r^2 + s)$.

Solution:

Let $u = \log(r^2 + s)$

$$\frac{\partial u}{\partial r} = \frac{2r}{r^2 + s}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{(r^2+s)2 - 2r(2r)}{(r^2+s)^2}$$
$$= \frac{2(s-r^2)}{(r^2+s)^2}.$$

Example 1.1.7 *If* $u = x^y$, *show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Solution: Given

$$u = x^y$$
$$\frac{\partial u}{\partial y} = x^y \log x$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x)$$
$$= x^y.\frac{1}{x} + \log x.x^{y-1}.y = x^{y-1}(1 + y\log x)$$
$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(x^{y-1}(1 + y\log x)\right)$$
$$= x^{y-1}\left(\frac{y}{x}\right) + (y-1)x^{y-2}(1 + y\log x)$$
$$= x^{y-2}(y + (y-1)(1 + y\log x))$$
$$= x^{y-2}(2y - 1 + (y-1)y\log x)$$
$$\frac{\partial u}{\partial x} = yx^{y-1}$$
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial y}\left(yx^{y-1}\right)$$
$$= yx^{y-1}\log x + x^{y-1}$$
$$= x^{y-1}(y\log x + 1)$$
$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 y}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left(x^{y-1}(y\log x + 1)\right)$$
$$= (y-1)x^{y-2}(y\log x + 1) + x^{y-1}\left(\frac{y}{x}\right)$$
$$= x^{y-2}\left((y-1)(y\log x + 1) + y\right)$$
$$= x^{y-2}(2y - 1 + (y-1)y\log x)$$

So $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

1.2 Functions of One variable

In this section, we recall some definitions and theorems without proof.

Limits and continuity

A function $f(x)$ approaches a limit A as x approaches a if, and only if, for each positive number ϵ there is another, δ, such that whenever

$$0 < |x - a| < \delta \text{ we have } |f(x) - A| < \epsilon.$$

In symbols we write

$$\lim_{x \to a} f(x) = A.$$

Note :

We make use of the following symbols / notations.

1. ϵ : belongs to or is a member of
2. $\Rightarrow$: implies
3. $\Leftrightarrow$: implies and implied by or if, and only if
4. C : the class of continuous functions

Definition 1.2.1 $f(x) \in C$ *at* $x = a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$.
This may be read as " $f(x)$ belongs to the class of functions continuous at $x = a$ or $f(x)$ is continuous at $x = a$ if, and only if the limit $f(x)$ is $f(a)$ as x approaches a."
Also, we have

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right).$$

Derivatives : We now introduce classes of functions, which have derivatives of certain order.

Definition 1.2.2 *The derivative of $f(x)$ at $x = a$ is*

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x},$$

the right and left derivatives of $f(x)$ *at* $x = a$ *are*

$$f'_+(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x) - f(a)}{\Delta x},$$

$$f'_-(a) = \lim_{\Delta x \to 0^-} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

Definition 1.2.3 .

$f(x) \in C^n \Leftrightarrow f^n(x) \in C,\ n = 1, 2, \ldots$ *When* $f'(x)$ *exist, then* $f(x) \in C$. *Hence if* $f(x) \in C^n$ *we also have* $f(x) \in C^k$ *for* $k = 0, 1, 2, \ldots, n-1$, $C^0 = C$.

Theorem 1.2.1 (Rolle's Theorem)

1. $f(x) \in C, \quad a \le x \le b$
2. $f'(x)$ *exists,* $a < x < b$ *and*
3. $f(a) = f(b) = 0$

$\Rightarrow f'(\xi) = 0$ *for some* $\xi, \quad a < \xi < b.$

Theorem 1.2.2 (Law of the Mean)

1. $f(x) \in C, \quad a \le x \le b$ *and*
2. $f'(x)$ *exists,* $a < x < b$

$\Rightarrow \quad f(b) - f(a) = f'(\xi)(b-a)$ *for some* $\xi, \quad a < \xi < b.$

1.3 Functions of several variables

In this unit we shall be mainly concerned with the applications of differential calculus to functions of more than one variable. The characteristic properties of a function of n independent variables may usually be understood by the study of a function of two or three variables and this restriction of two or three variables will be generally maintained. This restriction has the considerable advantage of simplifying the formulae and of reducing the mechanical labour.

1.3.1 Limits and continuity

We now define the limit of a function of two variables.

Definition 1.3.1 *We say that a function $f(x, y)$ approaches a limit A as x approaches a and y approaches b,*

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = A,$$

if, and only if, for each positive number ϵ there is another, δ, such that $|f(x, y)-A| < \epsilon$, whenever $|x-a| < \delta$, $|y-b| < \delta$ or $0 < (x-a)^2+(y-b)^2$.

In otherwords, a function tends to a limit A when (x, y) tends to (a, b) if to every positive number ϵ there corresponds a neighborhood with center at (a, b) such that $|f(x, y) - A| < \epsilon$ for every point (x, y) other than (a, b) of the neighborhood.

Example 1.3.1 *If $f(x, y) = x^2 + y^2$, find $\lim_{\substack{x\to 0\\ y\to 0}} f(x, y)$.*

Solution: Given $\epsilon > 0$ we may choose $\delta = \sqrt{\frac{\epsilon}{2}}$
so that $|x| < \sqrt{\frac{\epsilon}{2}}$ $|y| < \sqrt{\frac{\epsilon}{2}}$
Consider

$$\begin{aligned} x^2 + y^2 &= |x^2| + |y^2| \\ &< \left(\sqrt{\frac{\epsilon}{2}}\right)^2 + \left(\sqrt{\frac{\epsilon}{2}}\right)^2 = \epsilon. \end{aligned}$$

so $|x^2 + y^2 - 0| < \epsilon$ whenever $|x| < \sqrt{\frac{\epsilon}{2}}$, $|y| < \sqrt{\frac{\epsilon}{2}}$
Hence,

$$\lim_{\substack{x\to 0\\ y\to 0}} x^2 + y^2 = 0.$$

Example 1.3.2

$$Let \quad f(x, y) = \begin{cases} \frac{x-y}{x+y}, & x \neq -y \\ 1, & x = -y \end{cases} \tag{1.6}$$

Then $f(x, y)$ approaches no limit as (x, y) approaches the origin.

Solution: $f(x, y)$ approaches no limit as (x, y) approaches the origin. Because, $f(x, y)$ is as large as we like at points near the line $x = -y$.
On the other hand, we see that

$$\lim_{x\to 0} \lim_{y\to 0} f(x, y) = \lim_{x\to 0} \lim_{y\to 0} \frac{x-y}{x+y}$$

$$= \lim_{x\to 0} 1 = 1,$$

and

$$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = \lim_{y\to 0} \lim_{x\to 0} \frac{x-y}{x+y}$$

$$= \lim_{y\to 0} -1 = -1.$$

Note :

The iterated limits $\lim_{x\to x_0} \lim_{y\to y_0} f(x, y)$ and $\lim_{y\to y_0} \lim_{x\to x_0} f(x, y)$ are not necessarily equal. Although they must be equal if $\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$ is to exist, their equality does not guarantee the existence of this last limit. In the above example the iterated limits are not equal. Hence $\lim_{\substack{x\to 0\\ y\to 0}} f(x, y)$ does not exist.

Definition 1.3.2

$$f(x, y) \in C \text{ at } (a, b) \Leftrightarrow \lim_{\substack{x\to a\\ y\to b}} f(x, y) = f(a, b)$$

We note that three conditions must be satisfied in order for $f(x, y)$ to be continuous at (a, b).

(i) $\lim_{\substack{x\to a\\ y\to b}} f(x, y)$ *must exist*

(ii) $f(a, b)$ *must exist and*

(iii) $\lim_{\substack{x\to a\\ y\to b}} f(x, y) = f(a, b)$.

Example 1.3.3 *If*

$$f(x, y) = \begin{cases} 3xy, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases},$$

then $f(x, y)$ is not continuous at $(1, 2)$.

Solution: Given $f(1, 2) = 0$.

$$\lim_{\substack{x \to 1 \\ y \to 2}} f(x, y) = 6 \neq f(1, 2).$$

Hence $f(x, y)$ is not continuous at $(1, 2)$.

Note: If we redefine the function, so that $f(x, y) = 6$ for $(x, y) = (1, 2)$, then the function is continuous at (1,2).

NEIGHBORHOODS

Point set : Any collection of points (x, y) is called a point set.

$\delta-$ **neighborhood :** The set of points $|x - a| < \delta$, $|y - b| < \delta$, where $\delta > 0$ is called an open square or two-dimensional interval or a $\delta-$ neighborhood of the point (a, b).

Deleted $\delta-$ neighborhood : The set of points $0 < |x - a| < \delta$, $0 < |y - b| < \delta$, where $\delta > 0$ which excludes (a, b) is called a deleted $\delta-$ neighborhood of the point (a, b).

Circular $\delta-$ neighborhood : The set of points $(x - a)^2 + (y - b)^2 < \delta^2$, where $\delta > 0$, is called circular $\delta-$ neighborhood of (a, b).

Limit point : A point (a, b) is a limit point of a set S if every $\delta-$ neighborhood of (a, b) contains points of S.

Closed set : A set S is closed if it contains all its limit points.

REGIONS

Interior point : A point is an interior point of S, if it is the centre of a $\delta-$ neighborhood composed entirely of points of S.

Exterior point : A point is an exterior point of S if there exist a $\delta-$ neighborhood which does not contains any point of S.

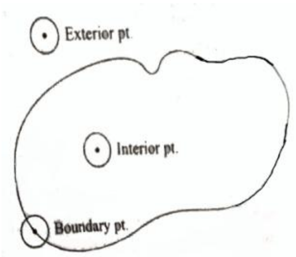

Figure 1.1

Boundary point : A point is a boundary point of S, if there exist a $\delta-$

neighborhood which contains points belonging to S and also points not belonging to S. The boundary of a set is the set of all limit points not interior points.

Open set : A set S is called an open set, if every point of S is an interior point of S. For example, if S is the set of points (x, y) for which $x^2+y^2 < a^2$, then S is open.

Domain : A domain is an open set,such that any two of whose points can be joined by a broken line having a finite number of segments, all of whose points belong to the set. We shall use the letter D to indicate a domain.

Region : A region is either a domain or a domain with some or all of its boundary.

If a region contains all of its boundary, it is a closed region. We use the letter R to indicate a region.

Remark :

1. $f(x, y) \in C$ in a domain D if and only if $f(x, y) \in C$ at every point of D.

2. $f(x, y) \in C$ at a boundary point (a, b) of a region R where $f(x, y)$ is defined if, and only if,

$$\lim_{\substack{x \to a \\ y \to b}} = f(a, b), \quad (x, y) \in R$$

   That is, the point (x, y) approaches (a, b) only through the points of R.

3. $f(x, y) \in C$ in R if $f(x, y) \in C$ at each point of R.

Uniform continuity : In the definition of continuity of $f(x, y)$ at (a, b) the choice of δ depends on ϵ and also on (a, b). If in a region R we can find a δ which depends only on ϵ but not on any particular point (a, b) in R, then $f(x, y)$ is said to be uniformly continuous in R.

1.3.2 Derivatives

We now define the classes C^n for functions of several variables.

Definition 1.3.3

$$f_1(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$
$$f_2(a,b) = \frac{\partial f}{\partial y}\Big|_{(a,b)} = \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$$

Definition 1.3.4

$$f(x,y) \in C^n \text{ in } R \Leftrightarrow \frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial x^{n-1}\partial y}, \dots, \frac{\partial^n f}{\partial y^n} \in C \text{ in } R.$$

Note: If $f(x,y)$ satisfies the condition of this definition, then $f(x,y) \in C^k, \quad k = 0, 1, 2, \dots, n-1.$

1.3.3 Basic mean value theorem

We now prove a theorem analogous to the law of the mean for functions of a single variable.This theorem is of fundamental importance in the theory of partial differentiation.

Theorem 1.3.1

1. $f(x,y) \in C^1$ *in* D *and*

2. *The circle* $(x-a)^2 + (y-b)^2 \leq \delta^2$ *lies in* D

$\Rightarrow f(a+\Delta x, b+\Delta y) - f(a,b) = f_1(a+\vartheta_1\Delta x, b)\Delta x + f_2(a+\Delta x, b+\vartheta_2\Delta y)\Delta y,$ *where* $\Delta x^2 + \Delta y^2 < \delta^2$ *and* $0 < \vartheta_1 < 1, 0 < \vartheta_2 < 1.$

Proof: Let

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b) \tag{1.7}$$

Then

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) + f(a+\Delta x, b) - f(a,b)$$
$$= [f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b)] + [f(a+\Delta x, b) - f(a,b)]$$

Applying law of the mean to the function $f(x, b)$ of the single variable x, its derivative is $f_1(x, b)$, and we have

$$f(a + \Delta x, b) - f(a, b) = f_1(a + \vartheta_1 \Delta x, b)\Delta x,$$

where $0 < \vartheta_1 < 1$.

Applying law of the mean to the function $f(a + \Delta x, y)$, we have

$$f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b) = f_2(a + \Delta x, b + \vartheta_2 \Delta y)\Delta y,$$

where $0 < \vartheta_2 < 1$. Thus,

$$\Delta f = f_2(a + \Delta x, b + \vartheta_2 \Delta y)\Delta y + f_1(a + \vartheta_1 \Delta x, b)\Delta x.$$

Hence,

$$f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a + \vartheta_1 \Delta x, b)\Delta x + f_2(a + \Delta x, b + \vartheta_2 \Delta y)\Delta y. \quad (1.8)$$

Here the two numbers ϑ_1 and ϑ_2 are different. Q

Remark : If we replace the hypothesis 2 by the hypothesis that (a, b) and $(a + \Delta x, b + \Delta y)$ are both points of D, then equation (1.8) might not be true . It will be clear from the following figure.

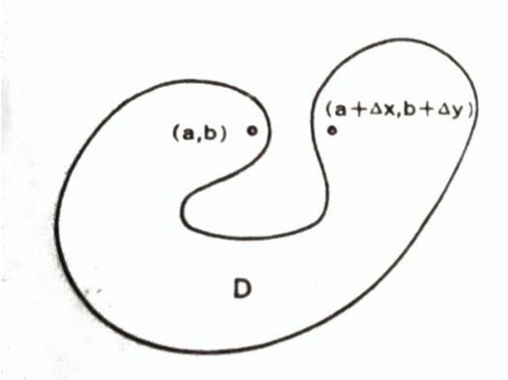

Figure 1.2

Example 1.3.4 *Using basic mean value theorem, find the numbers* ϑ_1 *and* ϑ_2, *if* $f(x, y) = x^2 + 3xy + y^2$, $a = b = 0, \Delta x = 1, \Delta y = -1$.

Solution: From basic mean value theorem,

$$f(a+\Delta x, b+\Delta y) - f(a,b) = f_1(a+\vartheta_1\Delta x, b)\Delta x + f_2(a+\Delta x, b+\vartheta_2\Delta y)\Delta y$$

Here

$$f(1,-1) - f(0,0) = f_1(\vartheta_1, 0) - f_2(1, -\vartheta_2) \tag{1.9}$$

$$f(x,y) = x^2 + 3xy + y^2$$

$$f_1(x,y) = 2x + 3y, \quad f_1(\vartheta_1, 0) = 2\vartheta_1$$

$$f_2(x,y) = 3x + 2y, \quad f_2(1-\vartheta_2) = 3 - 2\vartheta_2$$

$$f(1,-1) = -1, \quad f(0,0) = 0$$

Hence we have from (1.9)

$$-1 = 2\vartheta_1 - 3 + 2\vartheta_2$$

$$\Rightarrow 2(\vartheta_1 + \vartheta_2) = 2$$

$$\Rightarrow \vartheta_1 + \vartheta_2 = 1$$

$$\Rightarrow \vartheta_1 = \frac{1}{2}, \quad \vartheta_2 = \frac{1}{2}.$$

Example 1.3.5 *Using basic mean value theorem, find the numbers ϑ_1 and ϑ_2 if $f(x,y) = x^2 + y^2 + x^3$, $(a,b) = (1,2)$.*

Solution: From basic mean value theorem,

$$f(a+\Delta x, b+\Delta y) - f(a,b) = f_1(a+\vartheta_1\Delta x, b)\Delta x + f_2(a+\Delta x, b+\vartheta_2\Delta y)\Delta y$$

Here

$$f(1+\Delta x, 2+\Delta y) - f(1,2) = f_1(1+\vartheta_1\Delta x, 2)\Delta x + f_2(1+\Delta x, 2+\vartheta_2\Delta y)\Delta y \tag{1.10}$$

$$f(x,y) = x^2 + y^2 + x^3$$

$$f(1+\Delta x, 2+\Delta y) = (1+\Delta x)^2 + (2+\Delta y)^2 + (1+\Delta x)^3$$

$$= 1 + \Delta x^2 + 2\Delta x + 4 + 4\Delta y + \Delta y^2 + 1 + \Delta x^3 + 3\Delta x^2 + 3\Delta x$$

$$= \Delta x^3 + 4\Delta x^2 + \Delta y^2 + 5\Delta x + 4\Delta y + 6$$

$$f(1,2) = 6.$$

$$f_1(x,y) = 2x + 3x^2$$

$$\begin{aligned} f_1(1+\vartheta_1\Delta x, 2) &= 2(1+\vartheta_1\Delta x) + 3(1+\vartheta_1\Delta x)^2 \\ &= 2 + 2\vartheta_1\Delta x + 3 + 3\vartheta_1^2\Delta x^2 + 6\vartheta_1\Delta x \\ &= 3\vartheta_1^2\Delta x^2 + 8\vartheta_1\Delta x + 5 \end{aligned}$$

$$f_2(x,y) = 2y$$

$$f_2(1+\Delta x, 2+\vartheta_2\Delta y) = 4 + 2\vartheta_2\Delta y$$

Hence we have from (1.10)

$$\begin{aligned} &\Delta x^3 + 4\Delta x^2 + \Delta y^2 + 5\Delta x + 4\Delta y \\ &\qquad = (3\vartheta_1^2\Delta x^2 + 8\vartheta_1\Delta x + 5)\Delta x + (2\vartheta_2\Delta y + 4)\Delta y \\ &\Delta x(\Delta x^2 + 4\Delta x + 5) + \Delta y(\Delta y + 4) \\ &\qquad = (3\vartheta_1^2\Delta x^2 + 8\vartheta_1\Delta x + 5)\Delta x + (2\vartheta_2\Delta y + 4)\Delta y. \end{aligned}$$

We have

$$\Delta x^2 + 4\Delta x + 5 = 5 + 8\vartheta_1\Delta x + 3\vartheta_1^2\Delta x^2$$

$$\Delta x^2(3\vartheta_1^2 - 1) + \Delta x(8\vartheta_1 - 4) = 0 \Rightarrow \Delta x[(3\vartheta_1^2 - 1)\Delta x + 8\vartheta_1 - 4] = 0$$

$$\Rightarrow (3\vartheta_1^2 - 1)\Delta x + 8\vartheta_1 - 4 = 0 \Rightarrow 3\vartheta_1^2\Delta x - \Delta x + 8\vartheta_1 - 4 = 0$$

$$\vartheta_1 = \frac{-8 \pm \sqrt{64 + 12(4+\Delta x)\Delta x}}{6\Delta x} = \frac{-4 \pm \sqrt{16 + 3(4+\Delta x)\Delta x}}{3\Delta x}$$

$$\vartheta_1 = \frac{-4 + \sqrt{16 + 12\Delta x + 3\Delta x^2}}{3\Delta x}$$

Also, $2\vartheta_2 = 1$

$$\Rightarrow \vartheta_2 = \frac{1}{2}.$$

1.3.4 Composite functions

The Basic mean value theorem can be used to differentiate composite functions.

Theorem 1.3.2 $f(x, y), g(r, s), h(r, s) \in C^1$

$$\Rightarrow \frac{\partial}{\partial r} f(g, h) = f_1(g, h) g_1(r, s) + f_2(g, h) h_1(r, s)$$

$$\frac{\partial}{\partial s} f(g, h) = f_1(g, h) g_2(r, s) + f_2(g, h) h_2(r, s)$$

Proof: Given that g and h are functions of r and s, f is a function of x and y, but the regions in which the given function $\in C^1$ are not specified. So it is understood that the region for (r, s) and the region for (x, y) must be such that we can form a function $f(g(r, s), h(r, s))$ by substituting $x = g$ and $y = h$. From the definition of partial derivative we have

$$\frac{\partial f}{\partial r}\Big|_{(r_0, s_0)} = \lim_{\Delta r \to 0} \frac{\Delta f}{\Delta r},$$

where $\Delta f = f(g(r_0 + \Delta r, s_0), h(r_0 + \Delta r, s_0)) - f(g(r_0, s_0), h(r_0, s_0))$.

Let

$$g(r_0 + \Delta r, s_0) = x_0 + \Delta x, \quad x_0 = g(r_0, s_0)$$

$$\text{and} \quad h(r_0 + \Delta r, s_0) = y_0 + \Delta y, \quad y_0 = h(r_0, s_0).$$

Now, $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$.

Applying mean value theorem, we have

$$\Delta f = f_1(x_0 + \vartheta_1 \Delta x, y_0)\Delta x + f_2(x_0 + \Delta x, y_0 + \vartheta_2 \Delta y)\Delta y$$

where $0 < \vartheta_1, \vartheta_2 < 1$.

Since g and h are continuous Δx and Δy tends to zero as Δr tends to zero, We have

$$\frac{\Delta f}{\Delta r} = f_1(x_0 + \vartheta_1 \Delta x, y_0)\frac{\Delta x}{\Delta r} + f_2(x_0 + \Delta x, y_0 + \vartheta_2 \Delta y)\frac{\Delta y}{\Delta r}$$

where $0 < \vartheta_1, \vartheta_2 < 1$.

$$\lim_{\Delta r \to 0} \frac{\Delta f}{\Delta r} = \lim_{\Delta r \to 0} f_1(x_0 + \vartheta_1 \Delta x, y_0)\frac{\Delta x}{\Delta r} + \lim_{\Delta r \to 0} f_2(x_0 + \Delta x, y_0 + \vartheta_2 \Delta y)\frac{\Delta y}{\Delta r},$$

where $0 < \vartheta_1, \vartheta_2 < 1$.

$$\frac{\partial f}{\partial r}\Big|_{(r_0, s_0)} = \lim_{\Delta r \to 0} f_1(g(r_0 + \vartheta_1 \Delta r, s_0), h(r_0, s_0))\frac{g(r_0 + \Delta r, s_0) - g(r_0, s_0)}{\Delta r}$$

$$+ \lim_{\Delta r \to 0} f_2(g(r_0 + \Delta r, s_0), h(r_0 + \vartheta_2 \Delta r, s_0))\frac{h(r_0 + \Delta r, s_0) - h(r_0, s_0)}{\Delta r}$$

$$= f_1(x_0, y_0) g_1(r_0, s_0) + f_2(x_0, y_0) h_1(r_0, s_0), \quad 0 < \vartheta_1, \vartheta_2 < 1$$

Replacing x_0, y_0 by their values and omitting subscripts, we have

$$\frac{\partial}{\partial r} f(g, h) = f_1(g, h)g_1(r, s) + f_2(g, h)h_1(r, s).$$

Similarly, we can obtain

$$\frac{\partial}{\partial s} f(g, h) = f_1(g, h)g_2(r, s) + f_2(g, h)h_2(r, s).$$

Q

Remark : The above results can be put in a simpler form as follows:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

Example 1.3.6 *If* $\square = x^3 - xy + y^3, x = r\cos\vartheta, y = r\sin\vartheta,$ *find* $\frac{\partial \square}{\partial r}, \frac{\partial \square}{\partial \vartheta}.$

Solution:

$$\frac{\partial \square}{\partial r} = \frac{\partial \square}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial \square}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\vartheta + (3y^2 - x)\sin\vartheta$$

and $$\frac{\partial \square}{\partial \vartheta} = \frac{\partial \square}{\partial x}.\frac{\partial x}{\partial \vartheta} + \frac{\partial \square}{\partial y}\frac{\partial y}{\partial \vartheta} = (3x^2 - y)(-r\sin\vartheta) + (3y^2 - x)r\cos\vartheta.$$

Remark : We can prove the following **case** using the method used in theorem 1.3.2. If $u = f(x, y, \square), x = g(r, s), y = h(r, s), \square = k(r, s),$ then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial \square}\frac{\partial \square}{\partial r}.$$

Example 1.3.7 *If* $u = x^2 + y^2, x = r\cos\vartheta, y = r\sin\vartheta,$ *compute* $\frac{\partial u}{\partial r}$ *and* $\frac{\partial u}{\partial \vartheta}.$

Solution:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 2x\cos\vartheta + 2y\sin\vartheta$$
$$\frac{\partial u}{\partial \vartheta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \vartheta} = -2xr\sin\vartheta + 2ry\cos\vartheta.$$

Example 1.3.8 *If $u = f(x, y)$, $x = r\cos\vartheta$, $y = r\sin\vartheta$, show that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \vartheta}\right)^2.$$

Solution:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\vartheta + \frac{\partial u}{\partial y}\sin\vartheta$$

$$\frac{\partial u}{\partial \vartheta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \vartheta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \vartheta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\vartheta) + \frac{\partial u}{\partial y}(r\cos\vartheta)$$

Therefore, $$\frac{1}{r}\frac{\partial u}{\partial \vartheta} = -\frac{\partial u}{\partial x}\sin\vartheta + \frac{\partial u}{\partial y}\cos\vartheta$$

Then, $$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial u}{\partial \vartheta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Example 1.3.9 *If $u = z\sin\frac{y}{x}$, where $x = 3r^2 + 2s$, $y = 4r - 2s^3$ and $z = 2r^2 - 3s^2$, find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.*

Solution:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(z\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)6r + \left(z\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)4 + \left(\sin\frac{y}{x}\right)4r$$

$$= -\frac{6ryz}{x^2}\cos\frac{y}{x} + \frac{4z}{x}\cos\frac{y}{x} + 4r\sin\frac{y}{x}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left(z\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)2 + \left(z\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)(-6s^2) + \left(\sin\frac{y}{x}\right)(-6s)$$

$$= \frac{2yz}{x^2}\cos\frac{y}{x} - \frac{6s^2z}{x}\cos\frac{y}{x} - 6s\sin\frac{y}{x}.$$

1.3.5 Higher Derivatives

We can compute higher order derivatives of composite functions by the principles that we know already.

For clear understanding consider the function $u = f(\varphi(r, s), \psi(r, s))$, where the three functions involved belong to C^2.

We now compute three derivatives of order two for the above function.

$$\frac{\partial u}{\partial r} = f_1(\varphi(r, s), \psi(r, s))\varphi_1(r, s) + f_2(\varphi(r, s), \psi(r, s))\psi_1(r, s),$$

$$\frac{\partial u}{\partial s} = f_1(\varphi(r, s), \psi(r, s))\varphi_2(r, s) + f_2(\varphi(r, s), \psi(r, s))\psi_2(r, s).$$

Note: In each φ or ψ with any subscript, we omit the arguments (r, s) and in each f we omit the arguments $(\varphi(r, s), \psi(r, s))$.

Differentiating both $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$ partially with respectively to r and s.

we have,

$$\frac{\partial^2 u}{\partial r^2} = f_1\varphi_{11} + f_2\psi_{11} + \varphi_1[f_{11}\varphi_1 + f_{12}\psi_1] + \psi_1[f_{21}\varphi_1 + f_{22}\psi_1],$$

$$\frac{\partial^2 u}{\partial s \partial r} = f_1\varphi_{12} + f_2\psi_{12} + \varphi_1[f_{11}\varphi_2 + f_{12}\psi_2] + \psi_1[f_{21}\varphi_2 + f_{22}\psi_2],$$

$$\frac{\partial^2 u}{\partial r \partial s} = f_1\varphi_{21} + f_2\psi_{21} + \varphi_2[f_{11}\varphi_1 + f_{12}\psi_1] + \psi_2[f_{21}\varphi_1 + f_{22}\psi_1],$$

and $$\frac{\partial^2 u}{\partial s^2} = f_1\varphi_{22} + f_2\psi_{22} + \varphi_2[f_{11}\varphi_2 + f_{12}\psi_2] + \psi_2[f_{21}\varphi_2 + f_{22}\psi_2].$$

Example 1.3.10 *If* $u = f(x, y) = e^{xy}$, $x = \varphi(r, s) = r + s$, $y = \psi(r, s) = r - s$, *find* $\frac{\partial^2 u}{\partial r^2}$ *and* $\frac{\partial^2 u}{\partial r \partial s}$.

Solution: Given $u = f(x, y) = e^{xy}$, $x = \varphi(r, s) = r + s$, $y = \psi(r, s) = r - s$.

$f_1 = ye^{xy}$,	$\varphi_1 = 1$,	$\psi_1 = 1$
$f_2 = xe^{xy}$,	$\varphi_2 = 1$,	$\psi_2 = -1$
$f_{11} = y^2e^{xy}$,	$\varphi_{11} = \varphi_{12} = 0$,	$\psi_{11} = \psi_{12} = 0$
$f_{12} = f_{21} = (1 + xy)e^{xy}$,	$\varphi_{21} = \varphi_{22} = 0$,	$\psi_{21} = \psi_{22} = 0$
$f_{22} = x^2e^{xy}$		

$$\frac{\partial^2 u}{\partial r^2} = f_1\varphi_{11} + f_2\psi_{11} + \varphi_1[f_{11}\varphi_1 + f_{12}\psi_1] + \psi_1[f_{21}\varphi_1 + f_{22}\psi_1]$$

$$= y^2e^{xy} + (1 + xy)e^{xy} + (1 + xy)e^{xy} + x^2e^{xy}$$

$$= e^{xy}(y^2 + 2 + 2xy + x^2)$$

$$= e^{(r+s)(r-s)}[(r - s)^2 + 2 + 2(r + s)(r - s) + (r + s)^2]$$

$$= e^{r^2-s^2}[4r^2 + 2].$$

$$\frac{\partial^2 u}{\partial r \partial s} = f_1\varphi_{21} + f_2\psi_{21} + \varphi_2[f_{11}\varphi_1 + f_{12}\psi_1] + \psi_2[f_{21}\varphi_1 + f_{22}\psi_1]$$

$$= y^2e^{xy} + (1 + xy)e^{xy} - 1[(1 + xy)e^{xy} + x^2e^{xy}]$$

$$= e^{xy}(y^2 - x^2)$$

$$= e^{(r+s)(r-s)}((r - s)^2 - (r + s)^2)$$

$$= e^{r^2-s^2}(-4rs)$$

$$= -4rse^{r^2-s^2}.$$

1.3.6 Differentiable functions

The class of differentiable functions lie between C and C^1.

Definition 1.3.5 *$f(x, y)$ is differentiable at (a, b) if, and only if $f_1(a, b), f_2(a, b)$ exist and if*

$$f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a, b)\Delta x + f_2(a, b)\Delta y + \varphi(\Delta x, \Delta y)\Delta x + \psi(\Delta x, \Delta y)\Delta y, \tag{1.11}$$

where $\varphi(\Delta x, \Delta y)$ and $\psi(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Example 1.3.11 *Prove that $f(x, y) = 2 - y + 2x^2 - x^2y$ is differentiable at every point .*

Solution: Given $f(x, y) = 2 - y + 2x^2 - x^2y$

$f_1 = 4x - 2xy, f_2 = -1 - x^2$.

At $(0, 0)$, $f = 2, f_1 = 0, f_2 = -1$

$$f(\Delta x, \Delta y) - f(0, 0) = 2 - \Delta y + 2(\Delta x)^2 - (\Delta x)^2\Delta y - 2$$

$$= -\Delta y + (2\Delta x - \Delta x\Delta y)\Delta x$$

$$= (-1)\Delta y + (2\Delta x - \Delta x\Delta y)\Delta x$$

$$= f_2(0, 0)\Delta y + (2\Delta x - \Delta x\Delta y)\Delta x$$

$$= f_2(0, 0)\Delta y + (2\Delta x)\Delta x - \Delta x^2\Delta y.$$

Hence we may take

$$\varphi(\Delta x, \Delta y) = 2\Delta x - \Delta x\Delta y \text{ and } \psi(\Delta x, \Delta y) = 0$$

or

$$\varphi(\Delta x, \Delta y) = 2\Delta x \text{ and } \psi(\Delta x, \Delta y) = -(\Delta x)^2.$$

In either case $\varphi(\Delta x, \Delta y)$ and $\psi(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to 0$.

Hence f is differentiable at every point.

Result : If $f(x, y) \in C^1$ in D, then $f(x, y)$ is differentiable at every point of D.

Proof: Suppose that $f(x, y) \in C^1$ in D.

Then f_1 and f_2 are continuous.

Let

$$\varphi(\Delta x, \Delta y) = f_1(a + \vartheta_1\Delta x, b) - f_1(a, b)$$

$$\psi(\Delta x, \Delta y) = f_2(a + \Delta x, b + \vartheta_2\Delta y) - f_2(a, b),$$

By the basic mean value theorem ,we have

$$f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a + \vartheta_1\Delta x, b)\Delta x + f_2(a + \Delta x, b + \vartheta_2\Delta y)\Delta y,$$

where, $0 < \vartheta_1, \vartheta_2 < 1$.

Now

$$\begin{aligned} f(a + \Delta x, b + \Delta y) - f(a, b) &= [\varphi(\Delta x, \Delta y) + f_1(a, b)]\,\Delta x \\ &\quad + [\psi(\Delta x, \Delta y) + f_2(a, b)]\,\Delta y, \end{aligned}$$

which is same as equation (1.11).

Since f_1 and f_2 are continuous, the functions φ and ψ tends to zero as $(\Delta x, \Delta y) \to (0, 0)$.

Hence $f(x, y)$ is differentiable at every point of D.

Example 1.3.12 *Show by examples that continuity at a point need not imply differentiability at that point.*

Solution: (i). Consider $f(x, y) = |x|(1 + y)$.

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0).$$

So $f(x, y)$ is continuous at $(0, 0)$.

But $f_1(0, 0)$ does not exist.

So $f(x, y)$ is not differentiable at $(0, 0)$.

(ii). Consider the function $f(x, y) = \begin{cases} x, & \text{when } |y| < |x| \\ -x, & \text{when } |y| \geq |x| \end{cases}$

$f(x, y)$ is continuous at $(0, 0)$.

Also $f_1(0, 0) = 0, f_2(0, 0) = 0$. Assume that $f(x, y)$ is differentiable at $(0, 0)$.

When $\Delta y = \Delta x$, equation (1.11) would become

$$\begin{aligned} f(\Delta x, \Delta x) &= -\Delta x \\ &= \Delta x + \Delta x\varphi(\Delta x, \Delta x) + \Delta x\psi(\Delta x, \Delta x). \end{aligned}$$

But this is a contradiction,as one sees by canceling Δx and letting $\Delta x \to 0$.

So our assumption is wrong and hence $f(x, y)$ is not differentiable at $(0, 0)$.

Remark : From the above two examples we may conclude that there exist continuous functions which are not differentiable.

Example 1.3.13 *Show by an example that there exist differentiable functions not belonging to* C^1.

Solution: Consider the function $f(x, y) = g(\sqrt{x^2 + y^2})$, $g(x) = x^2 \sin \frac{1}{x}$, $g(0) = 0$.

It is easy to prove that $f_1(0, 0) = f_2(0, 0) = 0$.

Now

$$\begin{aligned} f(\Delta x, \Delta y) &= g\left(\sqrt{\Delta x^2 + \Delta y^2}\right) \\ &= (\Delta x^2 + \Delta y^2) \sin \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \end{aligned}$$

From equation (1.11) we have

$$\begin{aligned} f(\Delta x, \Delta y) &= (\Delta x^2 + \Delta y^2) \sin(\Delta x^2 + \Delta y^2)^{-\frac{1}{2}} \\ &= \varphi(\Delta x, \Delta y)\Delta x + \psi(\Delta x, \Delta y)\Delta y. \end{aligned}$$

If $\varphi = \Delta x \sin(\Delta x^2 + \Delta y^2)^{-\frac{1}{2}}, \quad \psi = \Delta y \sin(\Delta x^2 + \Delta y^2)^{-\frac{1}{2}}$,

then $\varphi(x), \psi(x) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Hence the function is differentiable at $(0,0)$.

To prove the function does not belongs to C^1.

It is enough to show that $f_1(x,x)$ has no limit as $x \to 0^+$.

We have $f_1(x,x) = \frac{1}{\sqrt{2}} g'(x\sqrt{2}), \quad x > 0.$

We can easily prove that $g'(0^+)$ does not

exist.So $f_1(x,x)$ does not belongs to C^1.

1.4 Equality of Cross Derivatives

In this section, we will find under what condition the cross derivatives $f_{12}(x, y)$ and $f_{21}(x, y)$ are equal. Also we shall show that the result is true for all functions of class C^2.

1.4.1 A preliminary Result

We define two operators Δ_x and Δ_y on a function $f(x,y)$ as follows:

$$\Delta_x f(x_0, y_0) = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

$$\Delta_y f(x_0, y_0) = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$$

Lemma 1.4.1 *Prove that for any function $f(x,y)$,*

$$\Delta_x \Delta_y f(x_0, y_0) = \Delta_y \Delta_x f(x_0, y_0).$$

Proof: Consider

$$\begin{aligned}\Delta_x \Delta_y f(x_0, y_0) &= \Delta_x \{f(x_0, y_0 + \Delta y) - f(x_0, y_0)\}\\ &= \Delta_x f(x_0, y_0 + \Delta y) - \Delta_x f(x_0, y_0)\\ &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)\\ &\quad + f(x_0, y_0).\end{aligned}$$

Consider

$$\Delta_y\Delta_x f(x_0, y_0) = \Delta_y \{f(x_0 + \Delta x, y_0) - f(x_0, y_0)\}$$
$$= \Delta_y f(x_0 + \Delta x, y_0) - \Delta_y f(x_0, y_0)$$
$$= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y)$$
$$+ f(x_0, y_0).$$

Hence $\quad \Delta_x\Delta_y f(x_0, y_0) = \Delta_y\Delta_x f(x_0, y_0).$ Q

1.4.2 The principle Result

Theorem 1.4.1

$$f(x, y) \in C^2 \Longrightarrow f_{12}(x, y) = f_{21}(x, y).$$

Proof: Let (x_0, y_0) be an arbitrary point in the domain where $f \in C^2$. Then by the previous lemma, we have

$$\Delta_x\Delta_y f(x_0, y_0) = \Delta_y\Delta_x f(x_0, y_0) \tag{1.12}$$

Let $\varphi(y) = f(x_0 + \Delta x, y) - f(x_0, y)$.

Then $\varphi(y_0) = \Delta_x f(x_0, y_0)$. Applying the law of the mean for functions of one variable,

$$\Delta_y\Delta_x f(x_0, y_0) = \Delta_y \varphi(y_0)$$
$$= \varphi(y_0 + \Delta y) - \varphi(y_0)$$
$$= \varphi'(y_0 + \vartheta_1 \Delta y)\Delta y, \quad 0 < \vartheta_1 < 1.$$

$$\Delta_y\Delta_x f(x_0, y_0) = f_2(x_0 + \Delta x, y_0 + \vartheta_1\Delta y)\Delta y - f_2(x_0, y_0 + \vartheta_1\Delta y)\Delta y \tag{1.13}$$

Let $\psi(x) = f(x, y_0 + \Delta y) - f(x, y_0)$.

Then $\psi(x_0) = \Delta_y f(x_0, y_0)$. Applying the law of the mean for functions of one variable,

$$\Delta_x\Delta_y f(x_0, y_0) = \Delta_x \psi(x_0)$$
$$= \psi(x_0 + \Delta x) - \psi(x_0)$$
$$= \psi'(x_0 + \vartheta_2 \Delta x)\Delta x, \quad 0 < \vartheta_2 < 1.$$

$$\Delta_x\Delta_y f\,(x_0,\ y_0) = f_1(x_0 + \vartheta_2\Delta x,\ y_0 + \Delta y)\Delta x\ -\ f_1(x_0 + \vartheta_2\Delta x,\ y_0)\Delta x \quad (1.14)$$

Applying law of the mean to the right side of equations (1.13) and (1.14), we have

$$\Delta_y\Delta_x f(x_0, y_0) = f_{12}(x_0 + \vartheta_3\Delta x, y_0 + \vartheta_1\Delta y)\Delta y\Delta x,$$

where $0 < \vartheta_3 < 1$ and

$$\Delta_x\Delta_y f(x_0, y_0) = f_{21}(x_0 + \vartheta_2\Delta x, y_0 + \vartheta_4\Delta y)\Delta x\Delta y,$$

where $0 < \vartheta_4 < 1$.

From equation (1.12), we have

$$f_{12}(x_0 + \vartheta_3\Delta x, y_0 + \vartheta_1\Delta y)\Delta y\Delta x = f_{21}(x_0 + \vartheta_2\Delta x, y_0 + \vartheta_4\Delta y)\Delta x\Delta y.$$

Then

$$f_{12}(x_0 + \vartheta_3\Delta x, y_0 + \vartheta_1\Delta y) = f_{21}(x_0 + \vartheta_2\Delta x, y_0 + \vartheta_4\Delta y).$$

Letting $\Delta x \to 0$ and $\Delta y \to 0$ and since f_{12} and f_{21} are continuous at (x_0, y_0), we have

$$f_{12}(x_0, y_0) = f_{21}(x_0, y_0).$$

As (x_0, y_0) is an arbitrary point in the domain, we have

$$f_{12}(x, y) = f_{21}(x, y).$$

Q

We shall now give an example of a function for which the cross derivatives are not equal.

Example 1.4.1 *If* $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 = 0 \\ 0, & (x, y) = (0, 0) \end{cases}$, *Prove that* $f_{12} \quad f_{21}$.

Solution: When $(x,\ y)$ is not the origin, then using formal rules of differentiation we can easily prove that $f_{12} = f_{21}$. When $(x,\ y)$ is the origin, then

$$f_1(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

$$f_2(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0.$$

If $(x, y) \neq (0, 0)$,

$$f_1(x,y) = 2y\frac{(x^2-y^2)}{(x^2+y^2)} + 2xy\frac{(x^2+y^2)2x - (x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= 2y\frac{(x^2-y^2)}{(x^2+y^2)} + 2xy\frac{4xy^2}{(x^2+y^2)^2}, \quad x^2+y^2 \neq 0$$

$$f_2(x,y) = 2x\frac{(x^2-y^2)}{x^2+y^2} - 2xy\frac{4x^2y}{(x^2+y^2)^2}, \quad x^2+y^2 \neq 0$$

$$f_{21}(0,0) = \lim_{\Delta x \to 0} \frac{f_2(\Delta x,0) - f_2(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2$$

$$f_{12}(0,0) = \lim_{\Delta y \to 0} \frac{f_1(0,\Delta y) - f_1(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-2\Delta y}{\Delta y} = -2.$$

Hence at $(0, 0)$, $f_{12}(0, 0) \neq f_{21}(0, 0)$.

Summary

- A function $f(x, y)$ approaches a limit A as x approaches a and y approaches b, that is $\lim_{\substack{x \to a \\ y \to b}} f(x, y) = A \iff$ for each positive number ϵ there is another δ, such that $|f(x, y) - A| < \epsilon$ whenever $|x - a| < \delta$, $|y - b| < \delta$ or $0 < (x - a)^2 + (y - b)^2$.

- $f(x, y) \in C$ at $(a, b) \iff \lim_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$

- $f(x, y) \in C^n$ in a region $R \iff \frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial x^{n-1}\partial y}, \ldots, \frac{\partial^n f}{\partial y^n} \in C$ in R

- **Basic mean value theorem:**

  1. $f(x, y) \in C^1$ in D and
  2. The circle $(x - a)^2 + (y - b)^2 \leq \delta^2$ lies in D

  $\Rightarrow f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a + \vartheta_1 \Delta x, b)\Delta x + f_2(a + \Delta x, b + \vartheta_2 \Delta y)\Delta y$, where $\Delta x^2 + \Delta y^2 < \delta^2$ and $0 < \vartheta_1 < 1$, $0 < \vartheta_2 < 1$.

- Basic mean value theorem can be used to differentiate composite functions.

- The class of differentiable function lie between C and C^1

- Continuity at a point need not imply differentiability at that point
- If $f \in C^1$ in D then f is differentiable at every point
- There exist differentiable functions not belonging to C^1
- f is differentiable $\Rightarrow f \in C$
- There exist nondifferentiable functions having partial derivatives
- $f(x, y) \in C^2 \Leftarrow\Rightarrow f_{12}(x, y) = f_{21}(x, y)$

Multiple Choice questions.

1. Choose the wrong statement.

   (a) Continuity at a point need not imply differentiablity at that point.

   (b) There exist differentiable functions not belonging to C^1.

   (c) Continuity at a point always imply differentiability at that point.

2. If $f(x, y), g(r, s), h(r, s) \in C^1$, then

   (a) $\frac{\partial f(g, h)}{\partial r} = f_1(g, h)g_1(r, s) + f_2(g, h)h_1(r, s)$

   (b) $\frac{\partial f(g, h)}{\partial r} = f_1(g, h)g_2(r, s) + f_2(g, h)h_2(r, s)$

   (c) $\frac{\partial f(g, h)}{\partial s} = f_1(g, h)g_1(r, s) + f_2(g, h)h_1(r, s)$

3. If $f(x, y) = x^2 + y^2$, then the value of $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ is

   a) -1 b) 1 c) 0

Ans: 1. (c) 2. (a) 3.(c)

Exercises 1

1. Define the following :

   (a) limit of a function of two variables.

   (b) continuity of a function of two variables.

(c) interior point, exterior point and boundary point.

(d) the classes C^n for functions of several variables.

(e) $\delta-$ neighbourhood and limit point.

(f) Domain and region.

2. State and prove basic mean value theorem.

3. Find the number ϑ_1 and ϑ_2 in Basic mean value theorem if $f = e^{xy}, a = b = \Delta x = \Delta y = 1$.

   Ans: $\vartheta_1 = \log(e-1), \vartheta_2 = \frac{1}{2}[\log(e^2-1) - \log 2]$.

4. Show that if $f(x, y)$ is differentiable at a point it is continuous there.

   Hint : let $(\Delta x, \Delta y) \to (0, 0)$ in equation (1.11).

5. $f(x, y) \in C^1$ in D, then prove that $f(x, y)$ is differentiable at every point of D.

6. If $f(x, y), g(r, s), h(r, s) \in C^1$ then prove that

$$\frac{\partial}{\partial r} f(g, h) = f_1(g, h)g_1(r, s) + f_2(g, h)h_1(r, s).$$
$$\frac{\partial}{\partial s} f(g, h) = f_1(g, h)g_2(r, s) + f_2(g, h)h_2(r, s).$$

7. If $u = e^v, v = \sin(xyz)$, find $\frac{\partial^2 u}{\partial y \partial z}$.

   Ans: $e^v x\cos(xyz) - e^v x^2yz\sin(xyz) + e^v x^2yz\cos^2(xyz)$.

8. For any function $f(x, y)$, prove that

   $\Delta x \Delta y f(x_0, y_0) = \Delta y \Delta x f(x_0, y_0)$.

9. If $f(x, y) \in C^2$, prove that $f_{12}(x, y) = f_{21}(x, y)$.

10. If $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}, xy \neq 0$ and $f(x, 0) = f(0, y) = 0$, prove that $f_{12}(0, 0) \neq f_{21}(0, 0)$.

11. If $f(x, y) = (x^2 + y^2)\tan^{-1}\frac{y}{x}$ when $x \neq 0$ and $f(x, y) = \frac{\pi}{2}y^2$ when $x = 0$, Show that $f_{12}(0, 0) \neq f_{21}(0, 0)$.

Unit 2

Homogeneous functions and Differentials

Learning Outcomes :

After studying this unit, students will be able

- F To explain homogeneous functions.
- F To understand the concept of total derivatives.
- F To describe the meaning of differentials.
- F To know about directional derivatives and gradient.

2.1 Homogeneous functions

A polynomial in x and y is said to be homogeneous if all its terms are of same degree. We generalize this property to functions of several variables.

Definition 2.1.1 *A function $f(x, y)$ is homogeneous of degree n in a region*

R if, and only if , for (x, y) in R and for every positive value of λ

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \tag{2.1}$$

The number n is positive, negative, or zero and need not be an integer. The region R must be such that $(\lambda x, \lambda y)$ is a point of it for all positive λ whenever (x, y) is a point of it.

Remark : The definition can be extended to a function of any number of variables.

Example 2.1.1 *Consider* $f(x, y) = x^4 + 2xy^3 - 5y^4$.
$f(\lambda x, \lambda y) = (\lambda x)^4 - 2(\lambda x)(\lambda y)^3 - 5(\lambda y)^4 = \lambda^4(x^4 + 2xy^3 - 5y^4) = \lambda^4 f(x, y)$.
Hence $f(x, y)$ is homogeneous of degree 4.

Example 2.1.2 *Consider* $f(x, y) = x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1} \frac{y}{x}$,
Here $n = 1$; R is any quadrant without the axes,
$f(\lambda x, \lambda y) = (\lambda x)^{\frac{1}{3}} (\lambda y)^{-\frac{4}{3}} \tan^{-1} \frac{\lambda y}{\lambda x} = \lambda^{-1} x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1} \frac{y}{x} = \lambda^{-1} f(x, y)$.
Hence $f(x, y)$ is homogeneous of degree -1.

Example 2.1.3 *Consider* $f(x, y) = x^{\frac{1}{2}} y^{-\frac{2}{3}} + x^{\frac{2}{3}} y^{-\frac{1}{3}}$.
$f(\lambda x, \lambda y) = (\lambda x)^{\frac{1}{2}} (\lambda y)^{-\frac{2}{3}} + (\lambda x)^{\frac{2}{3}} (\lambda y)^{-\frac{1}{3}} = \lambda^{-\frac{1}{3}} x^{\frac{1}{3}} y^{-\frac{2}{3}} + \lambda^{\frac{1}{3}} x^{\frac{2}{3}} y^{-\frac{1}{3}}$.
So the function is not homogeneous.

Example 2.1.4 *Consider* $f(x, y) = \sqrt{x^2 + y^2}^{\,3}$,
Here $n = 3$; R is the whole plane.
$f(\lambda x, \lambda y) = \sqrt{(\lambda x)^2 + (\lambda y)^2}^{\,3} = \lambda^3 \sqrt{x^2 + y^2}^{\,3} = \lambda^3 f(x, y) = |\lambda|^3 f(x, y)$.
If λ is a negative number, then equation (1.12) is not satisfied for this function.

Example 2.1.5 *Consider* $f(x, y) = 3 + \log \frac{y}{x}$,
Here $n = 0$; R is the first or third quadrant without the axes.
$f(\lambda x, \lambda y) = 3 + \log \frac{\lambda y}{\lambda x} = 3 + \log \frac{y}{x} = \lambda^0 f(x, y)$.
The function is homogeneous of order 0.

2.2 Euler's Theorem

Theorem 2.2.1 (Euler)

1. $f(x, y) \in C^1$, (x, y) *in* R

2. $f(x, y)$ *is homogeneous of degree* n *in* R.

$$\Rightarrow \quad f_1(x, y)x + f_2(x, y)y = nf(x, y), \quad (x, y) \text{ in } R \qquad (2.2)$$

Proof: Since $f(x, y)$ is homogeneous of degree n, we have

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

Differentiating partially with respect to λ

$$xf_1(\lambda x, \lambda y) + yf_2(\lambda x, \lambda y) = n\lambda^{n-1} f(x, y)$$

Let $\lambda = 1$, we have $xf_1(x, y) + yf_2(x, y) = nf(x, y)$. Q

Remark 1: Some of the authors may define homogeneity in a different way , demanding that equation (2.1) should hold for all real values of λ with this definition the function in example 2.1.4 is not homogeneous.But this definition would have two disadvantages that the converse of Euler's theorem would be false.

We now prove that converse of the Euler's theorem is valid under definition 2.1.1.

Theorem 2.2.2

1. $f(x, y) \in C^1$, (x, y) *in* R.

2. $xf_1 + yf_2 = nf$, (x, y) *in* R.

$\Rightarrow f(x, y)$ *is homogeneous of degree* n, (x, y) *in* R.

Proof: Let (x_0, y_0) be an arbitrary point of R. Then $(\lambda x_0, \lambda y_0) \in R$ for all positive values of λ.

Let $\varphi(\lambda) = f(\lambda x_0, \lambda y_0)$ be defined for all positive values of λ.

Differentiating partially with respect to λ we have

$$\varphi'(\lambda) = x_0 f_1(\lambda x_0, \lambda y_0) + y_0 f_2(\lambda x_0, \lambda y_0).$$

Since $(x_0, y_0) \in R$, from hypothesis 2, we have

$$x_0 f_1(x_0, y_0) + y_0 f_2(x_0, y_0) = nf(x_0, y_0).$$

Since $(\lambda x_0, \lambda y_0) \in R$, replace x_0 by λx_0 and y_0 by λy_0

$$\lambda x_0 f_1(\lambda x_0, \lambda y_0) + \lambda y_0 f_2(\lambda x_0, \lambda y_0) = nf(\lambda x_0, \lambda y_0)$$

$$\lambda\varphi'(\lambda) = n\varphi(\lambda).$$

Differentiating $\varphi(\lambda)\lambda^{-n}$ with respect to λ, we have

$$\begin{aligned}(\varphi(\lambda)\lambda^{-n})' &= \varphi'(\lambda)\lambda^{-n} - n\varphi(\lambda)\lambda^{-n-1}\\ &= \lambda^{-n-1}\,\lambda\varphi'(\lambda) - n\varphi(\lambda)\\ &= 0.\end{aligned}$$

Here $\varphi(\lambda)\lambda^{-n} = C$, a constant.

Let $\lambda = 1$, then $\varphi(1) = C$ and so $f(x_0, y_0) = C$.

Hence using $\varphi(\lambda) = f(\lambda x_0, \lambda y_0)$ in $\varphi(\lambda)\lambda^{-n} = C$, we have

$$f(\lambda x_0, \lambda y_0)\lambda^{-n} = f(x_0, y_0)$$

$$\Longrightarrow f(\lambda x_0, \lambda y_0) = \lambda^n f(x_0, y_0).$$

Since (x_0, y_0) is an arbitrary point of R, the theorem is true for all (x, y) in R. Q

Remark : If $f(x, y)$ is homogeneous of degree n,

we have $x^2 f_{11} + xyf_{12} + y^2 f_{22} = n(n-1)f$.

Example 2.2.1 *Verify Euler's theorem for the function*

$$a)\, u = x^2 + y^2 + 2xy \qquad b)\, u = x^3 + y^3 + z^3 + 3xyz$$

Solution: a) Given $u = x^2 + y^2 + 2xy$.

This is a homogeneous function of degree 2.

To verify Euler's theorem we have to prove

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$\frac{\partial u}{\partial x} = 2x + 2y$$

$$x\frac{\partial u}{\partial x} = 2x^2 + 2xy$$

$$\frac{\partial u}{\partial y} = 2y + 2x$$

$$y\frac{\partial u}{\partial y} = 2y^2 + 2xy$$

Hence, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(x^2 + y^2 + 2xy) = 2u.$$

b) Given $u = x^3 + y^3 + z^3 + 3xyz$.

This is a homogeneous function of degree 3.

To verify Euler's theorem we have to prove

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

$$\frac{\partial u}{\partial x} = 3x^2 + 3yz$$

$$x\frac{\partial u}{\partial x} = 3x^2 + 3xyz$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3xz$$

$$y\frac{\partial u}{\partial y} = 3y^3 + 3xyz$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3xy$$

$$z\frac{\partial u}{\partial z} = 3z^3 + 3xyz$$

Hence, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3(x^3 + y^3 + z^3 + 3xyz)$$

$$= 3u.$$

Example 2.2.2 *If* $u = \tan^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, *Show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u \cos u.$$

Solution:

$$u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\tan u = \frac{x^2+y^2}{x+y} = f(\text{say})$$

Then $\tan u$ is a homogeneous function of degree 1.

Hence by Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 1.\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \frac{\sin u\cos^2 u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u\cos u$$

Example 2.2.3 *If* $u = \sin^{-1}\left(\frac{x^2+y^2}{(x+y)}\right)$*, Show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$

Solution: Given $u = \sin^{-1}\left(\frac{x^2+y^2}{(x+y)}\right) = \Rightarrow \sin u = \frac{x^2+y^2}{(x+y)} = f$ (say).

f is homogeneous function of degree 1.

So by Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\text{Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

Example 2.2.4 *If* $u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$*, then prove that*

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}.$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{3}{2}.$

Solution: $u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow e^u = \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}} = f$ (say).

Then f is homogeneous function of degree $\frac{3}{2}$.

To prove (i). By Euler's theorem for f, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{3}{2}f$$

$$x\frac{\partial}{\partial x}e^u + y\frac{\partial}{\partial y}e^u = \frac{3}{2}e^u$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = \frac{3}{2}e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}.$$

Differentiating (i) partially with respect to x.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 0.$$

Multiply by x

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = 0.$$

Differentiating (i) partially with respect to y

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0.$$

Multiply by y

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 0.$$

Adding above two resulting equations, and using (i) we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{3}{2} = 0$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{3}{2}.$$

2.3 Total derivatives

Let $x = \varphi(t), y = \psi(t)$ define two functions for t, $t_0 < t < t_1$. Then the pair (x, y) define a corresponding region R in the $xy-$ plane.

Let u be the function of x and y defined on the region R such that $u = f(x, y)$ and $x = \varphi(t), y = \psi(t)$, for $t_0 \le t \le t_1$.

Now u may vary because of variations in t and $\frac{du}{dt}$ at a point, if it exists, is called the total derivative of u with respect to t.

Let Δx, Δy and Δu be the corresponding changes, for a change Δt in t.

We have,

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

and

$$\frac{du}{dt} = \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}.$$

Now $\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$.

Therefore,

$$\begin{aligned}
\frac{du}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta t \to 0}} \frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta t} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta t \to 0}} \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)}{\Delta t} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta t \to 0}} \left[\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \frac{\Delta x}{\Delta t} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \frac{\Delta y}{\Delta t} \right]
\end{aligned}$$

So,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

Remark 1: If u is a function of x, y, z which are functions of a single variable t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Remark 2: Suppose $t = x$, that is u is a function of x and y, where y is itself a function of x, then we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

Example 2.3.1 *Find* $\frac{du}{dx}$ *if* $u = \sin(x^2 + y^2)$ *where* $a^2x^2 + b^2y^2 = c^2$.

Solution:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

Given $u = \sin(x^2 + y^2)$

$$\Rightarrow \frac{\partial u}{\partial x} = \cos(x^2 + y^2)2x \text{ and } \frac{\partial u}{\partial y} = \cos(x^2 + y^2)2y.$$

Consider $a^2x^2 + b^2y^2 = c^2$. Differentiating with respect to x, we have

$$2a^2x + 2b^2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{a^2x}{b^2y}.$$

Now

$$\frac{du}{dx} = 2x\cos(x^2 + y^2) + 2y\cos(x^2 + y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2 + y^2)\left(1 - \frac{a^2}{b^2}\right).$$

Example 2.3.2 *Find* $\frac{du}{dt}$ *if* $u = x^3y^4$ *where* $x = t^3$ *and* $y = t^2$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 3x^2y^4(3t^2) + 4x^3y^3(2t)$$

$$= 3t^6t^8(3t^2) + 4t^9t^6(2t)$$

$$= 9t^{16} + 8t^{16}$$

$$= 17t^{16}.$$

Example 2.3.3 *If* $u = x^2y^3$ *where* $x = \log t$ *and* $y = e^t$, *find* $\frac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 2xy^3.\frac{1}{t} + 3x^2y^2.e^t$$

$$= 2\log t.e^{3t}.\frac{1}{t} + 3(\log t)^2e^{2t}.e^t$$

$$= \frac{2\log t.e^{3t}}{t} + 3(\log t)^2.e^{3t}$$

Example 2.3.4 *If* $u = \sin(e^x + y)$, $x = f(t)$, $y = g(t)$, *compute* $\dfrac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \cos(e^x + y)e^x f'(t) + \cos(e^x + y)g'(t).$$

Example 2.3.5 *If* $u = xy^z$, $x = t$, $y = t^2$, $z = t^3$, *compute* $\dfrac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = y^z + zxy^{z-1}(2t) + xy^z \log y(3t^2)$$

$$= y^z + 2zxty^{z-1} + 3xy^z t^2 \log y.$$

Example 2.3.6 *If* $u = f(x, y)$, $x = g(r, s)$, $y = h(r, s)$, $r = \varphi(t)$, $s = \psi(t)$, *find* $\dfrac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= f_1(x, y)\left(\frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial s}\frac{ds}{dt}\right) + f_2(x, y)\left(\frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial s}\frac{ds}{dt}\right)$$

$$= f_1\left(g_1\varphi' + g_2\psi'\right) + f_2\left(h_1\varphi' + h_2\psi'\right).$$

2.4 Differentials

We shall introduce briefly the idea of differential of a function of several variables. It will be sufficient to give our definitions for functions of two variables.

2.4.1 The Differential

Definition 2.4.1 *Let $u = f(x, y)$ be a function of C^1, where x and y are independent variables. Form the following function of four variables:*

$$\varphi(x, y, r, s) = f_1(x, y)r + f_2(x, y)s.$$

If $r = \Delta x, s = \Delta y$ are variables whose range is a neighborhood of $r = 0, s = 0$, then the differential du of u, is defined as $\varphi(x, y, \Delta x, \Delta y)$:

$$du = \varphi(x, y, \Delta x, \Delta y) = f_1(x, y)\Delta x + f_2(x, y)\Delta y \tag{2.3}$$

Thus there is associated with each point (x, y) where $f(x, y) \in C^1$, a differential which is itself a linear function of two variables $\Delta x, \Delta y$.

Example 2.4.1 *Compute the differential du for the function $u = f(x, y) = \frac{x}{y}$.*

Solution:

$$u = f(x, y) = \frac{x}{y},$$

$$f_1 = \frac{1}{y}, \quad f_2 = -\frac{x}{y^2}.$$

$$\varphi(x, y, r, s) = f_1(x, y)r + f_2(x, y)s = \frac{r}{y} - \frac{sx}{y^2}.$$

$$du = f_1(x, y)\Delta x + f_2(x, y)\Delta y = \frac{\Delta x}{y} - \frac{x\Delta y}{y^2}.$$

Example 2.4.2 *Compute the differential du for the function $u = f(g(x, y), h(x, y))$.*

Solution: $du = (f_1 g_1 + f_2 h_1)\Delta x + (f_1 g_2 + f_2 h_2)\Delta y.$

2.4.2 Meaning of the Differential

The equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) of the surface is

$$z - z_0 = f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)(y - y_0).$$

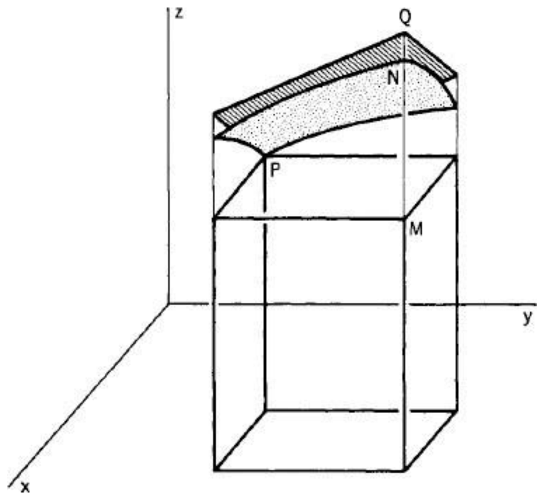

Figure 2.1

By the definition of differential,

$$dz = f_1(x, y)\Delta x + f_2(x, y)\Delta y.$$

dz at (x_0, y_0) is

$$dz = f_1(x_0, y_0)\Delta x + f_2(x_0, y_0)\Delta y.$$

The point Q lies on that plane and its coordinates are $(x_0 + \Delta x, y_0 + \Delta y, z_0 + dz)$. If

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0),$$

then the point $N(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$, lies on the surface $z = f(x, y)$. Hence $MN = \Delta z, MQ = dz$.

That is, $|dz| =$ the length of the ordinate $x = x_0 + \Delta x, y = y_0 + \Delta y$ cutoff between the tangent plane and the plane $z = z_0$.

It is clear from the property of a tangent plane that dz will be nearly equal to Δz for small values of Δx and Δy.

So for simplicity we have assumed that $f(x, y) \in C^1$. But if we assume only that $f(x, y)$ is differentiable at (a, b), the differential df is equally well defined at (a, b) by equation (2.3). Then from the equation

$$f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a, b)\Delta x + f_2(a, b)\Delta y + \varphi(\Delta x, \Delta y)\Delta x + \psi(\Delta x, \Delta y)\Delta y,$$

where $\varphi(\Delta x, \Delta y)$ and $\psi(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.
We have

$$\Delta u = \Delta f = du + \Delta x \varphi(\Delta x, \Delta y) + \Delta y \psi(\Delta x, \Delta y).$$

So du is nearly equal to Δu, when $(\Delta x, \Delta y)$ is near $(0, 0)$.
Thus,

$$\frac{|\Delta u - du|}{|\Delta x| + |\Delta y|} \leq |\varphi(\Delta x, \Delta y)| + |\psi(\Delta x, \Delta y)|$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u - du}{|\Delta x| + |\Delta y|} = 0.$$

Example 2.4.3 *Find approximately how much x^2+y^3 changes when (x, y) changes from $(1, 1)$ to $(1.1, 0.9)$.*

Solution: Let $u = x^2 + y^3$.
Then

$$f_1(x, y) = 2x, \quad f_2(x, y) = 3y^2 \quad \text{and}$$

$$\Delta x = 1.1 - 1 = 0.1, \quad \Delta y = 0.9 - 1 = -0.1.$$

$$du = f_1(x, y)\Delta x + f_2(x, y)\Delta y$$

$$\text{So } \; d(x^2 + y^3) = 2x\Delta x + 3y^2\Delta y.$$

$$d(x^2 + y^3)_{at\,(1,1)} = 2\Delta x + 3\Delta y.$$

Approximate change in $(x^2 + y^3) = |2(0.1) + 3(-0.1)| = 0.1$.
Actual change in $x^2 + y^3 = 2 - (1.1^2 + 0.9^3) = 0.061$.

2.5 Directional Derivatives

The partial derivatives describes the rate of change of a function in the direction of each coordinate axis. A natural generalization of partial derivatives is the directional derivative, which studies the rate of change of a function in an arbitrary direction.

In the definition of $f_1(x_0, y_0)$, the numerator of the difference quotient used involves the values of $f(x, y)$ at two points $(x_0 + \Delta x, y_0)$ and (x_0, y_0). As Δx approaches zero , the first point approaches the latter along the line $y = y_0$. For $f_2(x_0, y_0)$ a point $(x_0, y_0 + \Delta x)$ approaches (x_0, y_0) along the line $x = x_0$. In the definition of directional derivative we replace these two special lines by an arbitrary line through (x_0, y_0).

A direction ξ_α is defined as the direction of any directed line which makes an angle α with the positive $x-$ axis (positive angle measured in the counterclockwise sense). For example the line segment directed from the point (0, 0) to the point $(-1, -1)$ has the direction $\xi_{\frac{5\pi}{4}}$ or $\xi_{-\frac{3\pi}{4}}$.

Definition 2.5.1 *The directional derivative of $f(x, y)$ in the direction ξ_α at (a, b) is*

$$\frac{\partial f}{\partial \xi_\alpha}\Big|_{(a,b)} = \lim_{\Delta s \to 0} \frac{f(a + \Delta s\cos\alpha, b + \Delta s\sin\alpha) - f(a, b)}{\Delta s}.$$

Example 2.5.1 *Find directional derivative of $f(x, y) = x^2 - 2y$ in the direction $\xi_{\frac{3\pi}{4}}$ at* (1, 2).

Solution: Given $f(x, y) = x^2 - 2y$.

$$f(a + \Delta s\cos\alpha, b + \Delta s\sin\alpha) = (a + \Delta s\cos\alpha)^2 - 2(b + \Delta s\sin\alpha).$$

At $a = 1, b = 2, \alpha = \frac{3\pi}{4}$,

the above value becomes $\left(1 - \frac{\Delta s}{\sqrt{2}}\right)^2 - 2\left(2 + \frac{\Delta s}{\sqrt{2}}\right)$,

$f(a, b) = a^2 - 2b$, $f(1, 2) = 1 - 4 = -3$.

$$\frac{\partial f}{\partial \xi_{\frac{3\pi}{4}}}\Big|_{(1,2)} = \lim_{\Delta s \to 0} \frac{\left(1 - \frac{\Delta s}{\sqrt{2}}\right)^2 - 2\left(2 + \frac{\Delta s}{\sqrt{2}}\right) + 3}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{1 - \frac{2\Delta s}{\sqrt{2}} + \frac{(\Delta s)^2}{2} - 4 - \frac{2\Delta s}{\sqrt{2}} + 3}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \left(\frac{\Delta s}{2} - 2\sqrt{2}\right) = -2\sqrt{2}.$$

Remark 1: At each point (x, y) a function has infinitely many directional derivatives so that $\frac{\partial f}{\partial \xi_\alpha}$ is a function of the three variables x, y, α.

Remark 2: In computing a directional derivative of higher order, the variable α must be held constant.

For example, if $\frac{\partial f}{\partial \xi_\alpha} = x\cos\alpha + y\sin\alpha$, then

$$\frac{\partial^2 f}{\partial \xi_\alpha^2} = \frac{\partial}{\partial \xi_\alpha}\left(\frac{\partial f}{\partial \xi_\alpha}\right)$$

$$= \lim_{\Delta s \to 0} \frac{(x + \Delta s\cos\alpha)\cos\alpha + (y + \Delta s\sin\alpha)\sin\alpha - x\cos\alpha - y\sin\alpha}{\Delta s}$$

$$= \cos^2\alpha + \sin^2\alpha = 1.$$

We see that, $\frac{\partial f}{\partial \xi_0} = f_1$, $\frac{\partial f}{\partial \xi_{\frac{\pi}{2}}} = f_2$, $\frac{\partial f}{\partial \xi_\pi} = -f_1$, $\frac{\partial f}{\partial \xi_{\frac{3\pi}{2}}} = -f_2$.

Theorem 2.5.1 $f(x, y) \in C^1$

$$\Rightarrow \frac{\partial f}{\partial \xi_\alpha} = f_1(x, y)\cos\alpha + f_2(x, y)\sin\alpha.$$

Proof: By the Basic mean value theorem, we have

$$\frac{f(a + \Delta s\cos\alpha, b + \Delta s\sin\alpha) - f(a, b)}{\Delta s} = f_1(a + \vartheta_1\Delta s\cos\alpha, b)\cos\alpha$$
$$+ f_2(a + \Delta s\cos\alpha, b + \vartheta_2\Delta s\sin\alpha)\sin\alpha$$

where $0 < \vartheta_1 < 1, 0 < \vartheta_2 < 1$.

Taking limit as Δs approaches zero, we obtain,

$$\frac{\partial f}{\partial \xi_\alpha} = f_1(x, y)\cos\alpha + f_2(x, y)\sin\alpha.$$

Q

Remark : This theorem enables us to compute directional derivatives without reverting to the defining limiting process.

In example (2.5.1), we have $f(x, y) = x^2 - 2y, f_1(x, y) = 2x, f_2(x, y) = -2$, so for any point (x, y) and any direction α,

$$\frac{\partial f}{\partial \xi_\alpha} = f_1(x, y)\cos\alpha + f_2(x, y)\sin\alpha = 2x\cos\alpha - 2\sin\alpha.$$

In particular, for $x = 1, y = 2, \alpha = \frac{3\pi}{4}$,

$$\left.\frac{\partial f}{\partial \xi_\alpha}\right|_{(1,2)} = 2\cos\frac{3\pi}{4} - 2\sin\frac{3\pi}{4} = 2\left(-\frac{1}{\sqrt{2}}\right) - 2\left(\frac{1}{\sqrt{2}}\right) = \frac{-4}{\sqrt{2}} = -2\sqrt{2}.$$

Also, we have

$$\frac{\partial^2 f}{\partial \xi_\alpha^2} = 2\cos^2\alpha, \qquad \frac{\partial^3 f}{\partial \xi_\alpha^3} = 0.$$

Definition 2.5.2 *A vector $\vec{r}$ is a triple of numbers (r_1, r_2, r_3). Its length is $|\vec{r}| = |\bar{r}| = (r_1^2+r_2^2+r_3^2)^{\frac{1}{2}}$. The direction cosines of the vector are $\frac{r_1}{|\bar{r}|}, \frac{r_2}{|\bar{r}|}, \frac{r_3}{|\bar{r}|}$. Its components are r_1, r_2, r_3. It is clear that a vector is completely determined by its length and its direction cosines.*

We now define directional derivatives for functions of three variables.

Definition 2.5.3 *Let $\vec{a} = (a_1, a_2, a_3)$ be a given vector and $\vec{r} = (r_1, r_2, r_3)$ a given point. Let ξ_a be the direction of the vector. Let its direction cosines be $\cos\alpha_1, \cos\alpha_2, \cos\alpha_3$.*

Then the directional derivative of the function $F(x_1, x_2, x_3)$ at the point $\vec{r}$ in the direction ξ_a will be $\frac{\partial F}{\partial \xi_a}\Big|_{(r_1,r_2,r_3)} = \frac{\partial F}{\partial \xi_a}(r_1, r_2, r_3)$,

$$\frac{\partial F}{\partial \xi_a}(r_1, r_2, r_3) = \lim_{\Delta s \to 0} \frac{F(r_1 + \Delta s\cos\alpha_1, r_2 + \Delta s\cos\alpha_2, r_3 + \Delta s\cos\alpha_3) - F(r_1, r_2, r_3)}{\Delta s}.$$

For example if $\vec{a}$ is taken successively as (1, 0, 0), (0, 1, 0), (0, 0, 1), then $\frac{\partial F}{\partial \xi_a}$ is successively the partial derivatives $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}$.

Now we state a theorem analogous to that of theorem 2.5.1 without proof.

Theorem 2.5.2

$$F(x_1, x_2, x_3) \in C^1 \Rightarrow \frac{\partial F}{\partial \xi_a} = \frac{\partial F}{\partial x_1}\cos\alpha_1 + \frac{\partial F}{\partial x_2}\cos\alpha_2 + \frac{\partial F}{\partial x_3}\cos\alpha_3.$$

Example 2.5.2 *Compute* $\frac{\partial F}{\partial \xi_a}$ *at* $\vec{r} = (1, 1, -1)$ *for the function* $F = x_1^2 - x_2^2 + 2x_2x_3$ *in the direction* $\vec{a} = (1, 0, -2)$.

Solution: $F = x_1^2 - x_2^2 + 2x_2x_3 \quad \vec{r} : (1, 1, -1), \quad \vec{a} : (1, 0, -2).$

The direction cosines of $\vec{a}$ are $\frac{1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{-2}{\sqrt{5}}$.

Then

$$\frac{\partial F}{\partial \xi_a} = \frac{\partial F}{\partial x_1}\cos\alpha_1 + \frac{\partial F}{\partial x_2}\cos\alpha_2 + \frac{\partial F}{\partial x_3}\cos\alpha_3$$

$$= 2x_1\frac{1}{\sqrt{5}} + 2x_2\frac{0}{\sqrt{5}} + 2x_2\left(-\frac{2}{\sqrt{5}}\right)$$

$$\frac{\partial F}{\partial \xi_a}\Big|_{(1,1,-1)} = \frac{2}{\sqrt{5}} + 0 - \frac{4}{\sqrt{5}} = -\frac{2}{\sqrt{5}}.$$

That is F is decreasing at a rate $\frac{2}{\sqrt{5}}$ in the direction ξ_a.

Consider the function $\vec{f} : R^n \to R^m$. We shall find the directional derivative of this function at $\vec{a}$ in the direction $\vec{u}$.

Definition 2.5.4 *The directional derivative of $\vec{f} : S \to R^m$ where $S \subset R^n$, at $\vec{a}$ in the direction $\vec{u}$, denoted by the symbol $\vec{f}'(\vec{a}, \vec{u})$ is defined by the equation*

$$\vec{f}'(\vec{a}, \vec{u}) = \lim_{h \to 0} \frac{\vec{f}(\vec{a} + h\vec{u}) - \vec{f}(\vec{a})}{h},$$

whenever the limit on the right exists.

Definition 2.5.5 (Operator $\vec{\nabla}$.) *The operator $\vec{\nabla}$ is a symbolic vector with components $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$. It may be applied to a scalar function $F(x_1, x_2, x_3)$ or to a vector function $\vec{y}(x_1, x_2, x_3)$ with components $y_i(x_1, x_2, x_3)$, $i = 1, 2, 3$.*

Definition 2.5.6 *$\vec{\nabla}F(x_1, x_2, x_3)$ is a vector function with components $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}$. It is called gradient of F : $Grad\, F = \vec{\nabla}F$.*

Definition 2.5.7

$$\vec{\nabla}.\vec{y} = \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} + \frac{\partial y_3}{\partial x_3}.$$

This scalar function is called the divergence of the vector function $\vec{y}$: $Div\, \vec{y} = \vec{\nabla}.\vec{y}$.

Definition 2.5.8 *$\vec{\nabla} \times \vec{y}$ is a vector function with components*

$$\frac{\partial y_3}{\partial x_2} - \frac{\partial y_2}{\partial x_3}, \frac{\partial y_1}{\partial x_3} - \frac{\partial y_3}{\partial x_1}, \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}$$

This vector function is called the $Curl$ of the vector function $\vec{y}$: $Curl\, \vec{y} = \vec{\nabla} \times \vec{y}$.

2.5.1 The Gradient

Definition 2.5.9 *The gradient of $f(x, y)$ at a point (a, b), $Grad f(x, y)|_{(a,b)}$ is a vector of magnitude $\left(f_1(a, b)^2 + f_2(a, b)^2\right)^{\frac{1}{2}}$ in the direction ξ_{α_1}, defined by the equations*

$$\sin\alpha_1 = \frac{f_2}{\sqrt{f_1^2 + f_2^2}}, \quad \cos\alpha_1 = \frac{f_1}{\sqrt{f_1^2 + f_2^2}} \tag{2.4}$$

$$\sin\alpha_2 = -\frac{f_2}{\sqrt{f_1^2 + f_2^2}}, \quad \cos\alpha_2 = -\frac{f_1}{\sqrt{f_1^2 + f_2^2}} \tag{2.5}$$

Example 2.5.3 *Find $Grad f(x, y)|_{(1,3)}$ for the function*

$$f(x, y) = x^2 - xy + y^2.$$

Solution: Consider $f(x, y) = x^2 - xy + y^2$.

$$f_1(x, y) = 2x - y, f_2(x, y) = -x + 2y.$$

The magnitude of $Grad f(x, y)|_{(1,3)}$ is

$$\left(f_1(1, 3)^2 + f_2(1, 3)^2\right)^{\frac{1}{2}} = (1 + 25)^{\frac{1}{2}} = \sqrt{26}.$$

The direction ξ_{α_1} of $Grad f(x, y)|_{(1,3)}$ is

$$\sin\alpha_1 = \frac{f_2}{\sqrt{f_1^2 + f_2^2}} = \frac{5}{\sqrt{26}}, \quad \cos\alpha_1 = \frac{f_1}{\sqrt{f_1^2 + f_2^2}} = -\frac{1}{\sqrt{26}}.$$

Example 2.5.4 *Find the $Grad f(x, y)|_{(3,4)}$ for the function $f(x, y) = x^2 + y^2$.*

Solution: Given

$$f(x, y) = x^2 + y^2$$

$$f_1(x, y) = 2x, \quad f_2(x, y) = 2y$$

The magnitude of $Grad f(x, y)|_{(3,4)} = \left(f_1(3, 4)^2 + f_2(3, 4)^2\right)^{\frac{1}{2}}$

$$= (36 + 64)^{2} = 10.$$

The direction ξ_{α_1} of $Grad\, f(x,y)|_{(3,4)}$ is

$$\sin\alpha_1 = \frac{f_2}{\sqrt{f_1^2+f_2^2}} = \frac{4}{5}, \qquad \cos\alpha_1 = \frac{f_1}{\sqrt{f_1^2+f_2^2}} = \frac{3}{5}.$$

Theorem 2.5.3

1. $f(x,y) \in C^1$

2. $f_1(a,b)^2 + f_2(a,b)^2 \neq 0$

$$\Rightarrow \max_{0\le\alpha\le 2\pi} \frac{\partial f}{\partial \xi_\alpha}\Big|_{(a,b)} = (f_1(a,b)^2 + f_2(a,b)^2)^{\frac{1}{2}} = \frac{\partial f}{\partial \xi_\alpha}\Big|_{(a,b)},$$

where ξ_{α_1} is the direction of $Grad f(x,y)|_{(a,b)}$ defined by equation (2.4) and (2.5).

Proof: Since $f(x,y) \in C^1$,

$$\frac{\partial f}{\partial \xi_\alpha} = f_1(x,y)\cos\alpha + f_2(x,y)\sin\alpha.$$

For a fixed point (a,b), we determine the direction ξ_α which will make $\frac{\partial f}{\partial \xi_\alpha}$ a maximum.

Let $F(\alpha) = f_1(a,b)\cos\alpha + f_2(a,b)\sin\alpha.$

Then $F(\alpha)$ will have a maximum or minimum when $F'(\alpha) = 0.$

So

$$-f_1\sin\alpha + f_2\cos\alpha = 0.$$

Case (i). If f_1 and f_2 are not both zero.

Then the above equation will have just two distinct solutions α_1 and α_2 between ϑ and 2π determined by the equations

$$\sin\alpha_1 = \frac{f_2}{\sqrt{f_1^2+f_2^2}}, \qquad \cos\alpha_1 = \frac{f_1}{\sqrt{f_1^2+f_2^2}}$$

$$\sin\alpha_2 = -\frac{f_2}{\sqrt{f_1^2+f_2^2}}, \qquad \cos\alpha_2 = -\frac{f_1}{\sqrt{f_1^2+f_2^2}}$$

For these directions, we have

$$\frac{\partial f}{\partial \xi_{\alpha_1}} = \frac{f_1.f_1}{\sqrt{f_1^2+f_2^2}} + \frac{f_2.f_2}{\sqrt{f_1^2+f_2^2}} = \sqrt{f_1^2+f_2^2}$$

and

$$\frac{\partial f}{\partial \xi_{\alpha_2}} = \sqrt{f_1^2+f_2^2}.$$

Hence $\frac{\partial f}{\partial \xi_\alpha}$ is maximum in the direction ξ_{α_1}, and is minimum in the direction ξ_{α_2}.

Here α_1 and α_2 differ by π.

Case (ii). If $f_1 = f_2 = 0$,

Then the maximum and minimum values of $\frac{\partial f}{\partial \xi_\alpha}$ are both zero,since the directional derivative is constantly zero. Q

Summary

- A function $f(x, y)$ is homogeneous of degree n in a region $R \Longleftrightarrow$ for (x, y) in R and for every positive value of λ, $f(\lambda x, \lambda y) = \lambda^n f(x, y)$, the number n is positive, negative and need not be an integer

- **Euler's theorem :**

  1. $f(x, y) \in C^1$, (x, y) in R
  2. $f(x, y)$ is homogeneous of degree n in R.

  $$\Rightarrow \quad f_1(x, y)x + f_2(x, y)y = nf(x, y), \quad (x, y) \text{ in } R$$

- Let $x = \varphi(t), y = \psi(t)$ define two functions for t, $t_0 < t < t_1$. Then the pair (x, y) define a corresponding region R in the $xy-$ plane.

  Let u be the function of x and y defined on the region R such that $u = f(x, y)$ and $x = \varphi(t), y = \psi(t)$, for $t_0 \leq t \leq t_1$.

  Now u may vary because of variations in t and $\frac{du}{dt}$ at a point, if it exists, is called the total derivative of u with respect to t

- Let $u = f(x, y)$ be a function of C^1, where x and y are independent variables. Form the following function of four variables:

  $$\varphi(x, y, r, s) = f_1(x, y)r + f_2(x, y)s.$$

If $r = \Delta x, s = \Delta y$ are variables whose range is a neighborhood of $r = 0, s = 0$, then the differential du of u, is defined as $\varphi(x, y, \Delta x, \Delta y)$:

$$du = \varphi(x, y, \Delta x, \Delta y) = f_1(x, y)\Delta x + f_2(x, y)\Delta y$$

- There is associated with each point (x, y), where $f \in C^1$, a differential which itself is a linear function of two variables $\Delta x, \Delta y$.

- The directional derivative of $f(x, y)$ in the direction ξ_α at (a, b) is
$$\frac{\partial f}{\partial \xi_\alpha}\Big|_{(a,b)} = \lim_{\Delta s \to 0} \frac{f(a + \Delta s\cos\alpha, b + \Delta s\sin\alpha) - f(a, b)}{\Delta s}.$$

- $f(x, y) \in C^1 \Rightarrow \dfrac{\partial f}{\partial \xi_\alpha} = f_1(x, y)\cos\alpha + f_2(x, y)\sin\alpha.$

- The gradient of $f(x, y)$ at a point (a, b), $Grad\, f(x, y)|_{(a,b)}$ is a vector of magnitude $\left(f_1(a, b)^2 + f_2(a, b)^2\right)^{\frac{1}{2}}$ in the direction ξ_{α_1}, defined by the equations
$$\sin\alpha_1 = \frac{f_2}{\sqrt{f_1^2 + f_2^2}}, \quad \cos\alpha_1 = \frac{f_1}{\sqrt{f_1^2 + f_2^2}}$$
$$\sin\alpha_2 = -\frac{f_1}{\sqrt{f_1^2 + f_2^2}}, \quad \cos\alpha_2 = \frac{f_2}{\sqrt{f_1^2 + f_2^2}}$$

Multiple Choice questions.

1. The function $f(x, y) = x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\frac{y}{x}$ is homogeneous of degree.

   $a)\ \frac{1}{3}$ $\quad b)\ \frac{-4}{3}$ $\quad c)\ -1$

2. Suppose u is a function of x and y where y is itself a function of x then

   (a) $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}.\dfrac{dy}{dx}$

   (b) $\dfrac{du}{dx} = \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial x}.\dfrac{dy}{dx}$

   (c) $\dfrac{du}{dx} = \dfrac{\partial u}{\partial y}.\dfrac{dy}{dx}$

3. If $f(x, y) \in C^1$ then $\dfrac{\partial f}{\partial \xi_\alpha}$ is

   (a) $f_1(x, y) + f_2(x, y)$

(b) $f_1(x, y) \cos \alpha + f_2(x, y) \sin \alpha$

(c) $f_1(x, y) \sin \alpha + f_2(x, y) \cos \alpha$

4. What is the magnitude of the gradient of $f(x, y)$ at a point (a, b)

(a) $(f_1(a, b)^2 + f_2(a, b)^2)^{\frac{1}{2}}$

(b) $(f_1(a, b)^2 + f_2(a, b)^2)$

(c) $(f_1(a, b) + f_2(a, b))$

5. The directional derivatives describes

(a) The rate of change of a function in the direction of each coordinate axes.

(b) The rate of change of a function in an arbitrary direction.

(c) The rate of change of a function in fixed direction.

6. If $u = f(x, y) = \frac{x}{y}$, then the value of du is

(a) $\frac{\Delta x}{y} - \frac{x \Delta y}{y^2}$

(b) $\Delta x - x\Delta y$

(c) $\frac{\Delta x}{y^2} - \frac{x \Delta y}{y}$

Ans: 1. (c) 2. (a) 3. (b) 4. (a) 5. (b) 6. (a)

Exercises 2

1. Define homogeneous function of degree n.

2. Which of the following functions are homogeneous ? Also find its degree .

   a) $\sqrt{x} - \sqrt{y}$

   b) $\log y - \log x$

   c) $(x^3 + y^3)^{\frac{2}{3}}$

   d) $\left[\frac{x+y}{xy} + x^{\frac{2}{3}} e^{\frac{x}{y}}\right] y^{-\frac{5}{3}}$

   e) $xf\left(\frac{y}{x}\right) + yg\left(\frac{x}{y}\right)$

   Ans: a) $n = \frac{1}{2}$, **b)** $n = 0$, **c)** $n = 2$, **d) no, e) n=1.**

3. State and prove Euler's theorem and its converse.

4. Verify Euler's theorem for

   (i) $f(x, y) = 3 + \log\left(\frac{y}{x}\right)$.

   (ii) $f(x, y) = x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}\left(\frac{y}{x}\right)$.

5. Explain the concept of total derivatives.

6. If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

   Ans: $1 + \log(xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$

7. If $u = e^{mx}(y - z)$, $y = m\sin x$ and $z = \cos x$, find $\frac{du}{dx}$.

   Ans: $e^{mx}(m^2 + 1)\sin x$.

8. Explain the meaning of differential.

9. Define (i) directional derivative, (ii) gradient.

10. If $f(x, y) \in C^1$ prove that $\frac{\partial f}{\partial \xi_\alpha} = f_1(x, y)\cos\alpha + f_2(x, y)\sin\alpha$.

11. If

    (a) $f(x, y) \in C^1$

    (b) $f_1(a, b)^2 + f_2(a, b)^2 \neq 0$

prove that

$$\max_{0\leq\alpha\leq2\pi} \frac{\partial f}{\partial \xi_\alpha}|_{(a,b)} = (f_1(a,b)^2 + f_2(a,b)^2)^{\frac{1}{2}} = \frac{\partial f}{\partial \xi_\alpha}|_{(a,b)},$$

where ξ_{α_1} is the direction of $Grad\, f(x,y)|_{(a,b)}$

BLOCK II

Implicit Functions and Inverse Functions

Unit 3

Jacobians

Learning Outcomes :

Upon completion of this unit, students will acquire knowledge

- F To understand the notation of Jacobian.
- F To distinguish between dependent and independent variables.
- F To find inverse of a transformation.
- F To find the relationship between Jacobians of a transformation and its inverse
- F To use the change of variable property of Jacobians.

In this unit we discuss the method of finding the derivatives of the solutions of the system of equations, assumed to exist. We can use Cramer's rule for solving simultaneous linear equations.

In solving simultaneous equations we may come across determinants whose elements are partial derivatives. If the order of the determinants is higher than two it is worth having a notation for them. Hence the Jacobians were introduced.

3.1 Jacobians

Theorem 3.1.1

1. $F(u, v, x, y), G(u, v, x, y), f(x, y), g(x, y) \in C^1$

2. $F(f(x, y), g(x, y), x, y) \equiv 0, \quad G(f(x, y), g(x, y), x, y) \equiv 0$

3. $\Delta = \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} \neq 0$

$$\Rightarrow f_1 = -\frac{\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\Delta}, \quad g_1 = -\frac{\begin{vmatrix} F_1 & F_3 \\ G_1 & G_3 \end{vmatrix}}{\Delta},$$

$$f_2 = -\frac{\begin{vmatrix} F_4 & F_2 \\ G_4 & G_2 \end{vmatrix}}{\Delta}, \quad g_2 = -\frac{\begin{vmatrix} F_1 & F_4 \\ G_1 & G_4 \end{vmatrix}}{\Delta},$$

Proof: Consider $F(f(x, y), g(x, y), x, y) \equiv 0, G(f(x, y), g(x, y), x, y) \equiv 0$. Differentiating partially with respect to x we have

$$F_1(f(x, y), g(x, y), x, y)f_1(x, y) + F_2(f(x, y), g(x, y), x, y)g_1(x, y) + F_3(f(x, y), g(x, y), x, y) = 0$$

and

$$G_1(f(x, y), g(x, y), x, y)f_1(x, y) + G_2(f(x, y), g(x, y), x, y)g_1(x, y) + G_3(f(x, y), g(x, y), x, y) = 0.$$

We can solve the following equations for f_1 and g_1 by Cramer's rule.

$$F_1f_1 + F_2g_1 + F_3 = 0$$

$$\text{and} \quad G_1f_1 + G_2g_1 + G_3 = 0$$

Given that,

$$\Delta = \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} = F_1G_2 - F_2G_1 \neq 0.$$

Then $f_1 = \dfrac{\Delta_1}{\Delta}$, where $\Delta_1 = \begin{vmatrix} -F_3 & F_2 \\ -G_3 & G_2 \end{vmatrix} = -\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}.$

$g_1 = \dfrac{\Delta_2}{\Delta}$, where $\Delta_2 = \begin{vmatrix} F_1 & -F_3 \\ G_1 & -G_3 \end{vmatrix} = -\begin{vmatrix} F_1 & F_3 \\ G_1 & G_3 \end{vmatrix}.$

Hence $f_1 = -\dfrac{\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\Delta}$, $g_1 = -\dfrac{\begin{vmatrix} F_1 & F_3 \\ G_1 & G_3 \end{vmatrix}}{\Delta}$.

Similarly considering

$F(f(x, y), g(x, y), x, y) \equiv 0,\ G(f(x, y), g(x, y), x, y) \equiv 0.$

Differentiating partially with respect to y we have

$$F_1f_2 + F_2g_2 + F_4 = 0$$

$$G_1f_2 + G_2g_2 + G_4 = 0.$$

Solving for f_2 and g_2 by Cramer's rule we have

$$f_2 = -\frac{\begin{vmatrix} F_4 & F_2 \\ G_4 & G_2 \end{vmatrix}}{\Delta},\quad g_2 = -\frac{\begin{vmatrix} F_1 & F_4 \\ G_1 & G_4 \end{vmatrix}}{\Delta}.$$ Q

The notation for jacobian is illustrated below:

Illustration 1:

Consider three functions F, G, H of six variables $u, v, w, x, y,$ ⍰ appearing in that order. The Jacobian of F, G, H with respect to $u, w,$ ⍰ is

$$\frac{\partial(F, G, H)}{\partial(u, w, ⍰)} = \begin{vmatrix} F_1 & F_3 & F_6 \\ G_1 & G_3 & G_6 \\ H_1 & H_3 & H_6 \end{vmatrix}.$$

Illustration 2:

Suppose we add a fourth function K of the same six variables then

$$\frac{\partial(G, F, K, H)}{\partial(w, x, \square, u)} = \begin{vmatrix} G_3 & G_4 & G_6 & G_1 \\ F_3 & F_4 & F_6 & F_1 \\ K_3 & K_4 & K_6 & K_1 \\ H_3 & H_4 & H_6 & H_1 \end{vmatrix}$$

Remark 1: Jacobians are often prove useful in obtaining partial derivatives of implicit functions. We could express the results of theorem 2.4.1 in Jacobian notation.

Consider $F(u, v, x, y) = 0,\ G(u, v, x, y) = 0,\ u = f(x, y)$ and $v = g(x, y)$.

$$\text{Then } f_1 = \frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}}, \quad where \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} \; 0.$$

In Jocobian notation we have,

$$f_1 = \frac{\partial u}{\partial x} = \frac{-\dfrac{\partial(F, G)}{\partial(x, v)}}{\dfrac{\partial(F, G)}{\partial(u, v)}},$$

Similarly

$$f_2 = \frac{\partial u}{\partial y} = \frac{-\dfrac{\partial(F, G)}{\partial(y, v)}}{\dfrac{\partial(F, G)}{\partial(u, v)}}, \quad g_1 = \frac{\partial v}{\partial x} = \frac{-\dfrac{\partial(F, G)}{\partial(u, x)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} \quad \text{and} \quad g_2 = \frac{\partial v}{\partial y} = \frac{-\dfrac{\partial(F, G)}{\partial(u, y)}}{\dfrac{\partial(F, G)}{\partial(u, v)}}.$$

Remark 2: We now express the results of theorem 3.1.1 using the same rule. We have considered the function $F(x, y, \square) = 0$, where $\square = f(x, y)$.

$$f_1 = -\frac{F_1}{F_3}, \quad f_2 = -\frac{F_2}{F_3}, \quad where\ F_3 \neq 0.$$

That is $f_1 = \dfrac{\partial \square}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial \square}}$ and $f_2 = \dfrac{\partial \square}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial \square}}$

Illustration 3: Consider the system

$F(u, v, w, x) = 0, G(u, v, w, x) = 0, H(u, v, w, x) = 0.$

Let u, v, w be the dependent variables and x is the independent variable. Since there is single independent variable, the method gives us the total derivatives $\frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx}$.

We have, for $\frac{\partial(F,G,H)}{\partial(u,v,w)} \neq 0$,

$$\frac{du}{dx} = \frac{-\frac{\partial(F,G,H)}{\partial(x,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}},$$

$$\frac{dv}{dx} = \frac{-\frac{\partial(F,G,H)}{\partial(u,x,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}},$$

$$\frac{dw}{dx} = \frac{-\frac{\partial(F,G,H)}{\partial(u,v,x)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}.$$

Illustration 4: Consider the system

$$F(u, v, w, x, y, z) = 0$$

$$F(u, v, w, x, y, z) = 0$$

$$F(u, v, w, x, y, z) = 0$$

$$F(u, v, w, x, y, z) = 0$$

and let u, v, w, x be the dependent variables.

then

$$\frac{\partial x}{\partial z} = \frac{-\frac{\partial(F,G,H,K)}{\partial(u,v,w,z)}}{\frac{\partial(F,G,H,K)}{\partial(u,v,w,x)}}, \quad \text{where } \frac{\partial(F,G,H,K)}{\partial(u,v,w,x)} \quad 0.$$

Remark : We can observe that the number of dependent variables is equal to number of simultaneous equations.

The following results concerning Jacobians are found to be useful in the problems of change of variable.

Illustration 5:

Let $f(u, v, w, x, y, z) = 0, g(u, v, w, x, y, z) = 0, h(u, v, w, x, y, z) = 0$. The above equations define three functions u, v, w of the variables x, y, z. Then

$$f_1\frac{\partial u}{\partial x} + f_2\frac{\partial v}{\partial x} + f_3\frac{\partial w}{\partial x} = -f_4.$$

$$g_1\frac{\partial u}{\partial x} + g_2\frac{\partial v}{\partial x} + g_3\frac{\partial w}{\partial x} = -g_4.$$

$$h_1\frac{\partial u}{\partial x} + h_2\frac{\partial v}{\partial x} + h_3\frac{\partial w}{\partial x} = -h_4.$$

Solving these linear equations, we obtain

$$\frac{\partial u}{\partial x} = -\frac{\partial(f, g, h)}{\partial(x, v, w)} \cdot \frac{1}{\Delta}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial(f, g, h)}{\partial(u, x, w)} \cdot \frac{1}{\Delta}$$

$$\frac{\partial w}{\partial x} = -\frac{\partial(f, g, h)}{\partial(u, v, x)} \cdot \frac{1}{\Delta} \quad \text{where } \Delta = \frac{\partial(f, g, h)}{\partial(u, v, w)} \neq 0.$$

Example 3.1.1 *If* $u^2 - v = 3x + y$ *and* $u - 2v^2 = x - 2y$, *find* $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$.

Solution: The given equations are

$$F(u, v, x, y) = u^2 - v - 3x - y = 0,$$

$$G(u, v, x, y) = u - 2v^2 - x + 2y = 0.$$

Then we have

$$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}}$$

$$= \frac{-\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}}$$

$$= \frac{-\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4v \end{vmatrix}}$$

$$= \frac{1 - 12v}{1 - 8uv}, \quad 1 - 8uv \neq 0.$$

$$\frac{\partial v}{\partial x} = \frac{-\dfrac{\partial(F, G)}{\partial(u, x)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = \frac{-\begin{vmatrix} F_1 & F_3 \\ G_1 & G_3 \end{vmatrix}}{\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}}$$

$$= \frac{-\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4v \end{vmatrix}}$$

$$= \frac{2u - 3}{1 - 8uv}, \quad 1 - 8uv \quad 0.$$

Similarly we can find $\dfrac{\partial u}{\partial y} = \dfrac{-2 - 4v}{1 - 8uv}, \quad \dfrac{\partial v}{\partial y} = \dfrac{-4u - 1}{1 - 8uv}, \quad 1 - 8uv \quad 0.$

3.2 Dependent and independent variables

In the statement of a given problem involving several variables, it is not always possible to determine from the notation which variables are intended to be independent and which dependent. One must then state clearly which variables are dependent and which variables are independent or else one must treat all possible cases.

If a partial derivative, such as $\dfrac{\partial y}{\partial x}$, appears in the statement of a problem, we may be sure that one of the dependent variables is y and one of the independent variables is x. We shall illustrate by use of a number of examples.

3.2.1 Illustration 1:

Find $\frac{\partial u}{\partial x}$ if

$$u = f(x, y),$$

$$y = g(x, \Box).$$

Here u is the dependent variable and x is the independent variable.

We can have only two cases, since there must be two dependent variables corresponding to the two equations.

Case (i).

Dependent variables : $u, \Box$

Independent variables : x, y

Differentiating

$$\begin{aligned} u &= f(x, y) \\ y &= g(x, \Box) \end{aligned} \qquad (3.1)$$

partially with respect to x, we have

$$\frac{\partial u}{\partial x} = f_1, \quad 0 = g_1 + g_2 \frac{\partial \Box}{\partial x}.$$

So, we get

$$\frac{\partial u}{\partial x} = f_1,$$

$$\frac{\partial \Box}{\partial x} = -\frac{g_1}{g_2}, \quad where\ g_2 = 0.$$

Case (ii).

Dependent variables : u, y

Independent variables : $x, \Box$

Differentiating (2.4) partially with respect to x, we have

$$\frac{\partial u}{\partial x} = f_1 + f_2 \frac{\partial y}{\partial x},$$

$$\frac{\partial y}{\partial x} = g_1.$$

Hence

$$\frac{\partial u}{\partial x} = f_1 + f_2 g_1, \quad \frac{\partial y}{\partial x} = g_1.$$

Sometimes, we can use the following notations to distinguish between such cases:

Case (i). $\dfrac{\partial u_{x,y}}{\partial x}, \dfrac{\partial \square_{x,y}}{\partial x}.$

Case (ii). $\dfrac{\partial u_{x,\square}}{\partial x}, \dfrac{\partial y_{x,\square}}{\partial x}.$

The independent variables are used as subscripts against the dependent ones.

3.2.2 Illustration 2:

Consider $u = f(x, y), \quad y = g(x, \square)$.

Find $\dfrac{\partial u}{\partial y}$.

Case (i). $\dfrac{\partial u_{y,\square}}{\partial y}, \dfrac{\partial x_{y,\square}}{\partial y}.$

$$\frac{\partial u}{\partial y} = f_1 \frac{\partial x}{\partial y} + f_2,$$

$$1 = g_1 \frac{\partial x}{\partial y}$$

$$\frac{\partial u_{y,\square}}{\partial y} = f_2 + \frac{f_1}{g_1},$$

$$\frac{\partial x_{y,\square}}{\partial y} = \frac{1}{g_1}, \quad \text{where } g_1 \quad 0.$$

Case (ii). $\dfrac{\partial u_{y,x}}{\partial y}, \dfrac{\partial \square_{y,x}}{\partial y}.$

In this case the two equations are independent of each other. The first equation defines u and the second equation defines $\square$.

$$\frac{\partial u_{y,\square}}{\partial y} = f_2, \qquad \frac{\partial \square_{y,x}}{\partial y} = \frac{1}{g_2}, \quad g_2 \quad 0.$$

3.2.3 Illustration 3:

Find $\dfrac{\partial y}{\partial x}$, if

$$v = f(x, y, \square), x = g(y, u, v).$$

Case (i). $\dfrac{\partial y_{x,u,v}}{\partial x}$.

The second equation alone is sufficient.

$$1 = g_1 \frac{\partial y}{\partial x} \Rightarrow \frac{\partial y}{\partial x} = \frac{1}{g_1}, \quad g_1 \neq 0.$$

Case (ii). $\dfrac{\partial y_{x,z,v}}{\partial x}$

The first equation alone is sufficient

$$f_1 + f_2 \frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{\partial y}{\partial x} = -\frac{f_1}{f_2}, \quad f_2 \neq 0.$$

Case (iii). $\dfrac{\partial y_{x,z,u}}{\partial x}$

Both equations are necessary

$$f_2 \frac{\partial y}{\partial x} - \frac{\partial v}{\partial x} = -f_1, \quad g_1 \frac{\partial y}{\partial x} + g_3 \frac{\partial v}{\partial x} = 1$$

Then

$$\frac{\partial y_{x,z,u}}{\partial x} = \frac{1 - f_1 g_3}{g_1 + f_2 g_3}, \quad g_1 + f_2 g_3 \neq 0.$$

3.3 The inverse of a transformation

A set of equations of the form

$$u = f(x, y, z), v = g(x, y, z), w = h(x, y, z)$$

is known as a transformation. These equations transforms a point with coordinates (x, y, z) into another with coordinates (u, v, w). If these equations can be solved for x, y, z, we have three functions of u, v, w. The three corresponding equations constitute the inverse of the original transformation. They would give explicitly the point or points (x, y, z) from which (u, v, w) could have come in the original transformation.

By using the following method we obtain the derivatives of x, y, z with respect to u, v, w without actually knowing the inverse transformation.

Let

$$F(u, v, w, x, y, z) = u - f(x, y, z).$$

$$G(u, v, w, x, y, z) = v - g(x, y, z).$$

$$H(u, v, w, x, y, z) = w - h(x, y, z).$$

Now

$$\frac{\partial y}{\partial w} = \frac{-\dfrac{\partial(F, G, H)}{\partial(x, w, z)}}{\dfrac{\partial(F, G, H)}{\partial(x, y, z)}}$$

$$= \frac{\begin{vmatrix} f_1 & 0 & f_3 \\ g_1 & 0 & g_3 \\ h_1 & 1 & h_3 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}}$$

$$= \frac{-\dfrac{\partial(f, g)}{\partial(x, z)}}{\dfrac{\partial(f, g, h)}{\partial(x, y, z)}}, \quad \text{where } \frac{\partial(f, g, h)}{\partial(x, y, z)} \neq 0.$$

Example 3.3.1 *If* $x = 4u + 3v, \quad y = 3u + 2v$, *find* $\dfrac{\partial u}{\partial y}$ *and* $\dfrac{\partial v}{\partial y}$.

Solution:

Method (i). Instead of using the above formula, we can find the partial derivative by direct method.

Differentiating the above equations partially with respect to y we have,

$$0 = 4\frac{\partial u}{\partial y} + 3\frac{\partial v}{\partial y},$$

$$1 = 3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}$$

Solving for $\dfrac{\partial u}{\partial y}$, we have

$$8\frac{\partial u}{\partial y} + 6\frac{\partial v}{\partial y} = 0,$$

$$9\frac{\partial u}{\partial y} + 6\frac{\partial v}{\partial y} = 3$$

So

$$\frac{\partial u}{\partial y} = 3.$$

Solving for $\frac{\partial v}{\partial y}$, we have

$$\frac{\partial v}{\partial y} = -4.$$

Method (ii). To find the partial derivative by finding the inverse transformation

$$x = 4u + 3v$$

$$\Rightarrow 4u = x - 3v$$

$$\text{and } y = 3u + 2v$$

$$\Rightarrow 2v = y - 3u.$$

So

$$4u = x - 3 \; \frac{y - 3u}{2}$$

$$= x - \frac{3y}{2} + \frac{9u}{2}$$

$$\frac{-u}{2} = \frac{2x - 3y}{2}$$

$$\Rightarrow u = -2x + 3y$$

and

$$v = \frac{y}{2} - \frac{3}{2}(-2x + 3y) = 3x - 4y.$$

Then $\frac{\partial u}{\partial y} = 3$ and $\frac{\partial v}{\partial y} = -4.$

Example 3.3.2 *If* $F(u, v, g(u, v, x)) = 0$, $G(u, v, h(u, v, y)) = 0$, *find* $\frac{\partial u}{\partial y}$.

Solution: Differentiating the above equations partially with respect y, we have

$$F_1\frac{\partial u}{\partial y} + F_2\frac{\partial v}{\partial y} + F_3 \;\; g_1\frac{\partial u}{\partial y} + g_2\frac{\partial v}{\partial y} \;\; = 0.$$

$$G_1\frac{\partial u}{\partial y} + G_2\frac{\partial v}{\partial y} + G_3 \;\; h_1\frac{\partial u}{\partial y} + h_2\frac{\partial v}{\partial y} + h_3 \;\; = 0.$$

That is

$$(F_1 + F_3 g_1)\frac{\partial u}{\partial y} + (F_2 + F_3 g_2)\frac{\partial v}{\partial y} = 0.$$

$$(G_1 + G_3 h_1)\frac{\partial u}{\partial y} + (G_2 + G_3 h_2)\frac{\partial v}{\partial y} = -G_3 h_3.$$

To find $\frac{\partial u}{\partial y}$ we must solve the above equations. This will be possible if

$$\begin{vmatrix} F_1 + F_3 g_1 & F_2 + F_3 g_2 \\ G_1 + G_3 h_1 & G_2 + G_3 h_2 \end{vmatrix} \neq 0.$$

3.4 Relationship between Jacobians of a transformation and its inverse

Result 1: Consider the transformation

$$\left.\begin{aligned} u &= f(x, y) \\ v &= g(x, y) \end{aligned}\right\} \tag{3.2}$$

with Jacobian $J = \frac{\partial(u, v)}{\partial(x, y)} \neq 0$. The inverse of the transformation (3.2) have the Jacobian $j = \frac{\partial(x, y)}{\partial(u, v)}$.

We now find the relationship between the Jacobian of the given transformation (3.2) and its inverse. Computing the derivatives,we have

$$\frac{\partial x}{\partial u} = \frac{g_2}{J},$$

$$\frac{\partial y}{\partial u} = -\frac{g_1}{J},$$

$$\frac{\partial x}{\partial v} = -\frac{-f_2}{J},$$

$$\frac{\partial y}{\partial v} = \frac{f_1}{J}.$$

So that

$$j = \begin{vmatrix} g_2 & -g_1 \\ -f_2 & f_1 \end{vmatrix} \frac{1}{J^2} = \frac{J}{J^2} = \frac{1}{J}$$

$$J.j = 1.$$

Hence

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

The above can be generalized to three functions.

Result 2: Consider the transformation

$$u = f(x, y, z)$$
$$v = g(x, y, z), \quad J = \frac{\partial(u, v, w)}{\partial(x, y, z)}, \quad j = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$
and $$w = h(x, y, z)$$

For the determinant $J = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$,

the determinant of co-factors is given by $K = \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix}$.

For example, the co-factor of g_3 is G_3, $G_3 = -\begin{vmatrix} f_1 & f_2 \\ h_1 & h_2 \end{vmatrix}$.

Then $\frac{\partial x}{\partial u} = \frac{F_1}{J}$, $\frac{\partial y}{\partial u} = \frac{F_2}{J}$, $\frac{\partial z}{\partial u} = \frac{F_3}{J}$ with similar equations for the derivatives with respect to v and w.

Then

$$j = \frac{K}{J^3}.$$

But

$$JK = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{vmatrix} = J^3$$

we have $JK = J^3$
$$JK = \frac{K}{j}.$$

so that $Jj = 1$.

Example 3.4.1 *If* $x = r\cos\vartheta$, $y = r\sin\vartheta$, *find* $\frac{\partial(r, \vartheta)}{\partial(x, y)}$.

Solution:

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

$$\frac{\partial(x,y)}{\partial(r,\vartheta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \vartheta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \vartheta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix}$$

$$= r(\cos^2\vartheta + \sin^2\vartheta)$$

$$= r$$

we have,

$$\frac{\partial(x,y)}{\partial(r,\vartheta)} \cdot \frac{\partial(r,\vartheta)}{\partial(x,y)} = 1$$

Hence,

$$\frac{\partial(r,\vartheta)}{\partial(x,y)} = \frac{1}{r}.$$

Example 3.4.2 *If* $u = x + y + z$, $uv = y + z$, $uvw = z$, *then show that* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{u^2 v}$.

Solution: Given that $u = x + y + z$, $uv = y + z$, $uvw = z$

$$u = x + y + z$$

$$= x + uv$$

$$\Rightarrow x = u - uv$$

$$y = uv - z$$

$$y = uv - uvw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1 - v)(u^2v - u^2vw + u^2vw) + u(v^2u - v^2uw + uv^2w)$$

$$= (1 - v)u^2v + u^2v^2$$

$$= u^2v^2$$

Since $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = 1,$

we have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{u^2v}.$

Example 3.4.3 *If* $x = uv,\ y = \dfrac{u}{v}$, *prove that* $\dfrac{\partial(x, y)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(x, y)} = 1.$

Solution:

$$x = uv, \quad y = \frac{u}{v}$$

$$\Rightarrow xy = u^2$$

$$\Rightarrow u = \sqrt{xy} = (xy)^{\frac{1}{2}}$$

Also $\dfrac{y}{x} = \dfrac{1}{v^2}$

$$\Rightarrow v = \sqrt{\frac{x}{y}}$$

Now

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{2u}{v}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & -\frac{1}{2}\sqrt{\frac{x}{y}} \cdot \frac{1}{y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & -\frac{x}{2vy^2} \end{vmatrix}$$

$$= -\frac{2x}{4uvy}$$

$$= -\frac{x}{2xy}$$

$$= -\frac{1}{2y}$$

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = -\frac{2u}{v} \quad -\frac{1}{2y}$$

$$= \frac{y}{y}$$

$$= 1.$$

3.5 Change of Variable

If

$$u = f(x, y), v = g(x, y)$$

and

$$x = \varphi(r, s), y = \psi(r, s),$$

then u and v may be regarded as functions of r and s.
We compute the Jacobian $\frac{\partial(u, v)}{\partial(r, s)}$.
Direct computation gives

$$\frac{\partial(u, v)}{\partial(r, s)} = \begin{vmatrix} f_1\varphi_1 + f_2\psi_1 & g_1\varphi_1 + g_2\psi_1 \\ f_1\varphi_2 + f_2\psi_2 & g_1\varphi_2 + g_2\psi_2 \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(r, s)}.$$

This result is analogous with the formula for the differentiation of a composite function of one variable. It generalizes easily to functions of more variables.

Example 3.5.1 *Find the value of the Jacobian* $\frac{\partial(u, v)}{\partial(r, \vartheta)}$, *where* $u = x^2 - y^2$, $v = 2xy$ *and* $x = r\cos\vartheta$, $y = r\sin\vartheta$.

Solution:

$$u = x^2 - y^2, \quad v = 2xy$$

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2 + y^2) = 4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\vartheta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \vartheta} \end{vmatrix} = \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix}$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r.$$

Then $$\frac{\partial(u,v)}{\partial(r,\vartheta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\vartheta)}$$

$$= 4r^2.r$$

$$= 4r^3.$$

Summary

- If

1. $f(x,y), F(x,y,?) \in C^1$
2. $F(x,y,f(x,y)) \equiv 0, \quad (x,y)$ in D and
3. $F_3(x,y,f(x,y)) \neq 0, \quad (x,y)$ in D

$$\Rightarrow f_1(x,y) = -\frac{F_1(x,y,f(x,y))}{F_3(x,y,f(x,y))}$$

$$f_2(x,y) = -\frac{F_2(x,y,f(x,y))}{F_3(x,y,f(x,y))}$$

- Jacobian is a determinant whose constituents are the derivatives of a number of functions with respect to each of the same number of variables

- Jacobians are useful in obtaining partial derivatives of implicit functions

- Jacobian of three functions F, G, H of six variables u, v, w, x, y, z approaches in that order, respect to u, w, z is

$$\frac{\partial(F, G, H)}{\partial(u, w, z)} = \begin{vmatrix} F_1 & F_3 & F_6 \\ G_1 & G_3 & G_6 \\ H_1 & H_3 & H_6 \end{vmatrix}$$

- In the statement of a given problem involving several variables, it is not always possible to determine from the notation which variables are intended to be independent and which dependent. We must state clearly which variables are dependent and which are independent or else one must treat all possible cases.

- A set of equations of the form $u = f(x, y, z)$, $v = g(x, y, z)$, $w = h(x, y, z)$ transforms a point with coordinates (x, y, z) into another with coordinates (u, v, w). If these equations can be solved for x, y, z we have three functions of u, v, w. The three corresponding equations constitute the inverse of the original transformation.

- The relationship between the Jacobians J and j of a transformations and its inverse is $J.j = 1$.

- **Change of variable**

  If $u = f(x, y), v = g(x, y)$ and $x = \varphi(r, s), y = \psi(r, s)$ we have $\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}$

Multiple Choice questions

1. The relationship between the Jacobians J and j of a transformation and its inverse is

   a) $J - j = 1$ b) $J + j = 1$ c) $Jj = 1$

2. In the notation $\frac{\partial u_{x,y}}{\partial x}$ what does the subscripts represents

   a) x and y are dependent variables

b) x and y are independent variables

c) x is dependent variable and y is independent variable

3. If u and v are functions of x and y while x and y are functions of r and s, then which of the following is an example of a chain rule for Jacobians.

a) $\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}$

b) $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}$

c) Both (a) and (b)

4. If u and v are functions of x and y such that $\frac{\partial(u, v)}{\partial(x, y)} = -4$ then

(a) $\frac{\partial(x, y)}{\partial(u, v)} = -4$

(b) $\frac{\partial(x, y)}{\partial(u, v)} = 4$

(c) $\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{4}$

5. If $x = e^u \cos v, y = e^u \sin u$ then the value of $\frac{\partial(x, y)}{\partial(u, v)}$ at $u = 1$ and $v = 0$ is

(a) $\frac{1}{e}$

(b) e

(c) e^2

Ans: 1. c) 2. b) 3. a) 4. c) 5. c)

Exercises 3

1. If $F(u, v, x, y), G(u, v, x, y), f(x, y), g(x, y) \in C^1$,

   $F(f(x, y), g(x, y), x, y) \equiv 0, G(f(x, y), g(x, y), x, y) \equiv 0,$

   $\Delta = \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} \neq 0$, prove that

$$f_1 = -\frac{\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\Delta}, \quad g_1 = -\frac{\begin{vmatrix} F_1 & F_3 \\ G_1 & G_3 \end{vmatrix}}{\Delta},$$

$$f_2 = -\frac{\begin{vmatrix} F_4 & F_2 \\ G_4 & G_2 \end{vmatrix}}{\Delta}, \quad g_2 = -\frac{\begin{vmatrix} F_1 & F_4 \\ G_1 & G_4 \end{vmatrix}}{\Delta},$$

2. Consider five functions P, Q, R, S, T of six variables u, v, w, x, y, z appearing in that order. Find the Jacobian of P, Q, R, S, T with respect to u, w, x, z.

3. Find $\frac{\partial u}{\partial x}$ if $u = f(x, y), y = g(x, z)$, where u is the dependent variable and x is the independent variable.

4. If $x^2 + y^2 + z^2 + u^2 = 1, xy - zu = 2$ compute $\frac{\partial z_{xy}}{\partial x}$ and $\frac{\partial z_{xu}}{\partial x}$.

   Ans: $\frac{xz + uy}{u^2 - z^2}, \quad \frac{y^2 - x^2}{yu + xz}$

5. If J and j are Jacobians of a transformation and its inverse then prove that $J.j = 1$

6. If $u = f(x, y), v = g(x, y)$ and $x = \varphi(r, s), y = \psi(r, s)$ then prove that $\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}$

Unit 4

Inverse Functions and Implicit Functions

Learning Outcomes :

Upon completion of this unit, students will acquire knowledge

- F To differentiate implicit functions.
- F To understand the concept of functional dependence of two functions.
- F To state and prove inverse function theorem for single variable.
- F To state and prove the existence theorem for implicit functions.

4.1 Inverse functions

We shall confine ourselves to the existence theorem and inverse function theorem for single variable only.

So far we have seen situations in which a function has constructed as the inverse of an already known function.

For example, the equation $x = \sin y$ is used to define y as a function arcsin x of x. This illustration shows that for a given x, there are inifinitely many values of y and we are to keep to single valued functions we must impose some restriction on the permitted values of y.

We shall now give an existence theorem which assure us that, if certain simple conditions are fulfilled, we can obtain a new function inverse to a known function.

To proceed further we need the following definition and intermediate value theorem.

Definition 4.1.1 *f is increasing for $a \leq x \leq b$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 such that $a \leq x_1 \leq x_2 \leq b$. If $f(x_1) < f(x_2)$, we say that f is strictly increasing.*

Theorem 4.1.1 (Intermediate value theorem) *Suppose that f is continuous in the closed interval $[a, b]$ and that $f(a) \neq f(b)$. Then f takes every value which lies between $f(a)$ and $f(b)$.*

Theorem 4.1.2 (Inverse function theorem) *Let $y = f(x)$ be continuous and strictly increasing for $a \leq x \leq b$. If, for a given x in $a < x < b$, $f'(x) \neq 0$, then the inverse function $x = g(y)$ is differentiable for the corresponding value of y and $g'(y) = \frac{1}{f'(x)}$.*

Proof: We shall first prove the existence theorem:

Existence theorem:

Let f be continuous and strictly increasing for $a \leq x \leq b$. Let $f(a) = c$, $f(b) = d$. Then there is a function g, continuous and strictly increasing for $c \leq y \leq d$, such that $f(g(y)) = y$ so that $g(y)$ is the function inverse to $f(x)$.

Proof of existence theorem:

Let k be any number such that $c < k < d$.

Then by intermediate value theorem, there is a value h such that $f(h) = k$.

Since f is strictly increasing, there is only one such h corresponding to a given k.

The inverse function g is defined by $h = g(k)$.

To prove g is strictly increasing:

Let $y_1 < y_2$ and $y_1 = f(x_1)$, $y_2 = f(x_2)$.

Then x_1 and x_2 are uniquely defined.

If $x_1 < x_2$, then, since f is increasing, $f(x_2) \leq f(x_2)$ that is $y_1 \leq y_2$ which contradicts the assumption $y_1 < y_2$.

So $x_1 < x_2$ and g is strictly increasing.

To prove that g is continuous :

Given $\epsilon > 0$, let $f(h - \epsilon) = k_1$ and $f(h + \epsilon) = k_2$

Then, Since f is increasing, $k_1 < k < k_2$ and $h - \epsilon < g(y) < h + \epsilon$ if $k_1 < y < k_2$.

Since ϵ is arbitrary, g is continuous at $y = k$.

Here k is any number in the open interval (c, d). A similar argument establishes one sided continuity at the end points c and d.

Proof of the main theorem:

If $h \neq 0$ is given, define k by $y + k = f(x + h)$.

Then $k \; /= 0$ and, if k is given, h is determined uniquely from $g(y+k) = x+h$. This shows that

$$\frac{g(y + k) - g(y)}{k} = \frac{h}{k} = \frac{h}{f(x + h) - f(x)}.$$

Let $k \to 0$. Then, since g is continuous $h \to 0$.

Hence, we have

$$g'(y) = \frac{1}{f'(x)}.$$

Q

Remark : We now state the inverse function theorem (without proof) for vector valued functions which represents one of the most important consequences of analysis.

Theorem 4.1.3 *Assume $\vec{f} = (f_1, f_2, ..., f_n) \in C^1$ on a open set S in R^n and let $T = \vec{f}(S)$. If the Jacobian $J_{\vec{f}}(\vec{a}) \neq 0$ for some point $\vec{a}$ in S, then there are two open sets $X \subseteq S, Y \subseteq T$ and a uniquely determined function $\vec{g}$ such that*

a) $\vec{a} \in X$ *and* $\vec{f}(\vec{a}) \in Y$

b) $Y = \vec{f}(X)$

c) $\vec{f}$ *is one-one on* X

d) $\vec{g}$ *is defined on* Y, $\vec{g}(Y) = X$ *and* $\vec{g}\left[\vec{f}(\vec{x})\right] = \vec{x}$ *for every* $\vec{x} \in X$

e) $\vec{g} \in C^1$ *on* Y

4.2 Implicit Functions

In unit 1 we have studied briefly the method of obtaining the derivatives of functions defined implicitly. We now study the method in more detail. Consider an equation of the form

$$F(x, y, z) = 0 \tag{4.1}$$

It cannot necessarily be solved for one of the variables in terms of the other two . For example, the equation $x^2 + y^2 + z^2 + a^2 = 0$ has no solution if $a \neq 0$. Even if $a = 0$, the equation does not define z as a function of (x, y) in any domain but only at the point $(0, 0)$.

We shall give later a sufficient condition that there should be a solution. Here we shall discuss the method of finding the derivatives of the implicit function if it is known to exist. We shall assume that $z = f(x, y)$ exists and satisfy equation (4.1)

$$F(x, y, f(x, y)) = 0$$

We shall compute the partial derivatives of $f(x, y)$ in terms of F.

4.2.1 Differentiation of Implicit functions

Theorem 4.2.1

1. $f(x, y), F(x, y, z) \in C^1$
2. $F(x, y, f(x, y)) \equiv 0, \quad (x, y)$ *in* D *and*
3. $F_3(x, y, f(x, y)) \neq 0, \quad (x, y)$ *in* D

$$\Rightarrow f_1(x, y) = -\frac{F_1(x, y, f(x, y))}{F_3(x, y, f(x, y))}$$
$$f_2(x, y) = -\frac{F_2(x, y, f(x, y))}{F_3(x, y, f(x, y))}$$

Proof: Consider $F(x, y, f(x, y)) \equiv 0.$

Differentiating partially with respect to x and y, we have

$$F_1(x, y, f(x, y)) + F_3(x, y, f(x, y)).f_1(x, y) = 0$$
$$\text{and} \quad F_2(x, y, f(x, y)) + F_3(x, y, f(x, y)).f_2(x, y). = 0$$

We have

$$f_1(x, y) = -\frac{F_1(x, y, f(x, y))}{F_3(x, y, f(x, y))}$$
$$\text{and} \quad f(x, y) = -\frac{F_2(x, y, f(x, y))}{F_3(x, y, f(x, y))}$$

Q

Example 4.2.1 *If* $F(x, y, z) = x^2 + y^2 + z^2 - 6$, *compute* $\frac{\partial z}{\partial x}$ *at* $(1, -1, 2)$.

Solution: Equation (4.1) now defines the two explicit functions

$z = \sqrt{6 - x^2 - y^2}, \quad z = -\sqrt{6 - x^2 - y^2}.$

From Theorem 4.2.1, we have

$$f_1(x, y) = -\frac{F_1(x, y, f(x, y))}{F_3(x, y, f(x, y))}.$$

$F_1(x, y, f(x, y)) = 2x \Rightarrow F_1(1, -1, 2) = 2.$

$F_3(x, y, f(x, y)) = 2z \Rightarrow F_3(1, -1, 2) = 4.$

Now $\dfrac{\partial z}{\partial x} = -\frac{2}{4} = -\frac{1}{2}$.

By the explicit method,

$$z = \sqrt{6 - x^2 - y^2} = (6 - x^2 - y^2)^{\frac{1}{2}},$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(6 - x^2 - y^2\right)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{6 - x^2 - y^2}}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,-1,2)} = -\frac{1}{\sqrt{6-1-1}} = -\frac{1}{2}.$$

Remark : Consider the equation

$$F(x, y) = 0. \tag{4.2}$$

Suppose y is a function of x, then we can compute its derivative in terms of F.

We have

$$F_1 + F_2\frac{dy}{dx} = 0 \tag{4.3}$$

$$\frac{dy}{dx} = \frac{-F_1}{F_2}, \quad F_2 \neq 0$$

Example 4.2.2 *If* $u = f(x, u)$, *find* $\dfrac{du}{dx}$.

Solution: This is a special case of equation (4.2) where $F(x, u) = f(x, u) - u$.

Then

$$\frac{du}{dx} = -\frac{F_1}{F_2} = -\frac{f_1(x, u)}{f_2(x, u) - 1}, \quad f_2(x, u) \neq 1.$$

Example 4.2.3 *If* $u = \log(x + u)$, *find* $\dfrac{du}{dx}$.

Solution: Given $u = \log(x + u)$.

We have $F(x, u) = f(x, u) - u$.

$$\frac{du}{dx} = -\frac{f_1(x, u)}{f_2(x, u) - 1} = \frac{-\dfrac{1}{x+u}}{\dfrac{1}{x+u} - 1} = \frac{-1}{1 - x - u}.$$

Example 4.2.4 *If* $u = f(g(x,u), h(y,u))$, *find* $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$.

Solution: This is a special case of equation (4.1). We have

$$F(x,y,u) = f(g(x,u), h(y,u)) - u.$$

$$\frac{\partial u}{\partial x} = -\frac{f_1 g_1}{f_1 g_2 + f_2 h_2 - 1}, \quad f_1 g_2 + f_2 h_2 - 1 \neq 0$$

$$\frac{\partial u}{\partial y} = -\frac{f_2 h_1}{f_1 g_2 + f_2 h_2 - 1}, \quad f_1 g_2 + f_2 h_2 - 1 \neq 0.$$

4.2.2 Higher order derivatives

We may compute the higher derivatives of functions defined implicitly. Consider the equation $F(x,y) = 0$, where y is a function of x. Then

$$\frac{dy}{dx} = -\frac{F_1}{F_2}, \quad F_2 \neq 0$$

$$\frac{d^2y}{dx^2} = -\frac{F_2\left(F_{11} + F_{12}\frac{dy}{dx}\right) - F_1\left(F_{21} + F_{22}\frac{dy}{dx}\right)}{F_2^2}$$

$$= -\frac{F_2\left(F_{11} + F_{12}\left(-\frac{F_1}{F_2}\right)\right) - F_1\left(F_{21} + F_{22}\left(-\frac{F_1}{F_2}\right)\right)}{F_2^2}$$

$$= -\frac{F_2 F_{11} - F_{12}F_1 - F_1 F_{21} + \frac{F_1^2 F_{22}}{F_2}}{F_2^2}$$

$$= -\frac{F_{11}F_2^2 - (F_{12} + F_{21})F_1 F_2 + F_{22}F_1^2}{F_2^3}$$

Remark : Consider the equation $F(x,y,z) = 0$. If $\dfrac{\partial x}{\partial y}$ is required, we may be sure that x is the dependent variable and y and z are independent variables. Then we find

$$\frac{\partial x}{\partial y} = -\frac{F_2}{F_1}, \quad \frac{\partial x}{\partial z} = -\frac{F_3}{F_1}, \quad F_1 \neq 0$$

$$\frac{\partial y}{\partial x} = -\frac{F_1}{F_2}, \quad \frac{\partial y}{\partial z} = -\frac{F_3}{F_2}, \quad F_2 \neq 0.$$

4.3 Existence Theorem for Implicit Functions

Let $F(x, y)$ be a function of two variables and $y = f(x)$ be a function of x such that for every value of x for which $f(x)$ is defined, $F(x, f(x))$ vanishes identically, that is, $y = f(x)$ is a root of the functional equation $F(x, y) = 0$. Then $y = f(x)$ is an implicit function defined by the functional equation $F(x, y) = 0$.

It is only in elementary cases, such as those given above, that it may be possible to express y as a function of x (i.e., determine the implicit function). For more complicated functional equations no such determination of implicit function is possible. The difficulty of actual determination of an analytical expression does not rule out the possibility of the existence of the implicit function or functions, defined by a functional equation; the actual determination may demand new processes or may be, from a practical standpoint, too laborious. We now consider an existence theorem, a theorem that specify condition which guarantee that a functional equation does define an implicit function even though actual determination may not be possible. For many purposes, however, it is the fact that an equation does define a function, rather than an expression for the implicit function thus defined, that is of real importance; hence the significance of Existence theorem.

We shall show that if $F(x_0, y_0) = 0, F_2(x_0, y_0) \neq 0$, then the equation $F(x, y) = 0$ can be solved for y when x is in a two sided neighborhood of x_0.

Theorem 4.3.1 (Existence Theorem for implicit functions)

1. $F(x, y) \in C^1, \quad |x - x_0| \leq \delta, |y - y_0| \leq \delta$
2. $F(x_0, y_0) = 0$ *and*
3. $F_2(x_0, y_0) \neq 0$

$\Rightarrow$ *There exists a unique function* $f(x)$ *and a positive number* η *such that*

A. $y_0 = f(x_0)$

B. $F(x, f(x)) = 0, \quad |x - x_0| < \eta$

C. $f(x) \in C^1, \quad |x - x_0| < \eta$

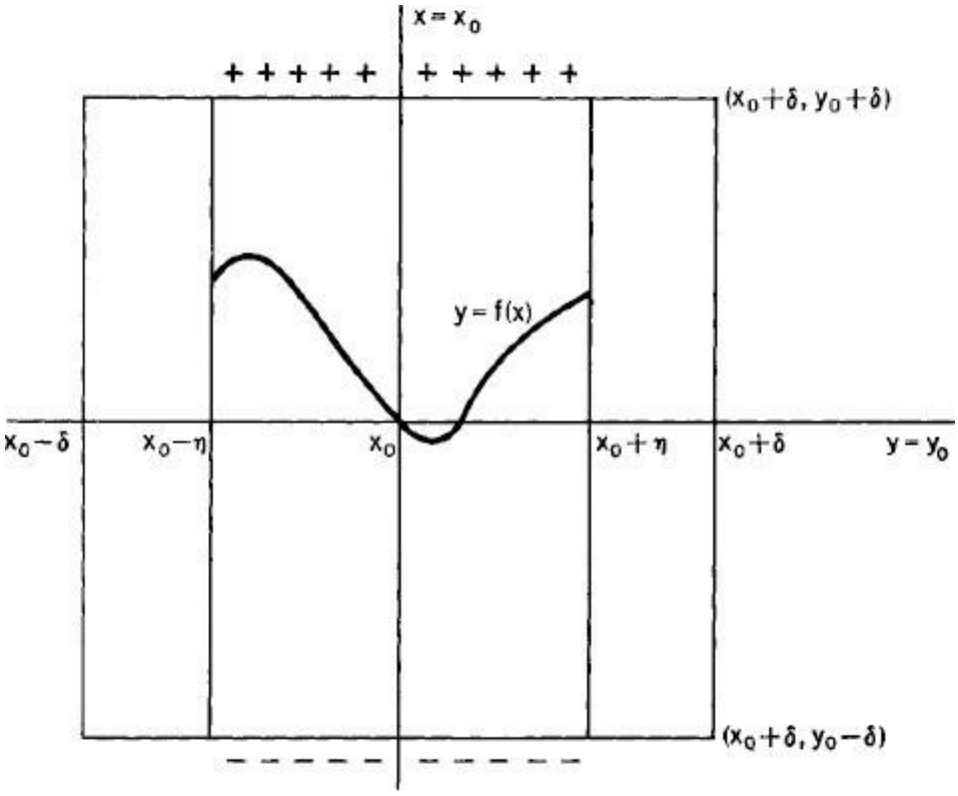

Figure 4.1

Proof: We now prove the existence of a function $f(x)$ and a positive number η satisfying hypothesis *A, B, C.*

Given that F_1 and F_2 are continuous in the neighbourhood

$|x - x_0| \leq \delta, |y - y_0| \leq \delta$ of the point (x_0, y_0).

Then F is differentiable and hence continuous in this neighbourhood.

Given $F_2(x_0, y_0) \neq 0$.

Suppose that $F_2(x_0, y_0) > 0$.

Since F_2 is continuous we have, $F_2(x, y) > \dfrac{F_2(x_0, y_0)}{2}$ in a whole neighborhood of (x_0, y_0).

Let us assume that neighbourhood to be original $\delta-$ neighbourhood.

Clearly $F(x_0, y)$ is a strictly increasing function for $|y - y_0| \leq \delta$.

Hence $F(x_0, y_0 + \delta) > F(x_0, y_0) = 0, F(x_0, y_0 - \delta) < F(x_0, y_0) = 0$.

By continuity of $F(x, y_0 + \delta)$ and of $F(x, y_0 - \delta)$, there exists a positive number η such that

$$F(x, y_0 + \delta) > 0, F(x, y_0 - \delta) < 0, \quad |x - x_0| < \eta.$$

A continuous function passing from positive to negative values must pass through zero.

Thus for each x in the interval $x_0 - \eta < x < x_0 + \eta$, there is just one value of y, which we call $f(x)$ between $y_0 - \delta$ and $y_0 + \delta$ where $F(x, y) = 0$.

To prove uniqueness

We now show that $y = f(x)$ is a unique solution of $F(x, y) = 0$. That is $F(x, y)$ cannot be zero for more than one value of y between $y_0 - \delta$ and $y_0 + \delta$.

Suppose there are two such values of y_1 and y_2 between $y_0 - \delta$ and $y_0 + \delta$ such that

$$F(x, y_1) = 0 \quad \text{and} \quad F(x, y_2) = 0.$$

Also $F(x, y)$ considered as a function of a single variable y is derivable between $y_0 - \delta$ and $y_0 + \delta$.

So that by Roll's theorem $F_2 = 0$ for a value between y_1 and y_2. This contradicts the fact that $F_2(x_0, y_0) \quad 0$.

Hence our assumption is wrong. There cannot be more than one such y.

From the definition of $f(x)$ we have

$$f(x_0) = y_0 \text{ and } F(x, f(x)) = 0, \quad |x - x_0| < \eta$$

To prove C. Let

$$y_1 = f(x_1), \quad x_0 - \eta < x_1 < x_0 + \eta.$$

$$y_1 + \Delta y = f(x_1 + \Delta x), \quad x_0 - \eta < x_1 + \Delta x < x_0 + \eta.$$

Then, by the law of the mean for functions of two variables,

$$F(x_1 + \Delta x, y_1 + \Delta y) = 0$$

$$= F_1(x_1 + \vartheta\Delta x, y_1 + \vartheta\Delta y)\Delta x + F_2(x_1 + \vartheta\Delta x, y_1 + \vartheta\Delta y)\Delta y, \; 0 < \vartheta < 1.$$

This equation shows first that $\Delta y \to 0$ as $\Delta x \to 0$.

For, the first term, and hence the second, approaches zero as Δx does.

But since the first factor of the second term is greater than $F_2(x_0, y_0)/2$, that term cannot approach zero unless Δy does.

Secondly, the above equation enables us to compute $\Delta y/\Delta x$.

Finally, using the continuity of F_1 and F_2, we obtain

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{F_1(x_1, y_1)}{F_2(x_1, y_1)}$$

This quotient is a continuous function of x_1, $[y_1 = f(x_1)]$, so that $f \in C^1$.

This completes the proof of the theorem. Q

Remark : The theorem can easily be generalized to include functions of more than two variables.

For example, the equation $F(x, y, z) = 0$ can be solved for z when (x, y) is near (x_0, y_0) if $F(x_0, y_0, z_0) = 0$, $F_3(x_0, y_0, z_0) \neq 0$.

We now state the implicit function theorem (without proof) for vector valued functions.

Theorem 4.3.2 *Let $\vec{f} = (f_1, f_2, ..., f_n)$ be a vector valued function defined on a open set S in R^{n+k} with values in R^n. Suppose that $\vec{f} \in C^1$ on S. Let $(\vec{x}_0; \vec{t}_0)$ be a point in S for which $\vec{f}(\vec{x}_0; \vec{t}_0) = \vec{0}$ and for which the $n \times n$ determinant $\det \left[D_j f_i(\vec{x}_0; \vec{t}_0) \right] \neq 0$. Then there exist a k dimensional open set T_0 containing $\vec{t}_0$ and one, and only one, vector valued function $\vec{g}$ defined on T_0 and having values in R^n such that*

a) $\vec{g} \in C^1$ *on* T_0

b) $\vec{g}(\vec{t}_0) = \vec{t}_0$

c) $\vec{f}(\vec{g}(\vec{t}); \vec{t}) = \vec{0}$

4.4 Functional Dependence

Two function $f(x, y)$ and $g(x, y)$ may be functionally dependent if there exists a function of a single variable $F(\square)$ such that

$$g(x, y) = F(f(x, y)).$$

Example:

Suppose

$$f(x, y) = \sin(x^2 + y^2), \quad g(x, y) = \cos(x^2 + y^2),$$

there exists a function of a single variable $F(\square)$ such that

$$g(x, y) = F(f(x, y)) \tag{4.4}$$

that is $F(\square) = \cos(\sin^{-1}\square)$.

The Jacobian

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} 2x\cos(x^2 + y^2) & 2y\cos(x^2 + y^2) \\ -2x\sin(x^2 + y^2) & -2y\sin(x^2 + y^2) \end{vmatrix} = 0$$

Note: We shall see that the vanishing of this Jacobian is a characteristic of functional dependence.

Remark:

We observe by virtue of equation (4.4) that if f and g are functionally dependent, then their Jacobian is identically zero:

$$g_1 = f_1 F', \quad g_2 = f_2 F', \quad \frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} f_1 & f_1 F' \\ f_2 & f_2 F' \end{vmatrix} = 0$$

We shall now prove the converse part, that is under certain conditions the vanishing of this Jacobian implies the functional dependence of f and g.

Theorem 4.4.1

1. $f(x, y), g(x, y) \in C^1 \quad |x - x_0| < \delta, |y - y_0| < \delta$
2. $\dfrac{\partial(f, g)}{\partial(x, y)} = 0, \quad |x - x_0| < \delta, |y - y_0| < \delta$ *and*

3. $f_2(x_0, y_0) \quad 0$

⇒ *There exists a function* $F(\square)$ *and a number* η *such that*

$$g(x, y) = F(f(x, y)), \qquad |x - x_0| < \eta, \ |y - y_0| < \eta.$$

Proof: Let $\square_0 = f(x_0, y_0)$. Then by the generalization of Theorem 4.3.1 to functions of three variables mentioned above, the equation

$$f(x, y) - \square = 0 \tag{4.5}$$

can be solved for y. That is, there exists a function $\varphi(x, \square)$ such that the equation

$$y = \varphi(x, \square) \tag{4.6}$$

is equivalent to (4.5) in an $\eta-$ neighborhood of the point $(x_0, y_0, \square_0)$. Also $\varphi_1(x, \square)$ can be computed in terms of f by the usual rule:

$$\varphi_1(x, \square) = \frac{-f_1(x, y)}{f_2(x, y)}, \qquad y = \varphi(x, \square) \tag{4.7}$$

We shall now compute the derivative of $g(x, \varphi(x, \square))$ with respect to x, using (4.7) and hypothesis 2:

$$\begin{aligned}
\frac{\partial}{\partial x} g(x, \varphi(x, \square)) &= g_1 + g_2 \varphi_1(x, \square) \\
&= g_1 - \frac{f_1 g_2}{f_2} \\
&= \frac{g_1 f_2 - f_1 g_2}{f_2} \\
&= -\frac{1}{f_2}[f_1 g_2 - g_1 f_2] \\
&= -\frac{1}{f_2}\frac{\partial(f, g)}{\partial(x, y)} = 0, \qquad |x - x_0| < \eta, \ |y - y_0| < \eta.
\end{aligned}$$

Integrating this equation, we obtain

$$g(x, \varphi(x, \square)) = F(\square), \qquad |x - x_0| < \eta, \ |\square - \square_0| < \eta$$

for some function $F(\square)$.

Finally, substituting $\square = f(x, y)$ in this equation

we have

$$g(x, \varphi(x, \square)) = F(f(x, y)).$$

Since equations (4.5) and (4.6) are equivalent near (x_0, y_0, z_0) we have

$$\varphi(x, z) = \varphi(x, f(x, y)) = y$$

$$\text{so} \quad g(x, \varphi(x, z)) = g(x, y) = F(f(x, y)), \qquad |x - x_0| < \eta, \; |y - y_0| < \eta.$$

This completes the proof. Q

Remark 1: Hypothesis 3 could be replaced by $f_1(x_0, y_0) \neq 0$. Then equation $f(x, y) - z = 0$ can be solved for x and we have the same conclusion.

Remark 2: Suppose $g_1(x_0, y_0) \neq 0$ or $g_2(x_0, y_0) \neq 0$ we would show that $f(x, y) = G(g(x, y))$ for some $G(z)$.

Example 4.4.1 *If* $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ *and* $w = x + y + z$, *determine whether there is a functional relationship between* u, v, w *and if so, find it.*

Solution:

$$u = xy + yz + zx$$

$$v = x^2 + y^2 + z^2$$

$$w = x + y + z$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y + z & z + x & x + y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x + y + z & x + y + z & x + y + z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x + y + z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0.$$

Hence the functional relationship exists between u, v and w.

Now,

$$w^2 = (x + y + z)^2$$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= v + 2u$$

$$\Rightarrow w^2 - v - 2u = 0$$

which is the required relationship.

Example 4.4.2 *Verify whether the following functions are functionally dependent, and if so, find the relation between them.*

$$u = \frac{x + y}{1 - xy}, \; v = \tan^{-1} x + \tan^{-1} y$$

Solution:

$$u = \frac{x + y}{1 - xy}, \; v = \tan^{-1} x + \tan^{-1} y$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1 + y^2}{(1 - xy)^2} & \frac{1 + x^2}{(1 - xy)^2} \\ \frac{1}{1 + x^2} & \frac{1}{1 + y^2} \end{vmatrix}$$

$$= \frac{1}{(1 - xy)^2} - \frac{1}{(1 - xy)^2}$$

$$= 0$$

Hence u, v are functionally dependent.

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x + y}{1 - xy} \right) = \tan^{-1} u$$

$$\Rightarrow v = \tan^{-1} u$$

$$\Rightarrow u = \tan v.$$

Example 4.4.3 *Verify whether the functions* $u = xy, \; v = x + y + z, \; w = xy + 2(x + y + z)$ *are functionally dependent , if so, find the relation between them.*

Solution:

$$u = xy,\ v = x + y + z,\ w = xy + 2(x + y + z)$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y & x & 0 \\ 1 & 1 & 1 \\ y+2 & x+2 & 2 \end{vmatrix}$$

$$= y(2 - x - 2) - x(2 - y - 2)$$

$$= 0$$

Hence u, v are functionally dependent.

$$u = xy$$

$$v = x + y + z$$

$$w = xy + 2(x + y + z)$$

$$w = u + 2v.$$

4.5 Simultaneous Equations

We denote a set of four numbers (u_o, v_o, x_o, y_o) as a point in four dimensions and to the set of values (u, v, x, y) for which $|u - u_o| < \delta$, $|v - v_o| < \delta$, $|x - x_o| < \delta$, $|y - y_o| < \delta$, as a $\delta-$ neighbourhood, $N_\delta(u_o, v_o, x_o, y_o)$, of that point.

Theorem 4.5.1

1. $F(u, v, x, y), G(u, v, x, y) \in C^1$ *in* $N_\delta(u_o, v_o, x_o, y_o)$
2. $F(u_o, v_o, x_o, y_o) = G(u_o, v_o, x_o, y_o) = 0$ *and*
3. $\frac{\partial(F, G)}{\partial(u, v)} \neq 0$ *at* (u_o, v_o, x_o, y_o)

⇒ *There exists a unique pair of functions* $f(x, y), g(x, y)$ *and a positive number* η *such that*

A $f(x, y), g(x, y) \in C^1, \quad |x - x_o| < \eta, |y - y_o| < \eta$

B $f(x_o, y_o) = u_o, g(x_o, y_o) = v_o$

C $F(f, g, x, y) = G(f, g, x, y) = 0, \quad |x - x_o| < \eta, |y - y_o| < \eta$

Proof: Given that $\frac{\partial(F, G)}{\partial(u, v)} \neq 0$ at (u_o, v_o, x_o, y_o). So not both F_u and F_v are zero at (u_o, v_o, x_o, y_o).

Assume that $F_u \neq 0$ there.

Then from the generalization of Theorem 4.3.1 , there exists a unique function $h(v, x, y)$ such that $h(v_o, x_o, y_o) = u_o$ and $F(h, v, x, y) = 0$ in some neighborhood of (v_o, x_o, y_o).

Differentiating partially with respect to v, we have

$$F_u h_v + F_v = 0$$
$$h_v = -\frac{F_v}{F_u}.$$

We now solve the following equation for v.

$$G(h(v, x, y), v, x, y) = 0 \qquad (4.8)$$

To solve this we need the derivative of the function with respect to v is not equal to zero at (v_o, x_o, y_o).

Differentiating partially with respect to v we have,

$$\begin{aligned} G_u h_v + G_v &= G_u \frac{-F_v}{F_u} + G_v \\ &= \frac{-G_u F_v - G_v F_u}{F_u} \\ &= \frac{1}{F_u}\frac{\partial(F, G)}{\partial(u, v)} \quad 0 \quad \text{at } (v_o, x_o, y_o) \end{aligned}$$

Hence, there exists a unique function $g(x, y)$ which is equal to v_o at (x_o, y_o). $g(x_o, y_o) = v_o$, which makes equation (4.8) an identity near (x_o, y_o) when it is substituted for v.

Now set $f(x, y) = h(g, x, y)$.

It is easy to prove that

$$f(x, y) \in C^1, \quad |x - x_0| < \eta, |y - y_0| < \eta$$

$$f(x_0, y_0) = u_0$$

$$\text{and} \quad F(f, g(x, y)) = 0.$$

Hence all the conclusions of the theorem are satisfied. Similarly we can prove the theorem if $F_v \neq 0$. Q

Summary

- We can compute higher derivatives of implicit functions
- We are given a functional relation $\varphi(x, y, \ldots, t) = 0$ connecting n variables and it is not in general possible to solve this equation to find an explicit function which expresses one of these variable say x in terms of the other $n - 1$ variables such function are called implicit functions
- **The inverse function theorem for single variable** Let $y = f(x)$ be continuous and strictly increasing for $a \leq x \leq b$. If, for a given x in $a < x < b$, $f'(x) \neq 0$, then the inverse function $x = g(y)$ is differentiable for the corresponding value of y and $g'(y) = \frac{1}{f'(x)}$.
- **Existence theorem for implicit functions**

  1. $F(x, y) \in C^1, \quad |x - x_0| \leq \delta, |y - y_0| \leq \delta$
  2. $F(x_0, y_0) = 0$ and
  3. $F_2(x_0, y_0) \neq 0$

  $\Rightarrow$ There exists a unique function $f(x)$ and a positive number η such that

  A) $y_0 = f(x_0)$

  B) $F(x, f(x)) = 0, \quad |x - x_0| < \eta$

C) $f(x) \in C^1, \quad |x - x_0| < \eta$

- Two variables $f(x, y)$ and $g(x, y)$ may be functionally dependent if there exists a function of a single variable $F(\square)$ such that $g(x, y) = F(f(x, y))$

- 1. $f(x, y), g(x, y) \in C^1 \quad |x - x_0| < \delta, |y - y_0| < \delta$
  2. $\frac{\partial(f, g)}{\partial(x, y)} = 0, \quad |x - x_0| < \delta, |y - y_0| < \delta$ and
  3. $f_2(x_0, y_0) \quad 0$

  $\Rightarrow$ There exists a function $F(\square)$ and a number η such that

$$g(x, y) = F(f(x, y)), \quad |x - x_0| < \eta, |y - y_0| < \eta.$$

- If f and g are functionally dependent, then their Jacobian is identically zero.

- If $f(x, y), g(x, y) \in C^1$, $|x - x_0| < \delta$, $|y - y_0| < \delta$ and $f_2(x_0, y_0) \neq 0$ then vanishing of the jocobian implies the functional dependence of f and g.

Multiple Choice questions

1. If $u = f(x, u)$, then $\frac{du}{dx}$ is

   a) $\frac{F_1}{F_2}$ b) $\frac{-F_2}{F_1}$ c) $\frac{-F_1}{F_2}$

2. Choose the correct statement.

   a) If f and g are functionally dependent, then their Jacobian is identically zero.

   b) If f and g are functionally dependent, then their Jacobian is never zero.

   c) If f and g are functionally dependent, then their Jacobian is either non-zero or 1

3. If $u = xy, v = x + y + \square, w = xy + 2(x + y + \square)$ then

   (a) $\frac{\partial(u, v, w)}{\partial(x, y, \square)} = 0$

(b) $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 1$

(c) $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 8$

Ans: 1. c) 2. a) 3. a)

Exercises 4

1. If $f(x, y), F(x, y, z) \in C^1$, $F(x, y, f(x, y)) = 0$, (x, y) in D and $F_3(x, y, f(x, y)) \neq 0, (x, y)$ in D, then prove that

$$f_1(x, y) = \frac{-F_1(x, y, f(x, y))}{F_3(x, y, f(x, y))}, \quad f_2(x, y) = \frac{-F_2(x, y, f(x, y))}{F_3(x, y, f(x, y))}$$

2. If $\log uy + y \log u = x$, find $\dfrac{\partial y}{\partial u}$ and $\dfrac{\partial y}{\partial x}$.

Ans: $-\dfrac{(y^2 + y)}{u + yu \log u}, \dfrac{y}{1 + y \log u}$

3. Find $\dfrac{d^2u}{dx^2}$ if $u = \log(x + u)$.

4. If $\sin \theta y = \cos \theta x$, compute $\dfrac{\partial \theta}{\partial x}$ when $\theta = \pi, x = \frac{1}{3}, y = \frac{1}{6}$.

Ans: -2π.

5. State and prove inverse function theorem for single variable.

6. State and prove existence theorem for implicit functions.

7. If $f(x, y), g(x, y) \in C^1$, $\dfrac{\partial(f, g)}{\partial(x, y)} = 0$, $|x - x_0| < \delta, |y - y_0| < \delta$, $f_2(x_0, y_0) \neq 0$, then prove that there exists a function $F(u)$ and a number η such that

$$g(x, y) = F(f(x, y)), \quad |x - x_0| < \eta, |y - y_0| < \eta.$$

8. If $F(u, v, x, y), G(u, v, x, y) \in C^1$ in $N_\delta(u_0, v_0, x_0, y_0)$, $F(u_0, v_0, x_0, y_0) = G(u_0, v_0, x_0, y_0) = 0$, $\dfrac{\partial(F, G)}{\partial(u, v)} \neq 0$ at (u_0, v_0, x_0, y_0), then prove that there exists a unique pair of functions $f(x, y), g(x, y)$ and a positive number η such that

A $f(x, y), g(x, y) \in C^1$, $|x - x_0| < \eta, |y - y_0| < \eta$

B $f(x_0, y_0) = u_0, g(x_0, y_0) = v_0$

C $F(f, g, x, y) = G(f, g, x, y) = 0$, $|x - x_0| < \eta, |y - y_0| < \eta$.

BLOCK III

Taylor's Theorem and Applications

Unit 5

Taylor's Theorem

Learning Outcomes :

After studying this unit, students will be able

- To derive Taylor's theorem for two variables.
- To find Taylor's expansion of given functions.
- To find the Maclaurin's expansion of given functions.

5.1 Taylor's Theorem for functions of a single variable

We state below Taylor's theorem for functions of a single variable and also the familiar Lagrange and Cauchy remainder.

Theorem 5.1.1 (Taylor's Theorem)

1. $f(x) \in C^{n+1}, \quad |x-a| \le h.$

$$\Rightarrow f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{n+1}(t)\frac{(x-t)^n}{n!}\,dt, \quad |x-a| \le h.$$

Lagrange remainder

1. $f(x) \in C^{n+1}, \quad |x-a| \leq h.$

$$\Rightarrow f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

where $R_n = f^{n+1}(X) \int_a^x \frac{(x-t)^n}{n!} dt = f^{(n+1)}(X) \frac{(x-a)^{n+1}}{(n+1)!}, \quad a < X < x.$

Cauchy remainder

1. $f(x) \in C^{n+1}, \quad |x-a| \leq h.$

$$\Rightarrow f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

where $R_n = f^{n+1}(X) \frac{(x-X)^n}{n!}(x-a), \quad a < X < x.$

5.2 Taylor's Theorem for Functions of two variables

Theorem 5.2.1

1. $f(x, y) \in C^{n+1}, \quad |x-a| \leq |h|, |y-b| \leq |k|$

$$\Rightarrow f(a+h, b+k) = \sum_{j=0}^{n} \frac{1}{j!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^j f(a, b) + R_n \tag{5.1}$$

where

$$\begin{aligned} R_n &= \int_0^1 \frac{(1-t)^n}{n!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(a+ht, b+kt)dt \\ &= \frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(a+\vartheta h, b+\vartheta k), \quad 0 < \vartheta < 1 \end{aligned}$$

Proof: Let (x, y) be a point in the domain under consideration such that

$$x = a + th, y = b + tk \tag{5.2}$$

where $0 \leq t \leq 1$ is a parameter.

Consider a new function,

$$F(t) = f(x, y) = f(a+th, b+tk) \tag{5.3}$$

Clearly $F(t)$ is a function of a single variable. Since the partial derivatives of $f(x, y)$ of order $(n+1)$ are continuous in the domain under consideration $F^{(n+1)}(t)$ is continuous in $[0, 1]$.

Expanding $F(t)$ in Taylor's series

$$F(t) = F(0) + tF'(0) + \frac{t^2}{2!}F''(0) + \dots + \frac{t^n}{n!}F^{(n)}(0) + \frac{t^{n+1}}{(n+1)!}F^{(n+1)}(\vartheta t), \quad (5.4)$$

$$F(1) = \sum_{j=0}^{n} \frac{F^{(j)}(0)}{j!} + \int_0^1 \frac{(1-t)^n}{n!} F^{(n+1)}(t)\,dt$$

$$= \sum_{j=0}^{n} \frac{F^{(j)}(0)}{j!} + \frac{F^{(n+1)}(\vartheta)}{(n+1)!},$$

$$= F(0) + F'(0) + \frac{1}{2!}F''(0) + \dots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\vartheta), \quad (5.5)$$

where $0 < \vartheta < 1$. Now

$$F(t) = f(a + th, b + tk)$$

$$F'(t) = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a + th, b + tk)$$

$$F''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a + th, b + tk)$$

$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + th, b + tk)$$

$$F^{(n+1)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a + th, b + tk)$$

Putting $t = 0$ in the above results upto n^{th} derivative

$$F(0) = f(a, b)$$

$$F'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$F''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$F^{(n)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a, b)$$

Putting $t = \vartheta$ in $F^{(n+1)}(t)$, $\quad 0 < \vartheta < 1$

$$F^{(n+1)}(\vartheta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a + \vartheta h, b + \vartheta k),$$

Also $F(1) = f(a+h, b+k)$.

Using all the above in (3.5)

$$\Rightarrow f(a+h, b+k) = \sum_{j=0}^{n} \frac{1}{j!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{j} f(a,b) + R_n$$

where

$$R_n = \int_0^1 \frac{(1-t)^n}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+ht, b+kt)dt$$

$$= \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\vartheta h, b+\vartheta k), \quad 0 < \vartheta < 1$$

Here R_n is called the remainder after n terms, the theorem is called Taylor's theorem with remainder or Taylor's expansion about the point (a, b). Q

Remark 1: Maclaurin's Theorem or Maclaurin's expansion

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{2} f(0,0)$$

$$+ \dots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n} f(0,0) + R_n$$

where $R_n = \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\vartheta x + \vartheta y), \quad 0 < \vartheta < 1.$

Remark 2: Taylor's Theorem can also be put in the form:

$$f(a+h, b+k) = f(a,b) + df(a,b) + \frac{1}{2!}d^2 f(a,b)$$

$$+ \dots + \frac{1}{n!}d^n f(a,b) + \frac{1}{(n+1)!}d^{n+1} f(a+\vartheta h, b+\vartheta k),$$

where $0 < \vartheta < 1$.

Remark 3: The Theorem can be extended to any number of variables.

Remark 4: Another form of Taylor's expansion about the point (a, b) in powers of $x - a$ and $y - b$.

Putting, $t = 1$ in (5.2) we have

$$x = a + h \Rightarrow h = x - a \quad y = b + h \Rightarrow k = y - b$$

From (5.1)

$$f(x,y) = f(a,b) + \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right) f(a,b)$$

$$+ \frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{2} f(a,b)$$

$$+ \dots + \frac{1}{n!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{n} f(a,b) + R_n$$

where

$$R_n = \frac{1}{(n+1)!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{n+1} f(a+(x-a)\vartheta, b+(y-b)\vartheta),$$

$0 < \vartheta < 1$.

That is

$$f(x,y) = \sum_{j=0}^{n} \frac{1}{j!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{j} f(a,b) + R_n$$

$$R_n = \frac{1}{(n+1)!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{n+1} f(r,s),$$

where $r = a+\vartheta(x-a), s = b+\vartheta(y-b)$, $0 < \vartheta < 1$ after the differentiation.

Remark 5: Taking $n = 0$ in equation (5.1), we have

$$f(a+h, b+k) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{0} f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a+\vartheta h, b+\vartheta k),$$

$0 < \vartheta < 1$.

$$f(a+h, b+k) = f(a,b) + hf_1(a+\vartheta h, b+\vartheta k) + kf_2(a+\vartheta h, b+\vartheta k)$$

$$f(a+h, b+k) - f(a,b) = hf_1(a+\vartheta h, b+\vartheta k) + kf_2(a+\vartheta h, b+\vartheta k) \quad (5.6)$$

where $0 < \vartheta < 1$. This equation is known as the law of the mean for function of two variables.

Note 1 : For third and more order derivatives are zero,

$$\begin{aligned} f(x,y) = {} & f(a,b) + (x-a)f_1(a,b) + (y-b)f_2(a,b) \\ & + \frac{1}{2!}\Big((x-a)^2 f_{11}(a,b) + 2(x-a)(y-b)f_{12}(a,b) \\ & + (y-b)^2 f_{22}(a,b)\Big) \end{aligned}$$

Note 2: For $n = 3$, (higher derivatives are not zero)

$$\begin{aligned} f(x,y) = {} & f(a,b) + (x-a)f_1(a,b) + (y-b)f_2(a,b) \\ & + \frac{1}{2!}\Big[(x-a)^2 f_{11}(a,b) + 2(x-a)(y-b)f_{12}(a,b) \\ & + (y-b)^2 f_{22}(a,b)\Big] \\ & + \frac{1}{3!}\Big[(x-a)^3 f_{111}(\vartheta x, \vartheta y) + 3(x-a)^2(y-b)f_{112}(\vartheta x, \vartheta y) \\ & + 3(x-a)(y-b)^2 f_{122}(\vartheta x, \vartheta y) + (y-b)^3 f_{222}(\vartheta x, \vartheta y)\Big] \end{aligned}$$

Example 5.2.1 *Expand* $f(x,y) = x^2 + xy + y^2$ *in powers of* $(x-2)$ *and* $(y-3)$.

Solution:

Function	**Value at** $(2,3)$
$f(x,y) = x^2 + xy + y^2$	$f(2,3) = 19$
$f_1 = 2x + y$	$f_1(2,3) = 7$
$f_2 = x + 2y$	$f_2(2,3) = 8$
$f_{11} = 2$	$f_{11}(2,3) = 2$
$f_{22} = 2$	$f_{22} = 2$
$f_{12} = 1$	$f_{12} = 1$

The values of third and higher order partial derivatives of f are zero.

By Taylor's expansion about the point $(2,3)$

$$f(x,y) = f(2,3) + (x-2)f_1(2,3) + (y-3)f_2(2,3)$$
$$+\frac{1}{2!}\ (x-2)^2 f_{11}(2,3) + (y-3)^2 f_{22}(2,3) + 2(x-2)(y-3)f_{12}(2,3)$$

we have

$$f(x,y) = 12+[7(x-2)+8(y-3)]+\frac{1}{2}\,2(x-2)^2 + 2(y-3)^2 + 2(x-2)(y-3)\ .$$

Example 5.2.2 *Obtain Taylor's formula for the function* e^{x+y} *at* $(0,0)$ *for* $n = 2$ *and write the remainder.*

Solution: The values of third and higher order partial derivatives are **not zero.**

For $n = 2$, the Taylor's formula about $(0,0)$ is

$$f(x,y) = f(0,0) + [xf_1(0,0) + yf_2(0,0)]$$
$$+\frac{1}{2!}\Big[x^2 f_{11}(0,0) + 2xy f_{12}(0,0) + y^2 f_{22}(0,0)\Big]$$
$$+\frac{1}{3!}\Big[x^3 f_{111}(\vartheta x,\vartheta y) + 3x^2 y f_{112}(\vartheta x,\vartheta y) + 3xy^2 f_{122}(\vartheta x,\vartheta y)$$
$$+y^3 f_{222}(\vartheta x,\ \vartheta y)\Big]$$

where $0 < \vartheta < 1$.

Function	Value at $(0, 0)$
$f(x, y) = e^{x+y}$	$f(0, 0) = 1$
$f_1 = e^{x+y}$	$f_1(0, 0) = 1$
$f_2 = e^{x+y}$	$f_2(0, 0) = 1$
$f_{11} = e^{x+y}$	$f_{11}(0, 0) = 1$
$f_{22} = e^{x+y}$	$f_{22}(0, 0) = 1$
$f_{12} = e^{x+y}$	$f_{12}(0, 0) = 1$

Function	Value at $(x = \vartheta x, y = \vartheta y)$
$f_{111} = e^{x+y}$	$f_{111}(\vartheta x, \vartheta y) = e^{(\vartheta x, \vartheta y)}$
$f_{222} = e^{x+y}$	$f_{222}(\vartheta x, \vartheta y) = e^{(\vartheta x, \vartheta y)}$
$f_{112} = e^{x+y}$	$f_{112}(\vartheta x, \vartheta y) = e^{(\vartheta x, \vartheta y)}$
$f_{122} = e^{x+y}$	$f_{122}(\vartheta x, \vartheta y) = e^{(\vartheta x, \vartheta y)}$

Hence,

$$e^{(x+y)} = 1 + (x + y) + \frac{1}{2!}(x^2 + 2xy + y^2) + \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3)e^{\vartheta(x+y)}$$

$$= 1 + (x + y) + \frac{1}{2!}(x + y)^2 + \frac{1}{3!}(x + y)^3 e^{\vartheta(x+y)}$$

Example 5.2.3 *If* $f(x, y) \in C^1$, $g(x, y) \in C^1$, $f(0, 0) = g(0, 0) = 0$, $g_1^2(0, 0) + g_2^2(0, 0)$ 0, *Find* $\lim \frac{f(x, y)}{g(x, y)}$ *as* (x, y) *approaches* $(0, 0)$ *along line* $y = \lambda x$.

Solution: By Taylor's theorem

$$f(x, y) = f(0, 0) + f_1(\vartheta x, \vartheta y)x + f_2(\vartheta x, \vartheta y)y$$

$$= f_1(\vartheta x, \vartheta y)x + f_2(\vartheta x, \vartheta y)y$$

$$g(x, y) = g(0, 0) + g_1(\vartheta_1 x, \vartheta_1 y)x + g_2(\vartheta_1 x, \vartheta_1 y)y, \quad 0 < \vartheta, \vartheta_1 < 1.$$

Taking limits $(x, y) \to (0, 0)$ along the line $y = \lambda x$.

$$\lim \frac{f(x, y)}{g(x, y)} = \lim_{x \to 0} \frac{f_1(\vartheta x, \vartheta \lambda x)x + f_2(\vartheta x, \vartheta \lambda x)\lambda x}{g_1(\vartheta_1 x, \vartheta_1 \lambda x)x + g_2(\vartheta_1 x, \vartheta_1 \lambda x)\lambda x}$$

$$= \frac{f_1(0, 0) + \lambda f_2(0, 0)}{g_1(0, 0) + \lambda g_2(0, 0)}.$$

where $g_1(0, 0) + \lambda g_2(0, 0) \neq 0$.

Example 5.2.4 *Expand* $\sin xy$ *in powers of* $(x-1)$ *and* $(y-\frac{\pi}{2})$ *upto second degree terms.*

Solution: By Taylor's expansion about the point $(1, \frac{\pi}{2})$ upto second order

$$f(x,y) = f(1,\tfrac{\pi}{2}) + (x-1)f_1(1,\tfrac{\pi}{2}) + (y-\tfrac{\pi}{2})f_2(1,\tfrac{\pi}{2})$$
$$+\frac{1}{2!}\Big[(x-1)^2 f_{11}(1,\tfrac{\pi}{2}) + (y-\tfrac{\pi}{2})^2 f_{22}(1,\tfrac{\pi}{2})$$
$$+2(x-1)(y-\tfrac{\pi}{2})f_{12}(1,\tfrac{\pi}{2})\Big] + \ldots$$

Function	**Value at** $(1, \frac{\pi}{2})$
$f(x,y) = \sin xy$	$f(1,\frac{\pi}{2}) = 1$
$f_1 = y\cos xy$	$f_1(1,\frac{\pi}{2}) = 0$
$f_2 = x\cos xy$	$f_2(1,\frac{\pi}{2}) = 0$
$f_{11} = -y^2 \sin xy$	$f_{11}(1,\frac{\pi}{2}) = -\frac{\pi^2}{4}$
$f_{22} = -x^2 \sin xy$	$f_{22}(1,\frac{\pi}{2}) = -1$
$f_{12} = \cos xy - xy\sin xy$	$f_{12}(1,\frac{\pi}{2}) = \frac{\pi}{2}$

Therefore,

$$\sin xy = 1 + (x-1)(0) + \left(y-\frac{\pi}{2}\right)(0)$$
$$+\frac{1}{2}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + \left(y-\frac{\pi}{2}\right)^2(-1) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)\right] + \ldots$$
$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) + \ldots$$

Example 5.2.5 *If* $f(x,y) = x^2 - 3xy + 2y^2$ *use the equation*

$$f(a+h, b+k) - f(a,b) = f_1(a+\vartheta h, b+\vartheta k)h + f_2(a+\vartheta h, b+\vartheta k)k, \qquad 0 < \vartheta < 1$$

to express the difference $f(1,2) - f(2,-1)$ *by partial derivatives and compute* ϑ.

Solution: Here $a+h = 1, b+k = 2, a = 2, b = -1$. So $h = -1, k = 3$.

$$a + \vartheta h = 2 - \vartheta$$
$$b + \vartheta k = 3\vartheta - 1.$$

$$f(x, y) = x^2 - 3xy + 2y^2$$

$$f_1(x, y) = 2x - 3y$$

$$f_2(x, y) = -3x + 4y$$

$$f(1, 2) = 3, f(2, -1) = 12$$

$$f(1, 2) - f(2, -1) = -9$$

By using the given equation

$$f(1, 2) - f(2, -1) = (-1)[2(2 - \vartheta) - 3(3\vartheta - 1)] + 3[-3(2 - \vartheta) + 4(3\vartheta - 1)]$$

$$= 4 - 2\vartheta + 9\vartheta - 3 - 18 + 9\vartheta + 36\vartheta - 12$$

$$= 56\vartheta - 37.$$

So $56\vartheta - 37 = -9, \quad \vartheta = \frac{28}{56} = \frac{1}{2}.$

Example 5.2.6 *Expand $x^3 - 2xy^2$ in Taylor's series ($a = 1, b = -1$) and check by algebra.*

Solution:

Function	**Value at** $(1, -1)$
$f(x, y) = x^3 - 2xy^2$	$f(1, -1) = -1$
$f_1 = 3x^2 - 2y^2$	$f_1(1, -1) = 1$
$f_2 = -4xy$	$f_2(1, -1) = 4$
$f_{11} = 6x$	$f_{11}(1, -1) = 6$
$f_{22} = -4x$	$f_{22}(1, -1) = -4$
$f_{12} = -4y$	$f_{12}(1, -1) = 4$
$f_{111} = 6$	$f_{111}(1, -1) = 6$
$f_{222} = 0$	$f_{222}(1, -1) = 0$
$f_{112} = 0$	$f_{112}(1, -1) = 0$
$f_{122} = -4$	$f_{122}(1, -1) = -4$

All the partial derivatives of order four are zero. Hence the remainder $R_4 = 0$.

The required Taylor's expansion will be

$$\begin{aligned} f(x,y) =& f(1,-1) + [(x-1)f_1(1,-1) + (y+1)f_2(1,-1)] \\ &+\frac{1}{2!}\Big[(x-1)^2 f_{11}(1,-1) + 2(x-1)(y+1)f_{12}(1,-1) \\ &+(y+1)^2 f_{22}(1,-1)\Big] \\ &+\frac{1}{3!}\Big[(x-1)^3 f_{111}(1,-1) + 3(x-1)^2(y+1)f_{112}(1,-1) \\ &+3(x-1)(y+1)^2 f_{122}(1,-1) + (y+1)^3 f_{222}(1,-1)\Big] \end{aligned}$$

Substituting all the values we have,

$$\begin{aligned} f(x,y) =& -1 + (x-1) + 4(y+1) + \frac{1}{2}\Big[6(x-1)^2 + 8(x-1)(y+1) - 4(y+1)^2\Big] \\ &+ \frac{1}{6}\Big[6(x-1)^3 - 12(x-1)(y+1)^2\Big] \end{aligned}$$

We can check the above result by algebra.

RHS of the above equation

$$= -1 + x - 1 + 4y + 4 + 3(x^2 - 2x + 1) + 4(xy - y + x - 1) - 2(y^2 + 2y + 1)$$

$$+ x^3 - 3x^2 + 3x - 1 - 2xy^2 - 4xy - 2x + 2y^2 + 4y + 2$$

$$= x^3 - 2xy^2$$

$$= f(x,y).$$

Example 5.2.7 *Expand the function* $f(x,y) = x^3 + 3x^2y + 4xy^2 + y^3$ *by Taylor's theorem in powers of* $(x-1)$ *and* $(y-1)$ *and check by algebra.*

Solution:

Function	**Value at** $(1,1)$
$f(x,y) = x^3+3x^2y+4xy^2+y^3$	$f(1,1) = 9$
$f_1 = 3x^2 + 6xy + 4y^2$	$f_1(1,1) = 13$
$f_2 = 3x^2 + 8xy + 3y^2$	$f_2(1,1) = 14$
$f_{11} = 6x + 6y$	$f_{11}(1,1) = 12$
$f_{22} = 8x + 6y$	$f_{22}(1,1) = 14$

Function	Value at $(1, 1)$
$f_{12} = 6x + 8y$	$f_{12}(1, 1) = 14$
$f_{111} = 6$	$f_{111}(1, 1) = 6$
$f_{222} = 6$	$f_{222}(1, 1) = 6$
$f_{112} = 8$	$f_{112}(1, 1) = 8$
$f_{122} = 6$	$f_{122}(1, 1) = 6$

All the partial derivatives of order four are zero. Hence the remainder $R_4 = 0$.

The required Taylor's expansion will be

$$f(x, y) = f(1, 1) + [(x-1)f_1(1, 1) + (y-1)f_2(1, 1)]$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{11}(1, 1) + 2(x-1)(y-1)f_{12}(1, 1) + (y-1)^2 f_{22}(1, 1)\right]$$
$$+\frac{1}{3!}\left[(x-1)^3 f_{111}(1, 1)\right.$$
$$\left.+3(x-1)^2(y-1)f_{112}(1, 1) + 3(x-1)(y-1)^2 f_{122}(1, 1) + (y-1)^3 f_{222}(1, 1)\right]$$

Substituting all the values we have,

$$f(x, y) = 9 + [13(x-1) + 14(y-1)] + \frac{1}{2!}\left[12(x-1)^2 + 28(x-1)(y-1) + 14(y-1)^2\right]$$
$$+\frac{1}{3!}\left[6(x-1)^3 + 18(x-1)^2(y-1) + 24(x-1)(y-1)^2 + 6(y-1)^3\right]$$
$$= 9 + 13(x-1) + 14(y-1) + 6(x-1)^2 + 14(x-1)(y-1) + 7(y-1)^2$$
$$+ (x-1)^3 + 3(x-1)^2(y-1) + 4(x-1)(y-1)^2 + (y-1)^3$$

We can check the above result by algebra.

RHS of the above equation

$$= 9 + 13x - 13 + 14y - 14 + 6x^2 - 12x + 6 + 14xy - 14x - 14y + 14 + 7y^2 - 14y + 7$$
$$+ x^3 - 1 + 3x - 3x^2 + y^3 - 1 - 3y^2 + 3y + 3x^2y - 6xy + 3y - 3x^2 + 6x - 3 + 4xy^2$$
$$- 8xy + 4x - 4y^2 + 8y - 4$$
$$= x^3 + 3x^2y + 4xy^2 + y^3.$$

Example 5.2.8 *Expand* $(1 - 3x + 2y)^3$ *in powers of* x *and* y.

Solution:

Function	**Value at** $(0,0)$
$f(x,y) = (1-3x+2y)^3$	$f(0,0) = 1$
$f_1 = -9(1-3x+2y)^2$	$f_1(0,0) = -9$
$f_2 = 6(1-3x+2y)^2$	$f_2(0,0) = 6$
$f_{11} = 54(1-3x+2y)$	$f_{11}(0,0) = 54$
$f_{22} = 24(1-3x+2y)$	$f_{22}(0,0) = 24$
$f_{12} = -36(1-3x+2y)$	$f_{12}(0,0) = -36$
$f_{111} = -162$	$f_{111}(0,0) = -162$
$f_{222} = 48$	$f_{222}(0,0) = 48$
$f_{112} = 108$	$f_{112}(0,0) = 108$
$f_{122} = -72$	$f_{122}(0,0) = -72$

All the partial derivatives of order four are zero, so the remainder $R_4 = 0$.

The required Taylor's expansion will be

$$\begin{aligned} f(x,y) &= f(0,0) + [xf_1(0,0) + yf_2(0,0)] \\ &\quad + \frac{1}{2!}\left[x^2 f_{11}(0,0) + 2xy f_{12}(0,0) + y^2 f_{22}(0,0)\right] \\ &\quad + \frac{1}{3!}\left[x^3 f_{111}(0,0) + 3x^2 y f_{112}(0,0) + 3xy^2 f_{122}(0,0) + y^3 f_{222}(0,0)\right] \end{aligned}$$

$$\begin{aligned} f(x,y) &= 1 - 9x + 6y + \frac{1}{2}\left[54x^2 - 72xy + 24y^2\right] \\ &\quad + \frac{1}{6}\left[-162x^3 + 324x^2 y - 216xy^2 + 48y^3\right] \end{aligned}$$

Example 5.2.9 *Expand* $(1-3x+2y)^3$ *in powers of* $(x+1)$ *and* $(y+1)$.

Solution:

Function	**Value at** $(-1,-1)$
$f(x,y) = (1-3x+2y)^3$	$f(-1,-1) = 8$
$f_1 = -9(1-3x+2y)^2$	$f_1(-1,-1) = -36$
$f_2 = 6(1-3x+2y)^2$	$f_2(-1,-1) = 24$
$f_{11} = 54(1-3x+2y)$	$f_{11}(-1,-1) = 108$
$f_{22} = 24(1-3x+2y)$	$f_{22}(-1,-1) = 48$

$f_{12} = -36(1 - 3x + 2y)$	$f_{12}(-1, -1) = -72$
$f_{111} = -162$	$f_{111}(-1, -1) = -162$
$f_{222} = 48$	$f_{222}(-1, -1) = 48$
$f_{112} = 108$	$f_{112}(-1, -1) = 108$
$f_{122} = -72$	$f_{122}(-1, -1) = -72$

All the partial derivatives of order four are zero, so the remainder $R_4 = 0$.

The required Taylor's expansion will be

$$\begin{aligned} f(x, y) = {} & f(-1, -1) + [(x + 1)f_1(-1, -1) + (y + 1)f_2(-1, -1)] \\ & + \frac{1}{2!}\Big[(x + 1)^2 f_{11}(-1, -1) + 2(x + 1)(y + 1)f_{12}(-1, -1) + (y + 1)^2 f_{22}(-1, -1)\Big] \\ & + \frac{1}{3!}\Big[(x + 1)^3 f_{111}(-1, -1) + 3(x + 1)^2(y + 1)f_{112}(-1, -1) \\ & + 3(x + 1)(y + 1)^2 f_{122}(-1, -1) + (y + 1)^3 f_{222}(-1, -1)\Big] \end{aligned}$$

$$\begin{aligned} f(x, y) = {} & 8 + [-36(x + 1) + 24(y + 1)] + \frac{1}{2}\Big[108(x + 1)^2 - 144(x + 1)(y + 1) + 48(y + 1)^2\Big] \\ & + \frac{1}{6}\Big[-162(x + 1)^3 + 324(x + 1)^2(y + 1) - 216(x + 1)(y + 1)^2 + 48(y + 1)^3\Big] \end{aligned}$$

Example 5.2.10 *Expand $e^x \sin y$ in powers of x and y upto second degree terms and write the remainder.*

Solution: By Taylor's expansion about $(0, 0)$ for $n = 2$, we have

$$\begin{aligned} f(x, y) = {} & f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) \\ & + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(\vartheta x, \vartheta x), \quad \text{where } 0 < \vartheta < 1. \end{aligned}$$

Function	**Value of f at** $(0, 0)$
$f(x, y) = e^x \sin y$	0
$f_1(x, y) = e^x \sin y$	0
$f_2(x, y) = e^x \cos y$	1
$f_{11}(x, y) = e^x \sin y$	0
$f_{22}(x, y) = -e^x \sin y$	0
$f_{12}(x, y) = e^x \cos y$	1

Function	Value of f at $x = \vartheta x, y = \vartheta y$
$f_{111}(x, y) = e^x \sin y$	$e^{\vartheta x} \sin \vartheta y$
$f_{222}(x, y) = -e^x \cos y$	$-e^{\vartheta x} \cos \vartheta y$
$f_{122}(x, y) = -e^x \sin y$	$-e^{\vartheta x} \sin \vartheta y$
$f_{112}(x, y) = e^x \cos y$	$e^{\vartheta x} \cos \vartheta y$

$$
\begin{aligned}
f(x, y) =& f(0, 0) + (xf_1(0, 0) + yf_2(0, 0)) \\
&+ \frac{1}{2!}\Big[x^2 f_{11}(0, 0) + 2xy f_{12}(0, 0) + y^2 f_{22}(0, 0)\Big] \\
&+ \frac{1}{3!}\Big[x^3 f_{111}(\vartheta x, \vartheta y) + 3x^2 y f_{112}(\vartheta x, \vartheta y) + 3xy^2 f_{122}(\vartheta x, \vartheta y) \\
&+ y^3 f_{222}(\vartheta x, \vartheta y)\Big] \\
=& 0 + [x(0) + y(1)] + \frac{1}{2}[x^2(0) + 2xy + y^2(0)] \\
&+ \frac{1}{6}[x^3 e^{\vartheta x} \sin \vartheta y + 3x^2 y e^{\vartheta x} \cos \vartheta y - 3xy^2 e^{\vartheta x} \sin \vartheta y - y^3 e^{\vartheta x} \cos \vartheta y] \\
=& y + xy + \frac{e^{\vartheta x}}{6}[\sin \vartheta y(x^3 - 3xy^2) + \cos \vartheta y(3x^2 y - y^3)].
\end{aligned}
$$

Example 5.2.11 *Find the first six terms of the expansion of the function* $e^x \log(1 + y)$ *in a Taylor's series in the neighbourhood of the point* $(0, 0)$.

Solution: The required Taylor's expansion will be

$$
\begin{aligned}
f(x, y) =& f(0, 0) + (xf_1(0, 0) + yf_2(0, 0)) \\
&+ \frac{1}{2!}\Big[x^2 f_{11}(0, 0) + 2xy f_{12}(0, 0) + y^2 f_{22}(0, 0)\Big] + \ldots
\end{aligned}
$$

Function	Value of f at $(0, 0)$
$f(x, y) = e^x \log(1 + y)$	0
$f_1(x, y) = e^x \log(1 + y)$	0
$f_2(x, y) = \frac{e^x}{1 + y}$	1
$f_{11}(x, y) = e^x \log(1 + y)$	0
$f_{22}(x, y) = \frac{-e^x}{(1 + y)^2}$	-1
$f_{12}(x, y) = \frac{e^x}{1 + y}$	1

$$\Rightarrow e^x \log(1+y) = y + xy - \frac{y^2}{2} + \dots$$

Summary

- **Taylor's theorem for function of two variables**

  1. $f(x,y) \in C^{n+1}, \quad |x-a| \le |h|, |y-b| \le |k|$

$$\Rightarrow f(a+h, b+k) = \sum_{j=0}^{n} \frac{1}{j!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^j f(a,b) + R_n$$

  where

$$R_n = \int_0^1 \frac{(1-t)^n}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+ht, b+kt)dt$$

$$= \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\vartheta h, b+\vartheta k), \quad 0 < \vartheta < 1$$

- **Maclaurin's Theorem or Maclaurin's expansion**

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)$$
$$+ \dots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0,0) + R_n$$

  where $R_n = \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\vartheta x + \vartheta y), \quad 0 < \vartheta < 1.$

- Taylor's expansion about the point (a,b) in powers of $(x-a)$ and $(y-b)$

$$f(x,y) = \sum_{j=0}^{n} \frac{1}{j!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^j f(a,b) + R_n$$

$$R_n = \frac{1}{(n+1)!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^{n+1} f(r,s),$$

  where $r = a + \vartheta(x-a), s = b + \vartheta(y-b), \quad 0 < \vartheta < 1$ after the differentiation.

- Taylor's expansion for third and more order derivatives are zero,

$$f(x,y) = f(a,b) + (x-a)f_1(a,b) + (y-b)f_2(a,b)$$
$$+ \frac{1}{2!}\Big[(x-a)^2 f_{11}(a,b) + 2(x-a)(y-b)f_{12}(a,b)$$
$$+ (y-b)^2 f_{22}(a,b)\Big]$$

- Taylor's expansion for $n = 3$, (higher derivatives are not zero)

$$f(x,y) = f(a,b) + (x-a)f_1(a,b) + (y-b)f_2(a,b)$$
$$+\frac{1}{2!}\Big[(x-a)^2 f_{11}(a,b) + 2(x-a)(y-b)f_{12}(a,b) + (y-b)^2 f_{22}(a,b)\Big]$$
$$+\frac{1}{3!}\Big[(x-a)^3 f_{111}(\vartheta x,\vartheta y) + 3(x-a)^2(y-b)f_{112}(\vartheta x,\vartheta y) + 3(x-a)(y-b)^2 f_{122}(\vartheta x,\vartheta y) + (y-b)^3 f_{222}(\vartheta x,\vartheta y)\Big]$$

- Putting $n = 0$ in the Taylor's theorem we get the Law of mean for function of two variables

$$f(a+h,b+k) - f(a,b) = hf_1(a+\vartheta h, b+\vartheta k) + kf_2(a+\vartheta h, b+\vartheta k)$$

where $0 < \vartheta < 1$.

Multiple Choice questions

1. Law of the mean for functions of two variables

   (a) $f(a+h,b+k) - f(a,b) = f_1(a+\vartheta h, b+\vartheta k)k + f_2(a+\vartheta h, b+\vartheta k)h,\ 0 < \vartheta < 1$.

   (b) $f(a+h,b+k) - f(a,b) = f_1(a+\vartheta h, b+\vartheta k)h + f_2(a+\vartheta h, b+\vartheta k)k,\ 0 < \vartheta < 1$.

   (c) $f(a+h,b+k) - f(a,b) = f_{11}(a+\vartheta h, b+\vartheta k)k + f_{22}(a+\vartheta h, b+\vartheta k)h,\ 0 < \vartheta < 1$.

2. If the Taylor's series expansion of $f(x,y)$ in powers of $(x-a)$ and $(x-b)$ is

$$f(a,b) + \beta f(a,b) + \frac{1}{2}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^2 f(a,b)$$

then β is

   (a) $(x-b)\frac{\partial}{\partial x} + (y-a)\frac{\partial}{\partial y}$

   (b) $(x-a)\frac{\partial}{\partial y} + (y-b)\frac{\partial}{\partial x}$

(c) $(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}$

3. Constant term in Taylor's series expansion of $f(x, y) = 3y^3 - 4xy + x^3 - 2x$ in powers $(x - 1)$ and $(y - 1)$ is

   (a) 0

   (b) −2

   (c) 9

4. Taylor's series expansion of the function $f(x, y) = e^{x+y}$ at (0, 0) upto second degree term is

   (a) $1 + (x - y) + \frac{1}{2!}(x^2 - 2xy + y^2)$

   (b) $1 + (x + y) + \frac{1}{2!}(x^2 + 2xy + y^2)$

   (c) $1 - (x - y) + \frac{1}{2!}(x^2 - 2xy + y^2)$

Ans: 1. b) 2. c) 3. b) 4. b)

Exercises 5

1. State and prove Taylor's theorem for function of two variables.

2. Expand $f(x, y) = x^2 + xy - y^2$ in Taylor's series $(a = 1, b = -2)$.

   Ans: $f(x, y) = -5+5(y+2)+\frac{1}{2}\,2(x-1)^2 + 2(x-1)(y+2) - 2(y+2)^2$

3. Expand $(1 - 3x + 2y)^3$ in powers of $(x - 1)$ and $(y + 1)$.

4. Expand $x^2y + \sin y + e^x$ in powers of $(x - 1)$ and $(y - \pi)$ through quadratic terms and write the remainders, without computing ϑ.

   Ans : $x^2y+\sin y+e^x = (\pi+e)+(2\pi+e)(\pi-1)+(\pi+\frac{e}{2})(\pi-1)^2+2(x-1)(y-\pi)+\frac{1}{6}(x-1)^3e^{\vartheta x-\vartheta+1}+(x-1)^2(y-\pi)+\frac{1}{6}(y-\pi)^3\cos(\vartheta y-\vartheta\pi)$.

Unit 6

Maximum and Minimum of functions of two and three variables

Learning Outcomes :

After studying this unit, students will be able

- F To understand, define and identify saddle points, extremum points.
- F To fit a straight line by the method of least squares.
- F To determine extrema of functions of two and three variables.

6.1 Maxima and Minima of functions of two variables

In this unit we shall discuss certain applications of partial differentiation. We shall prove a result for functions of two variables which provides a sufficient condition for the existence of an absolute maximum or minimum at an interior point of the region of definition.

6.1.1 Absolute maximum or minimum

Definition 6.1.1 *A function* $f(x, y)$ *has an absolute maximum at a point* (X, Y) *of a region* $R \iff f(X, Y) \geq f(x, y)$ *for all* (x, y) *in* R.

Definition 6.1.2 *A function* $f(x, y)$ *has a relative maximum at a point* (X, Y) *of a region* $R \iff$ *there exists a positive number* δ *such that*
$f(X, Y) > f(x, y)$ *for all* (x, y) *of* R
at which $0 < (x - X)^2 + (y - Y)^2 < \delta$.

Definition 6.1.3 *A function* $f(x, y)$ *has an absolute minimum at a point* (X, Y) *of a region* $R \iff f(X, Y) \leq f(x, y)$ *for all* (x, y) *in* R.

Definition 6.1.4 *A function* $f(x, y)$ *has a relative minimum at a point* (X, Y) *of a region* R *there exists a positive number* δ *such that*
$f(X, Y) < f(x, y)$ *for all* (x, y) *of* R *at which* $0 < (x - X)^2 + (y - Y)^2 < \delta$.

Theorem 6.1.1

1. $f(x, y) \in C^1$ *in a bounded region* R *consisting of a domain* D *and a boundary curve* Γ.

2. $f(a, b) > f(x, y)$ *for some* $(a, b) \in D$ *and all* $(x, y) \in \Gamma$.

$\Rightarrow$ *There exists a point* $(X, Y) \in D$ *such that*

A. $f(x, y) \leq f(X, Y)$ *for all* $(x, y) \in R$.

B. $f_1(X, Y) = f_2(X, Y) = 0$.

Proof: Since $f(x, y)$ is continuous in the closed region R it has a maximum there. Let it attains its maximum at (X, Y).

By hypothesis (2), $(X, Y) \in D$,

we have $f(X, Y) \geq f(x, y)$ for all $(x, y) \in R$.

Then we have $f(X + \Delta x, Y) \leq f(X, Y)$ for some $(X + \Delta x, Y) \in R$.
Then

$$\frac{f(X + \Delta x, Y) - f(X, Y)}{\Delta x} \geq 0 \quad if\ \Delta x < 0$$
$$\frac{f(X + \Delta x, Y) - f(X, Y)}{\Delta x} \leq 0 \quad if\ \Delta x > 0$$

Let $\Delta x \to 0$ then we have

$$\lim_{\Delta x \to 0} \frac{f(X + \Delta x, Y) - f(X, Y)}{\Delta x} \geq 0 \quad if\ \Delta x < 0$$
$$\lim_{\Delta x \to 0} \frac{f(X + \Delta x, Y) - f(X, Y)}{\Delta x} \leq 0 \quad if\ \Delta x > 0$$

Hence $f_1(X, Y) \geq 0$ and $f_1(X, Y) \leq 0$.
So $f_1(X, Y) = 0$.

We also have $f(X, Y + \Delta y) \leq f(X, Y)$ for some $(X, Y + \Delta y) \in R$. Then

$$\frac{f(X, Y + \Delta y) - f(X, Y)}{\Delta y} \geq 0 \quad if\ \Delta y < 0$$
$$\frac{f(X, Y + \Delta y) - f(X, Y)}{\Delta y} \leq 0 \quad if\ \Delta y > 0.$$

Let $\Delta y \to 0$. We have

$$\lim_{\Delta y \to 0} \frac{f(X, Y + \Delta y) - f(X, Y)}{\Delta y} \geq 0 \quad if\ \Delta y < 0$$
$$\lim_{\Delta y \to 0} \frac{f(X, Y + \Delta y) - f(X, Y)}{\Delta y} \leq 0 \quad if\ \Delta y > 0.$$

Hence $f_2(X, Y) \leq 0$ and $f_2(X, Y) \geq 0$ implies $f_2(X, Y) = 0$. Q

Example 6.1.1 *If* $f(x, y) = \sqrt{4 - x^2 - y^2}$, $x^2 + y^2 \leq 1$, *find the absolute maximum or minimum.*

Solution: Choose $a = b = 0$. Then $f(0, 0) = 2 > f(x, y)|_{x^2+y^2=1} = \sqrt{3}$.
Hence the absolute maximum exists at an interior point (X, Y).

To find the absolute maximum

We have,

$$f_1(x, y) = \frac{-x}{\sqrt{4 - x^2 - y^2}} = 0$$

$$f_2(x, y) = \frac{-y}{\sqrt{4 - x^2 - y^2}} = 0$$

This is true for $(x, y) = (0, 0)$. Hence the absolute maximum for $f(x, y)$ occurs at the origin. At the origin $(0, 0)$, we have $f(x, y) = 2$ at $(0, 0)$.

Example 6.1.2 *If* $f(x, y) = 1 - \sqrt{x^2 + y^2}$, $x^2 + y^2 \leq 1$, *find the absolute maximum or minimum.*

Solution: We cannot apply theorem (6.1.1) because $f(x, y) \notin C^1$.
But inspection we can see

$$f(0, 0) = 1 > f(x, y)|_{x^2+y^2=1} = 0.$$

So the function has the absolute maximum value at $(0, 0)$.

Example 6.1.3 *If* $f(x, y) = x + y$, $x^2 + y^2 \leq 1$, *find the absolute maximum or minimum.*

Solution: $f(x, y) = x + y \Rightarrow f_1(x, y) = 1, f_2(x, y) = 1$.
So the first order partial derivatives does not vanish. Hypothesis 2 fails here. The function attains an absolute maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example 6.1.4 *If* $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$, $x^2 + y^2 < \infty$, *find the absolute maximum or minimum.*

Solution: Using polar coordinates $x = r\cos\vartheta, y = r\sin\vartheta, 0 \leq \vartheta \leq 2\pi$, we have

$$\begin{aligned} f(r\cos\vartheta, r\sin\vartheta) &= r^4\cos^4\vartheta + r^4\sin^4\vartheta - r^2\cos^2\vartheta - r^2\sin^2\vartheta + 1 \\ &= r^4(\cos^4\vartheta + \sin^4\vartheta) - r^2(\cos^2\vartheta + \sin^2\vartheta) + 1 \\ &= r^4(\cos^4\vartheta + \sin^4\vartheta) - r^2 + 1 \end{aligned}$$

On the circle $r = r_0$, we have

$$r^4(\cos^4\vartheta + \sin^4\vartheta) - r^2 + 1 = r_0^4(\cos^4\vartheta + \sin^4\vartheta) - r_0^2 + 1$$

So the first term should be $\geq \frac{r_0^4}{2}$.

Hence on the circle $r = 2$,

$f \geq 8 - 4 + 1 = 5$ and $f(0, 0) = 1$.

So that the absolute minimum exists.

To find absolute minimum

$$f_1(x, y) = 4x^3 - 2x, \quad f_2(x, y) = 4y^3 - 2y.$$

For absolute minimum

$$4X^3 - 2X = 0 \Rightarrow 2X(2X^2 - 1) = 0 \Rightarrow X = 0, X = \pm\sqrt{\frac{1}{2}}$$

$$4Y^3 - 2Y = 0 \Rightarrow 2Y(2Y^2 - 1) = 0 \Rightarrow Y = 0, Y = \pm\sqrt{\frac{1}{2}}$$

There are nine points where both equations hold $(0, 0)$, $(0, \frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, 0)$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Point	**Value of f**
$(0, 0)$	1
$(0, \frac{1}{\sqrt{2}})$	$\frac{3}{4}$
$(0, -\frac{1}{\sqrt{2}})$	$\frac{3}{4}$
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{2}$
$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\frac{1}{2}$
$(-\frac{1}{\sqrt{2}}, 0)$	$\frac{3}{4}$
$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{2}$
$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\frac{1}{2}$
$(\frac{1}{\sqrt{2}}, 0)$	$\frac{3}{4}$

Hence we find that there are absolute minimum at four points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and value of the function $= \frac{1}{2}$

So $f \geq \frac{1}{2}$ in all the plane.

6.2 Sufficient conditions

A sufficient condition for relative maxima and minima is obtained in this section.

6.2.1 Relative Extrema

Theorem 6.2.1

1. $f(x, y) \in C^2$
2. $f_1 = f_2 = 0$ *at* (X, Y)

1. $f_{12}^2 - f_{12}f_{22} < 0$ *at* (X, Y)

4. $f_{11} < 0$ *at* (X, Y)

$\Rightarrow f(x, y)$ *has a relative maximum at* (X, Y).

Proof: Suppose $f_1(X, Y) = 0$ and $f_2(X, Y) = 0$.
Suppose that $f(x, y)$ possesses continuous second order partial derivatives in a certain neighborhood of (X, Y) and these derivatives at (X, Y) are $f_{11}(X, Y), f_{22}(X, Y), f_{12}(X, Y)$ and not all zero.
Choose a point $(X + h, Y + k)$ in the neighborhood of (X, Y).
By Taylor's theorem with remainder we have for $0 < \vartheta < 1$,

$$f(X+h, Y+k) = f(X, Y)+[hf_1(X, Y) + kf_2(X, Y)]+\frac{1}{2!}\Big[h^2 f_{11}(X + \vartheta h, Y + \vartheta k)$$
$$+2hkf_{12}(X + \vartheta h, Y + \vartheta k) + k^2 f_{22}(X + \vartheta h, Y + \vartheta k)\Big].$$

Since $f_1(X, Y) = 0, f_2(X, Y) = 0$

$$f(X + h, Y + k) - f(X, Y) = \frac{1}{2}\Big[h^2 f_{11}(X + \vartheta h, Y + \vartheta k)$$
$$+2hkf_{12}(X + \vartheta h, Y + \vartheta k) + k^2 f_{22}(X + \vartheta h, Y + \vartheta k)\Big].$$

Let $$\Delta f = f(X + h, Y + k) - f(X, Y) = \frac{1}{2}\Big[Ah^2 + 2Bhk + Ck^2\Big] \qquad (6.1)$$

where

$$\begin{aligned} A &= f_{11}(X + \vartheta h,\ Y + \vartheta k) \\ B &= f_{12}(X + \vartheta h,\ Y + \vartheta k) \\ C &= f_{22}(X + \vartheta h,\ Y + \vartheta k), \quad 0 < \vartheta < 1 \end{aligned} \tag{6.2}$$

Suppose $A \neq 0$, then

$$\begin{aligned} \Delta f &= \frac{1}{2}(Ah^2 + 2Bhk + Ck^2) \\ &= \frac{1}{2A}\left[A^2h^2 + 2ABhk + ACk^2 + B^2k^2 - B^2k^2\right] \\ &= \frac{1}{2A}\left[(Ah + Bk)^2 + (AC - B^2)k^2\right] \end{aligned}$$

Since $f(x, y) \in C^2$, inequalities $f_{12}^2 - f_{11}f_{22} < 0$ and $f_{11} < 0$ will also hold in some circle of radius δ and center at (X, Y),

This circle will contain in its interior the point $(X+\vartheta h, Y+\vartheta k)$ if $h^2+k^2 < \delta^2$ and hence $A < 0, AC - B^2 > 0$. Then

$$\Delta f = \frac{1}{2A}\left[(Ah + Bk)^2 + (AC - B^2)k^2\right] < 0.$$

So $f(X + h, Y + K) - f(X, Y) < 0$ implies $f(X + h, Y + k) < f(X, Y)$.

That is $f(x, y)$ has a relative maximum at (X, Y). Q

Remark : For relative minimum we have following theorem.

Theorem 6.2.2

1. $f(x, y) \in C^2$
2. $f_1 = f_2 = 0$ *at* (X, Y)
3. $f_{12}^2 - f_{11}f_{22} < 0$ *at* (X, Y)
4. $f_{11} > 0$ *at* (X, Y)

$\Rightarrow f(x, y)$ *has a relative minimum at* (X, Y).

6.2.2 Saddle points

Definition 6.2.1 *A function* $f(x, y)$ *has a saddle point at* (X, Y) *if* $f_1(X, Y) = f_2(X, Y) = 0$ *and if* $\Delta f = f(X + h, Y + k) - f(X, Y)$ *will have* *both positive and negative values in every neighborhood of* (X, Y), *where* $(X + h, Y + k)$ *is a point in the neighborhood of* (X, Y).

Theorem 6.2.3

1. $f(x, y) \in C^2$
2. $f_1 = f_2 = 0$ *at* (X, Y)
3. $f_{12}^2 - f_{11}f_{22} > 0$ *at* (X, Y)

$\Rightarrow f(x, y)$ *has a saddle point at* (X, Y).

Proof: Suppose $f(x, y) \in C^2$, $f_1 = f_2 = 0$ at (X, Y).

By Taylor's theorem with remainder we have for $0 < \vartheta < 1$,

$$f(X+h, Y+k) = f(X, Y) + [hf_1(X, Y) + kf_2(X, Y)] + \frac{1}{2!}\Big[h^2 f_{11}(X + \vartheta h, Y + \vartheta k)$$
$$+ 2hkf_{12}(X + \vartheta h, Y + \vartheta k) + k^2 f_{22}(X + \vartheta h, Y + \vartheta k)\Big].$$

Since $f_1(X, Y) = 0, f_2(X, Y) = 0$

$$f(X + h, Y + k) - f(X, Y) = \frac{1}{2}\Big[h^2 f_{11}(X + \vartheta h, Y + \vartheta k)$$
$$+ 2hkf_{12}(X + \vartheta h, Y + \vartheta k) + k^2 f_{22}(X + \vartheta h, Y + \vartheta k)\Big].$$

$$\Delta f = f(X + h, Y + k) - f(X, Y) = \frac{1}{2}[Ah^2 + 2Bhk + Ck^2]$$

where

$$A = f_{11}(X + \vartheta h, Y + \vartheta k)$$

$$B = f_{12}(X + \vartheta h, Y + \vartheta k)$$

$$C = f_{22}(X + \vartheta h, Y + \vartheta k), \quad 0 < \vartheta < 1.$$

Let $a = f_{11}(X, Y)$, $b = f_{12}(X, Y)$, $c = f_{22}(X, Y)$. As h and k approach zero, A, B and C approach a, b and c respectively.

We have three cases.

Case (i). Suppose $a \neq 0$. First set $h = \lambda, k = 0$ then,

$$\Delta f = \frac{1}{2}[A\lambda^2]$$
$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{\frac{1}{2}A\lambda^2}{\lambda^2} = \lim_{\lambda \to 0} \frac{A}{2} = \frac{a}{2}.$$

Now set $h = -\lambda b, k = \lambda a$ then

$$\Delta f = \frac{1}{2}[A(-\lambda b)^2 + 2B(-\lambda b)(\lambda a) + C(\lambda a)^2]$$
$$= \frac{1}{2}[A\lambda^2 b^2 - 2B\lambda^2 ab + C\lambda^2 a^2]$$

Now

$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{\frac{\lambda^2}{2}[Ab^2 - 2Bab + Ca^2]}{\lambda^2}$$
$$= \lim_{\lambda \to 0} \frac{\frac{1}{2}[Ab^2 - 2Bab + Ca^2]}{\lambda^2}$$
$$= \frac{1}{2}\lim_{\lambda \to 0} Ab^2 - \lim_{\lambda \to 0} Bab + \frac{1}{2}\lim_{\lambda \to 0} Ca^2$$
$$= \frac{1}{2}b^2 a - b^2 a + \frac{1}{2}ca^2 = -\frac{1}{2}b^2 a + \frac{1}{2}ca^2 = \frac{a}{2}(ac - b^2).$$

By hypothesis 3, $f_{12}^2 - f_{11}f_{22} > 0$ that is $b^2 - ac > 0$ we have $ac - b^2 < 0$.

$$\text{If } a < 0, \quad \text{then } \frac{a}{2}(ac - b^2) > 0.$$
$$\text{If } a > 0, \quad \text{then } \frac{a}{2}(ac - b^2) < 0.$$

So the above two limits have opposite signs. Since $f(x, y) \in C^2$, Δf will have opposite signs for small λ in the two cases.

Case (ii). Suppose $c \neq 0$

Set $h = 0, k = \lambda,$ then

$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{C}{2} = \frac{c}{2}$$

Now set, $h = \lambda c, k = -\lambda b.$

$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{\lambda^2}{2} \frac{Ac^2 - 2Bcb + Cb^2}{\lambda^2}$$
$$= \frac{ac^2 - b^2 c}{2}$$
$$= \frac{c}{2}(ac - b^2).$$

So as in case (i) Δf will have opposite signs for small λ in the two cases $c < 0$ and $c > 0$.

Case (iii). Suppose $a = c = 0$, then $b \neq 0$

First set $h = k = \lambda$. Then

$$\Delta f = \frac{1}{2}[A\lambda^2 + 2B\lambda^2 + C\lambda^2] = \frac{\lambda^2}{2}(A + 2B + C)$$

$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{1}{2}(A + 2B + C) = \frac{1}{2}(a + 2b + c) = b$$

Then set $h = -k = \lambda$ then

$$\lim_{\lambda \to 0} \frac{\Delta f}{\lambda^2} = \lim_{\lambda \to 0} \frac{1}{2}(A - 2B + C) = \frac{1}{2}(a - 2b + c) = -b$$

Again from case (i) we have the desired result. So in all cases Δf will have both positive and negative values in every neighborhood of (X, Y).

Hence $f(x, y)$ has a saddle point at (X, Y). Q

Example: Consider $f(x, y) = xy$. $f_1 = y, f_2 = x, f_{11} = 0, f_{12} = 1$,

$f_{12}^2 - f_{11}f_{22} = 1 > 0$. The origin is a saddle point.

Summary of the results :

Suppose that

1. $f(x, y) \in C^2$
2. $f_1 = f_2 = 0$ at (X, Y)

(i) If $f_{12}^2 - f_{11}f_{22} < 0$ at (X, Y) and $f_{11} < 0$ at (X, Y), then $f(x, y)$ has a relative maximum at (X, Y).

(ii) If $f_{12}^2 - f_{11}f_{22} < 0$ at (X, Y) and $f_{11} > 0$ at (X, Y), then $f(x, y)$ has a relative minimum at (X, Y).

(iii) If $f_{12}^2 - f_{11}f_{22} > 0$ at (X, Y) and $f_1 = f_2 = 0$ at (X, Y), then $f(x, y)$ has a saddle point at (X, Y).

Example 6.2.1 *Find the maximum and minimum of the function*

$f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$.

Solution:

$f_1 = 4x^3 - 2x$ and $f_2 = 4y^3 - 2y$.

For stationary points $f_1 = 0, f_2 = 0$

$$4x^3 - 2x = 0 \Rightarrow 2x(2x^2 - 1) = 0 \Rightarrow x = 0, x = \pm\sqrt{\frac{1}{2}}$$

$$4y^3 - 2y = 0 \Rightarrow 2y(2y^2 - 1) = 0 \Rightarrow y = 0, y = \pm\sqrt{\frac{1}{2}}$$

There are nine points $(0,0)$, $(0, \sqrt{\frac{1}{2}})$, $(0, -\sqrt{\frac{1}{2}})$, $(\sqrt{\frac{1}{2}}, 0)$, $(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$, $(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}})$, $(-\sqrt{\frac{1}{2}}, 0)$, $(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$, $(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}})$

$f_{11} = 12x^2 - 2, \quad f_{22} = 12y^2 - 2, \quad f_{12} = 0, \quad , f_{21} = 0$

$f_{11}^2 - f_{11}f_{22} = -(12x^2 - 2)(12y^2 - 2)$

Point	$f_{12}^2 - f_{11}f_{22}$	f_{11}	Extreme value	**Value of f**
$(0,0)$	$-4 < 0$	$-2 < 0$	Relative maximum	1
$(0, \sqrt{\frac{1}{2}})$	8	$-2 < 0$	Saddle point	$-\frac{3}{4}$
$(0, -\sqrt{\frac{1}{2}})$	8	$-2 < 0$	Saddle point	$-\frac{3}{4}$
$(\sqrt{\frac{1}{2}}, 0)$	8	$4 > 0$	Saddle point	$-\frac{3}{4}$
$(-\sqrt{\frac{1}{2}}, 0)$	8	$4 > 0$	Saddle point	$-\frac{3}{4}$
$(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	-16	4	Relative minimum	$\frac{1}{2}$
$(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}})$	-16	4	Relative minimum	$\frac{1}{2}$
$(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	-16	4	Relative minimum	$\frac{1}{2}$
$(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}})$	-16	4	Relative minimum	$\frac{1}{2}$

Hence there is relative maximum at $(0,0)$ and there are relative minimum at the four points $(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}})$ and $(0, \pm\sqrt{\frac{1}{2}})$, $(\pm\sqrt{\frac{1}{2}}, 0)$ are saddle points.

6.2.3 Least Squares

For fitting a straight line $y = ax+b$ through the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by the method of least squares, we have to determine constants a and b so that $f(a, b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2$ should be minimum. We have

$$f_1(a, b) = 2\sum_{i=1}^{n}(ax_i + b - y_i)x_i = 0$$

$$f_2(a, b) = 2\sum_{i=1}^{n}(ax_i + b - y_i) = 0$$

$$f_{11}(a, b) = 2\sum_{i=1}^{n} x_i^2$$

$$f_{12}(a, b) = 2\sum_{i=1}^{n} x_i$$

$$f_{22}(a, b) = 2\sum_{i=1}^{n} 1 = 2n$$

Also

$$f_{11}f_{22} - f_{12}^2 = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} 1 - \left(\sum_{i=1}^{n} x_i\right)^2$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2 > 0.$$

So we can apply theorem 6.2.1 and theorem 6.1.1 is also applicable, because the unique relative minimum thus assumed must also be an absolute minimum.

If $n = 3$,

$$\sum_{i=1}^{3} 1 \sum_{i=1}^{3} x_i^2 - \left(\sum_{i=1}^{3} x_i\right)^2 = 3(x_1^2 + x_2^2 + x_3^2) - (x_1 + x_2 + x_3)^2$$

$$= 3(x_1^2 + x_2^2 + x_3^2) - x_1^2 - x_2^2 - x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$$

$$= 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1).$$

We now solve equations $f_1(a, b) = 0$ and $f_2(a, b) = 0$ for a and b and substitute these values in the equation of the line. We obtain

$$\begin{vmatrix} x & y & 1 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} 1 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_iy_i & \sum_{i=1}^{n} x_i \end{vmatrix} = 0.$$

Example 6.2.2 *Find the line through the points* (1, 2), (0, 0), (2, 2).

Solution: Given $x_1 = 1, y_1 = 2, x_2 = 0, y_2 = 0, x_3 = 2, y_3 = 2.$

$$\begin{vmatrix} x & y & 1 \\ \sum_{i=1}^{3} x_i & \sum_{i=1}^{3} y_i & \sum_{i=1}^{3} 1 \\ \sum_{i=1}^{3} x_i^2 & \sum_{i=1}^{3} x_i y_i & \sum_{i=1}^{3} x_i \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ 3 & 4 & 3 \\ 5 & 6 & 3 \end{vmatrix} = 0$$

$$x(12 - 18) - y(9 - 15) + (18 - 20) = 0$$

$$-6x + 6y - 2 = 0$$

$$3x - 3y + 1 = 0.$$

6.3 Functions of Three Variables

6.3.1 Quadratic forms

Definition 6.3.1 *Quadratic form in three variables is defined by*

$$\begin{aligned} F(x_1, x_2, x_3) &= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j, \quad a_{ij} = a_{ji} \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 \\ &\quad + a_{31}x_3x_1 + a_{32}x_2x_3 + a_{33}x_3^2 \end{aligned} \tag{6.3}$$

(i) Positive definite A quadratic form is positive definite if, and only if $F(x_1, x_2, x_3) > 0$ except when $x_1 = x_2 = x_3 = 0$.

Clearly $F(0, 0, 0) = 0$.

(ii) Positive semi- definite A quadratic form is positive semi-definite if, and only if, $F(x_1, x_2, x_3) \geq 0$, the equality holds for certain values of x_1, x_2, x_3 not all zero.

Example: Consider $F = x_1^2 + x_2^2 + x_3^2, \quad G = x_1^2 + x_2^2$.

Here $F(x_1, x_2, x_3) > 0$ except when $x_1 = x_2 = x_3 = 0$.

So F is positive definite.

If G is considered as a form of three variables then $G(0, 0, 1) = 0$. So G is positive semi-definite. But if we consider G as a form of two variables x_1, x_2, then G is positive definite.

Remark 1: We can also define negative definite forms.

Remark 2: The form in two variables $Ax_1^2 + 2Bx_1x_2 + Cx_2^2$ is positive definite if, and only if,

$$A > 0, \quad \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0 \qquad (6.4)$$

that is $A > 0$ and $AC - B^2 > 0$.

Lemma 6.3.1 *The form*

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j, \quad a_{ij} = a_{ji}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3$$

$$+ a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3$$

$$+ a_{31}x_3x_1 + a_{32}x_2x_3 + a_{33}x_3^2$$

is positive definite $\Leftarrow\Rightarrow$

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

Proof: We only prove the sufficient part.

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \; a_{11} > 0, \; \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0.$

$\vdots a_{31}$.

A_{ij} = co- factor of a_{ij}.

By using the formula for the product of two determinants, we have

$$\Delta \begin{vmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12}A_{22} + a_{13}A_{32} & a_{22}A_{22} + a_{23}A_{32} & a_{32}A_{22} + a_{33}A_{32} \\ a_{12}A_{23} + a_{33}A_{13} & a_{22}A_{23} + a_{23}A_{33} & a_{32}A_{23} + a_{33}A_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ -a_{11}A_{21} & \Delta - a_{21}A_{21} & -a_{31}A_{21} \\ -a_{11}A_{31} & -a_{21}A_{31} & \Delta - a_{31}A_{31} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$$

$$= a_{11}\Delta^2.$$

So $$\Delta \begin{vmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{vmatrix} = a_{11}\Delta^2$$

Hence $$\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = a_{11}\Delta$$

Now collecting the terms in x_1^2 and x_1 in the form (6.3) as

$$F = x_1^2(a_{11}) + x_1(a_{12}x_2 + a_{13}x_3 + a_{21}x_2 + a_{31}x_3) + a_{22}x_2^2 + a_{23}x_2x_3 + a_{32}x_3x_2 + a_{33}x_3^2$$

$$F = Ax_1^2 + 2Bx_1 + C$$

where, $A = a_{11}$, $B = a_{12}x_2 + a_{13}x_3$, $C = a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2$.

To prove that F is positive definite. we have to prove $F > 0$ except when $x_1 = x_2 = x_3 = 0$.

For this we shall prove that $AC - B^2 > 0$ unless $x_2 = x_3 = 0$, by (6.4) this will imply $F > 0$.

Suppose $x_2 = x_3 = 0$. Then $A = a_{11}$, $B = 0$, $C = 0$.
So $F = Ax_1^2 = a_{11}x_1^2$. This is positive unless $x_1 = 0$.

So F is positive definite.

Consider $AC - B^2$ and collect the terms with x_2^2, x_2x_3 and x_3^2.

$$AC - B^2 = a_{11}(a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2) - a_{12}^2x_2^2 - a_{13}^2x_3^2 - 2a_{12}a_{13}x_2x_3$$
$$= x_2^2[a_{11}a_{22} - a_{12}a_{21}] + 2x_2x_3[a_{11}a_{23} - a_{12}a_{13}] + x_3^2[a_{11}a_{33} - a_{13}a_{31}]$$

$$AC - B^2 = A_{33}x_2^2 - 2A_{23}x_2x_3 + A_{22}x_3^2$$

To show that this is always positive, unless $x_2 = x_3 = 0$, we use (6.4), we need

$$A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = a_{11}\Delta = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

This is true by hypothesis. Hence F is positive definite. Q

6.3.2 Relative Extrema

Theorem 6.3.1

1. $f(x, y, z) \in C^2$

2. $f_1 = f_2 = f_3 = 0$ *at* (X, Y, Z)

3. $f_{11} > 0, \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0 \quad at \quad (X, Y, Z)$

$\Rightarrow f(x, y, z)$ *has a relative minimum at* (X, Y, Z).

Proof: Suppose that $f(x, y, z) \in C^2$ and $f_1 = f_2 = f_3 = 0$ at (X, Y, Z) Choose a point $(X + h_1, Y + h_2, Z + h_3)$ in the neighbourhood of (X, Y, Z).

By Taylor's formula with remainders we have for $0 < \vartheta < 1$,

$$f(X + h_1, Y + h_2, Z + h_3)$$
$$= f(X, Y, Z) + [h_1 f_1(X, Y, Z) + h_2 f_2(X, Y, Z) + h_3 f_3(X, Y, Z)]$$
$$+ \frac{1}{2!}\sum_{i=1}^{3}\sum_{j=1}^{3} f_{ij}(X + \vartheta h_1, Y + \vartheta h_2, Z + \vartheta h_3) h_i h_j$$

Since $f_1 = f_2 = f_3 = 0$ at (X, Y, Z),

$$\Delta f = f(X + h_1, Y + h_2, Z + h_3) - f(X, Y, Z)$$
$$= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} f_{ij}(X + \vartheta h_1, Y + \vartheta h_2, Z + \vartheta h_3) h_i h_j$$

where $0 < \vartheta < 1$. Since $f(x, y, z) \in C^2$, it is clear that inequalities $f_{11} > 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$ also hold in some neighbourhood of (X, Y, Z).

If the point $(X + h_1, Y + h_2, Z + +h_3)$ is in this neighbourhood, the coefficients of the quadratic form Δf will satisfy the conditions of lemma 6.3.1, so that $\Delta f > 0$ throughout the neighbourhood, except at $h_1 = h_2 = h_3 = 0$ where $\Delta f = 0$.

Hence, f has a relative minimum at (X, Y, Z). Q

Remark : For relative maximum we have the following theorem.

Theorem 6.3.2

1. $f(x, y, z) \in C^2$
2. $f_1 = f_2 = f_3 = 0$ *at* (X, Y, Z)
3. $f_{11} < 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0$

$\Rightarrow f(x, y, z)$ *has a relative minimum at* (X, Y, Z).

Example 6.3.1 *Prove that the function* $f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$ *has a relative minimum at* $(0, 0, 0)$.

Solution:

$$f_1 = 2x - y + 2z,\ f_{11} = 2,\quad f_{12} = -1,\quad f_{13} = 2.$$

$$f_2 = -x + 2y + z,\ f_{21} = -1,\quad f_{22} = 2,\quad f_{23} = 1.$$

$$f_3 = 2x - y + 6z,\ f_{31} = 2,\ f_{32} = 1,\ f_{33} = 6.$$

$f(0, 0, 0) = 0.$

Now we will check the conditions, $f_{11} > 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$

for $X = Y = Z = 0$.

$$f_{11} = 2 > 0,\quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0,$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 2(12-1) + 1(-6-2) + 2(-1-4)$$

$$= 22 - 8 - 10 = 4 > 0$$

Hence $f(x, y, z) \geq f(0, 0, 0) = 0$.

Therefore $f(x, y, z)$ has a relative minimum at $(0, 0, 0)$.

Example 6.3.2 *Show that* $f(x, y, z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$ *has a relative minima at* $(1, 1, 1)$ *and relative maxima at* $(-1, -1, -1)$.

Solution: We have

$$f_1 = 3(x + y + z)^2 - 24yz - 3$$

$$f_2 = 3(x + y + z)^2 - 24zx - 3$$

$$f_3 = 3(x + y + z)^2 - 24xy - 3$$

The stationary points are given by

$$(x + y + z)^2 - 8yz - 1 = 0$$

$$(x + y + z)^2 - 8zx - 1 = 0$$

$$(x + y + z)^2 - 8xy - 1 = 0$$

Subtracting second equation from the first, $z(x - y) = 0$.

Similarly $x(y - z) = 0$, $y(z - x) = 0$.

$\Rightarrow$ Either $x = 0, y = 0, z = 0$ or $x = y = z$.

Therefore stationary points are (0, 0, 0), (1, 1, 1), (−1, −1, −1).

Again, we have

$$f_{11} = 6(x + y + z) = f_{22} = f_{33}$$

$$f_{12} = 6(x + y + z) - 24z = f_{21}$$

$$f_{23} = 6(x + y + z) - 24x = f_{32}$$

$$f_{31} = 6(x + y + z) - 24y = f_{13}.$$

At (1, 1, 1),

$$f_{11} = f_{22} = f_{33} = 18$$

$$f_{12} = f_{23} = f_{31} = -6$$

$$f_{11} = 18 > 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288 > 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix} = 5184 - 864 - 864 = 3456 > 0.$$

Therefore the function has a relative minimum at (1, 1, 1).

At (−1, −1, −1)

$$f_{11} = f_{22} = f_{33} = -18$$

$$f_{12} = f_{23} = f_{31} = 6$$

$$f_{11} = -18 < 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288 > 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -5184 + 864 + 864 = -3456 < 0.$$

Hence the function has a relative maximum at (−1, −1, −1).

Summary

- A function $f(x, y)$ has an absolute maximum at a point (X, Y) of a region $R \Longleftrightarrow f(X, Y) \geq f(x, y)$ for all (x, y) in R.

- A function $f(x, y)$ has a relative maximum at a point (X, Y) of a region $R \Longleftrightarrow$ there exists a positive number δ such that
  $f(X, Y) > f(x, y)$ for all (x, y) of R
  at which $0 < (x - X)^2 + (y - Y)^2 < \delta$.

- A function $f(x, y)$ has an absolute minimum at a point (X, Y) of a region $R \Longleftrightarrow f(X, Y) \leq f(x, y)$ for all (x, y) in R.

- A function $f(x, y)$ has a relative minimum at a point (X, Y) of a region R there exists a positive number δ such that
  $f(X, Y) < f(x, y)$ for all (x, y) of R at which $0 < (x-X)^2 + (y-Y)^2 < \delta$.

- A function $f(x, y)$ has a saddle point at (X, Y) if
  $f_1(X, Y) = f_2(X, Y) = 0$ and if $\Delta f = f(X + h, Y + k) - f(X, Y)$ will have both positive and negative values in every neighborhood of (X, Y), where $(X + h, Y + k)$ is a point in the neighborhood of (X, Y).

- Suppose that

  1. $f(x, y) \in C^2$
  2. $f_1 = f_2 = 0$ at (X, Y)

  (i) If $f_{12}^2 - f_{11}f_{22} < 0$ at (X, Y) and $f_{11} < 0$ at (X, Y), then $f(x, y)$ has a relative maximum at (X, Y).

  (ii) If $f_{12}^2 - f_{11}f_{22} < 0$ at (X, Y) and $f_{11} > 0$ at (X, Y), then $f(x, y)$ has a relative minimum at (X, Y).

  (iii) If $f_{12}^2 - f_{11}f_{22} > 0$ at (X, Y) and $f_1 = f_2 = 0$ at (X, Y), then $f(x, y)$ has a saddle point at (X, Y).

- For fitting a straight line $y = ax+b$ through the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by the method of least squares. We have to determine constants a and b so that $f(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$ should be minimum

- By the method of least square, the straight line $y = ax + b$ through the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is given by

$$\begin{vmatrix} x & y & 1 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} 1 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \end{vmatrix} = 0$$

- Quadratic form in three variables

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j, \quad a_{ij} = a_{ji}$$

- A quadratic form is

(i) Positive definite if, and only if $F(x_1, x_2, x_3) > 0$ expect when $x_1 = x_2 = x_3 = 0$.

(ii) Positive semi-definite if, and only if $F(x_1, x_2, x_3) \geq 0$, the equality holding for certain values of x_1, x_2, x_3 not all zero.

- The quadratic form in two variables $Ax_1^2 + 2Bx_1x_2 + Cx_2^2$ is positive definite if, and only if $A > 0$, $\begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0$

- The quadratic form $F(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j, \quad a_{ij} = a_{ji}$ is positive definite if, and only if

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$$

- If $f(x, y, z) \in C^2$, $f_1 = f_2 = f_3 = 0$ at (X, Y, Z) then at (X, Y, Z)

(i) if $f_{11} > 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$

then $f(x, y, z)$ has a relative minimum at (X, Y, Z)

(ii) if $f_{11} < 0$, $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0$, $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0$

then $f(x, y, z)$ has a relative maxi mum at (X, Y, Z)

Multiple Choice questions

1. Choose the incorrect statement.

   (a) A function $f(x, y)$ has an absolute maximum at a point (X, Y) of a region if and only if $f(X, Y) \leq f(x, y)$ for all (x, y) in R.

   (b) A function $f(x, y)$ has an absolute minimum at a point (X, Y) of a region if and only if $f(X, Y) \leq f(x, y)$ for all (x, y) in R.

   (c) A function $f(x, y)$ has a relative maximum at a point (X, Y) of a region R iff there exists a positive number δ such that $f(X, Y) < f(x, y)$ for all (x, y) of R at when $0 < (x - X)^2 + (y - Y)^2 < \delta$.

2. The saddle point of function $f(x, y) = xy$ is

   a) $(-1. -1)$ b) $(1, 1)$ c) Origin

3. A quadratic form in three variables is positive definite if and only if

   (a) $F(x_1, x_2, x_3) > 0$, except when $x_1 = x_2 = x_3 = 0$.

   (b) $F(x_1, x_2, x_3) \geq 0$ for certain values of x_1, x_2, x_3 not all zero.

   (c) $F(x_1, x_2, x_3) < 0$ for certain values if x_1, x_2, x_3 not all zero.

4. The form in two variables $Ax_1^2 + 2Bx_1x_2 + Cx_2^2$ is positive definite if, and only if

   (a) $A < 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} < 0$

(b) $A > 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} > 0$.

(c) $A < 0$, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} > 0$.

Ans: 1. a) 2. c) 3. a) 4. b)

Exercises 6

1. Define absolute maximum and absolute minimum.

2. If

   (a) $f(x, y) \in C^1$ in a bounded region R consisting of a domain D and a boundary curve Γ.

   (b) $f(a, b) > f(x, y)$ for some $(a, b) \in D$ and all $(x, y) \in \Gamma$.

   then prove that there exists a point $(X, Y) \in D$ such that

   A. $f(x, y) \le f(X, Y)$ for all $(x, y) \in R$.

   B. $f_1(X, Y) = f_2(X, Y) = 0$.

3. Find the absolute maximum and absolute minimum of $f(x, y) = x^2 + 2y^2 - x$ on the set $x^2 + y^2 \le 1$.

4. Show that the function $x^4+y^4-2x^2+8y^2+4$ has an absolute minimum.

5. Define relative maximum and relative minimum.

6. Define saddle point.

7. Explain principle of least squares.

8. Find the relative maxima and minima of

   $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

   Ans: Relative maximum at $(1, -2)$, Relative minimum at $(1, 2)$, Saddle points: $(-1, 2)$, $(-1, -2)$.

9. Test the following functions for relative maxima, relative minima and saddle points.

   (i) $x^2 + 2xy + 2y^2 + 4x$

   (ii) $x^3 - y^3 + 3x^2 + 3y^2 - 9x$

10. Pass a line through the following points by least squares:

$$(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 2).$$

11. Define quadratic forms.

12. Define positive definite and positive semi-definite.

13. Prove that the form

$$F(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j, \quad a_{ij} = a_{ji}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2^2$$

$$+ a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_2x_3 + a_{33}x_3^2$$

is positive definite if and only if

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

14. If

   (a) $f(x, y, z) \in C^2$

   (b) $f_1 = f_2 = f_3 = 0$ at (X, Y, Z)

   (c) $f_{11} > 0, \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0$

   then prove that $f(x, y, z)$ has a relative minimum at (X, Y, Z).

Unit 7

Lagrange's Multipliers

Learning Outcomes :

After studying this unit, students will be able

- F To determine extrema of function using Lagrange's multipliers.

7.1 Lagrange's Multipliers

A problem of considerable importance for applications is that of maximizing or minimizing a function of several variables, where the variables are related by one or more equations. To handle such problems, if possible we can eliminate some of the variables by using the given conditions and reduce the problem to an ordinary maximum and minimum problem.
This procedure is not always feasible and the following procedure by introducing extreneous parameters, known as Lagrange's multipliers is often more convenient. It also treats the variables in a more symmetrical way, so that various simplifications may be possible. We shall illustrate the method in several cases.

7.1.1 One relation between two variables

To maximize a function

$$u = f(x, y) \tag{7.1}$$

where x and y are connected by an equation

$$g(x, y) = 0 \tag{7.2}$$

Let us suppose $f, g \in C^1$ and $g_1^2 + g_2^2 > 0$ in a region of the $xy-$ plane.

Case:(i). Suppose g_2 is not zero, we solve equation (7.2) for y and substitute in equation (7.1) thus treating x as the independent variable.

In this case a necessary condition for maximum or minimum is

$$\frac{du}{dx} = 0$$

$$f_1 - f_2\frac{g_1}{g_2} = 0$$

The simultaneous solutions of the equations

$$\frac{\partial(f, g)}{\partial(x, y)} = 0, \quad g(x, y) = 0$$

gives the desired points.

Case:(ii). Suppose $g_1 \neq 0$, we take y as the independent variable. But in this case also we are led to the same pair of equation

$$\frac{\partial(f, g)}{\partial(x, y)} = 0, g(x, y) = 0.$$

To solve the same problem by the method of Lagrange, introduce the Lagrange multiplier λ, forming the function

$$\text{Let } V = f(x, y) + \lambda g(x, y).$$

$$\frac{\partial V}{\partial x} = f_1 + \lambda g_1, \quad \frac{\partial V}{\partial y} = f_2 + \lambda g_2.$$

For maxima and minima , we have $\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0$. We must solve the three equations $g(x, y) = 0, f_1 + \lambda g_1 = 0, f_2 + \lambda g_1 = 0$ for x, y and λ.

Theorem 7.1.1

1. *$f(x, y), g(x, y) \in C^1$ in a domain D*
2. *$g_1^2 + g_2^2 > 0$ in D*

$\Rightarrow$ *The set of points (x, y) on the curve $g(x, y) = 0$, where $f(x, y)$ has maxima or minima, is included in the set of simultaneous solutions (x, y, λ) of the equations*

$$f_1(x, y) + \lambda g_1(x, y) = 0$$
$$f_2(x, y) + \lambda g_2(x, y) = 0$$
$$g(x, y) = 0$$

Proof: Given $f(x, y), g(x, y) \in C^1$.
To find the maxima or minima of the function

$$u = f(x, y) \tag{7.3}$$

where x and y are connected by an equation

$$g(x, y) = 0 \tag{7.4}$$

To find the solution of these we use Lagrange multipliers λ.

Let $V = f(x, y) + \lambda g(x, y)$

$\frac{\partial V}{\partial x} = f_1 + \lambda g_1$, $\frac{\partial V}{\partial y} = f_2 + \lambda g_2$.

For maxima and minima , we have $\frac{\partial V}{\partial x} = 0$, $\frac{\partial V}{\partial y} = 0$. So

$$f_1(x, y) + \lambda g_1(x, y) = 0 \tag{7.5}$$

$$f_2(x, y) + \lambda g_2(x, y) = 0 \tag{7.6}$$

We can solve atleast one of these equations for λ and substitute in the other equation. Combining the result with $g(x, y) = 0$, we arrive at equations $\frac{\partial(f, g)}{\partial(x, y)} = 0$, $g(x, y) = 0$. Also we must solve the three equations (7.4),(7.5) and (7.6) for x, y and λ. Q

Example 7.1.1 *Find the rectangle of perimeter l which has maximum area.*

Solution: Let x and y be the lengths of the sides of the rectangle.

$$\text{Area of the rectangle} = xy = f(x, y)$$

$$\text{Perimeter of the rectangle } l = 2x + 2y = g(x, y)$$

Then if λ is the Lagrange multiplier, we have

$$V = f(x, y) + \lambda g(x, y) = xy + \lambda(2x + 2y - l)$$

$$\frac{\partial V}{\partial x} = y + 2\lambda$$

$$\frac{\partial V}{\partial y} = x + 2\lambda$$

$$\frac{\partial V}{\partial \lambda} = 2x + 2y - l$$

For extremum values $\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial \lambda} = 0,$ we have

$$y + 2\lambda = 0 \tag{7.7}$$

$$x + 2\lambda = 0 \tag{7.8}$$

$$2x + 2y - l = 0 \tag{7.9}$$

Solving (7.7) and (7.8), we have $x = y$.

Using in (7.9) we have

$$x + y = \frac{l}{2},$$

$$2x = \frac{l}{2},$$

$$x = \frac{l}{4}.$$

From (7.8), $\frac{l}{4} + 2\lambda = 0$, so $\lambda = -\frac{l}{8}$.

Hence if $x = y = \frac{l}{4}$ and $\lambda = -\frac{l}{8}$ the rectangle of maximum area is obtained, which is a square.

Example 7.1.2 *Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 255$, ⍰ $= 0$.*

Solution: Distance from (0, 0) to a point (x, y) on the hyperbola $= \sqrt{x^2 + y^2}$.

$f(x, y) = x^2 + y^2, \quad g(x, y) = x^2 + 8xy + 7y^2 - 225.$

We must minimize the function $f(x, y) = x^2 + y^2$ subject to the condition $x^2 + 8xy + 7y^2 = 225$.

Consider the function

$$V = f(x, y) + \lambda g(x, y)$$

$$= x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225),$$ where λ is the Lagrangian multiplier.

$$\frac{\partial V}{\partial x} = 2x + 2\lambda x + 8\lambda y$$

$$\frac{\partial V}{\partial y} = 2y + 8\lambda x + 14\lambda y$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0 \text{ and } g(x, y) = 0$$

$$(1 + \lambda)x + 4\lambda y = 0$$

$$4\lambda x + (1 + 7\lambda)y = 0, \quad x^2 + 8xy + 7y^2 = 225$$

Solving we have $\lambda = 1, -\frac{1}{9}$.

when $\lambda = 1$, $x = -2y$, using in $x^2 + 8xy + 7y^2 = 225$

we have $y^2 = -45$, which is not real solution.

when $\lambda = -\frac{1}{9}$, $y = 2x$, using in $x^2 + 8xy + 7y^2 = 225$

we have $x^2 = 5$, $y^2 = 20$, so $x^2 + y^2 = 25$.

So the shortest distance from the origin to the given hyperbola = 5.

Example 7.1.3 *Find the shortest distance from the point* $(1, 0)$ *to the parabola* $y^2 = 4x$.

Solution: Distance from $(1, 0)$ to a point (x, y) on the parabola $y^2 = 4x$ is $\sqrt{(x - 1)^2 + y^2}$.

$$f(x, y) = (x - 1)^2 + y^2, \quad g(x, y) = y^2 - 4x.$$

We must minimize the function $(x - 1)^2 + y^2$ subject to the condition $y^2 = 4x$.

Form the following function using Lagrange's multiplier λ.

$$V = f(x, y) + \lambda g(x, y)$$

$$= (x - 1)^2 + y^2 + \lambda(y^2 - 4x)$$

$$\frac{\partial V}{\partial x} = 2(x - 1) - 4\lambda$$

$$\frac{\partial V}{\partial y} = 2y + 2y\lambda$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0 \text{ and } y^2 = 4x.$$

$$\Rightarrow 2(x - 1) - 4\lambda = 0$$

$$2y + 2y\lambda = 0$$

$$y^2 - 4x = 0$$

From $2y + 2\lambda y = 0$ we have

$$2y(1 + \lambda) = 0$$

$$\Rightarrow \text{ either } y = 0 \text{ or } \lambda = -1.$$

$$\text{when } \lambda = -1,$$

$$2(x - 1) - 4\lambda = 0$$

$$\Rightarrow x = -1$$

The parabola has no real point with negative abscissa. The valid range is $x \geq 0$.

Therefore $\lambda = -1$ must be rejected.

Hence, the only real solution is $x = 0, y = 0, \lambda = -\frac{1}{2}$ and the required minimum distance is attained at (0, 0) and the minimum distance = 1.

This is shortest because any other point say (1, 2) on $y^2 = 4x$ gives distance $= \sqrt{0 + 4} = 2 > 1$.

7.1.2 One relation between three variables

Now we consider the case

$$u = f(x, y, z)$$

$$g(x, y, z) = 0$$

$$g_1^2 + g_2^2 + g_3^2 > 0$$

It is easily seen by elimination that the desired extrema will lie among the simultaneous solutions of one the three systems:

Suppose

$$g = 0$$

$$\frac{\partial(f, g)}{\partial(x, y)} = 0$$

$$\frac{\partial(f, g)}{\partial(x, z)} = 0,$$

$$g = 0$$

$$\frac{\partial(f, g)}{\partial(y, x)} = 0$$

$$\frac{\partial(f, g)}{\partial(y, z)} = 0,$$

$$g = 0$$

$$\frac{\partial(f, g)}{\partial(z, x)} = 0$$

$$\frac{\partial(f, g)}{\partial(z, y)} = 0$$

according as it is g_1, g_2, or g_3 which is different from zero.

To solve the same problem by the method of Lagrange, we introduce the Lagrange multiplier λ, forming the function

$$V = f(x, y, z) + \lambda g(x, y, z)$$

Treating x, y, z as independent variables and set $\frac{\partial V}{\partial x} = 0$, $\frac{\partial V}{\partial y} = 0$, $\frac{\partial V}{\partial z} = 0$, we are led to the system

$$g = 0$$

$$f_1 + \lambda g_1 = 0$$

$$f_2 + \lambda g_2 = 0$$

$$f_3 + \lambda g_3 = 0$$

We can solve at least one of these for λ and thus arrive at one of the above systems.

Example 7.1.4 *Find the rectangular paralleopiped of surface area* a^2 *and maximum volume.*

Solution:

The volume of the rectangular paralleopiped = xyz.

Surface area of the rectangular paralleopiped = $a^2 = 2xy + 2yz + 2zx$.

We form the following function using Lagrange's multiplier λ

$$V = xyz + \lambda(2xy + 2yz + 2zx - a^2).$$

Now the extremum of xyz is given by $\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0$ and $2xy + 2yz + 2zx = 0$.

$$\frac{\partial V}{\partial x} = yz + \lambda(2y + 2z) = 0$$
$$\frac{\partial V}{\partial y} = xz + \lambda(2x + 2z) = 0$$
$$\frac{\partial V}{\partial z} = xy + \lambda(2x + 2y) = 0$$

Since the variables x, y, z must all be positive, no coefficient of λ is zero, so that

$$\frac{x}{y} = \frac{x + z}{y + z}, \quad \frac{y}{z} = \frac{x + y}{x + z}$$

Consider

$$\frac{x}{y} = \frac{x + z}{y + z}$$

$$x(y + z) - y(x + z) = 0$$

$$x(x - y) = 0$$

$$z = 0, x = y$$

z cannot be zero, so $x = y$. Similarly we have $y = z$. so $x = y = z$.

Now $2xy + 2yz + 2zx = a^2$

gives

$$6x^2 = a^2$$

$$x = \sqrt{\frac{a}{6}}$$

Also

$$y = \frac{a}{\sqrt{6}},\ z = \frac{a}{\sqrt{6}},\ \lambda = -\frac{a}{4\sqrt{6}}.$$

Then the box is a cube.

Example 7.1.5 *Find the miminum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$.*

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2$.

Given $ax + by + cz = p$.

Let $g(x, y, z) = ax + by + cz - p$.

Form the following function using the Lagrange's multiplier λ.

$$V = f(x, y, z) + \lambda g(x, y, z)$$

$$= x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$$

For extremum values,

$$\frac{\partial V}{\partial x} = 0,\ \frac{\partial V}{\partial y} = 0,\ \frac{\partial V}{\partial z} = 0 \text{ and } ax + by + cz = p$$

$$\Rightarrow 2x + \lambda a = 0$$

$$2y + \lambda b = 0$$

$$2z + \lambda c = 0$$

$$ax + by + cz = p$$

Solving the above equations we have

$$x = \frac{-\lambda a}{2},\ y = \frac{-\lambda b}{2},\ z = \frac{-\lambda c}{2}$$

Using the above values of x, y, z in $ax + by + cz = p$, we have

$$a\left(\frac{-\lambda a}{2}\right) + b\left(\frac{-\lambda b}{2}\right) + c\left(\frac{-\lambda c}{2}\right) = p$$

$$-\lambda(a^2 + b^2 + c^2) = 2p$$

$$\lambda = \frac{-2p}{a^2 + b^2 + c^2}$$

Therefore, we have

$$x = \frac{ap}{a^2 + b^2 + c^2},\ y = \frac{bp}{a^2 + b^2 + c^2},\ z = \frac{cp}{a^2 + b^2 + c^2}$$

Using the above values of x, y, z in $f(x, y, z) = x^2 + y^2 + z^2$

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$= \frac{a^2p^2 + b^2p^2 + c^2p^2}{a^2 + b^2 + c^2}$$
$$= \frac{p^2}{a^2 + b^2 + c^2}$$

Hence, the minimum value of $x^2 + y^2 + z^2$ is $\frac{p^2}{a^2 + b^2 + c^2}$.

Example 7.1.6 *Divide* 24 *into three positive numbers* x, y, z *such that* xy^2z^3 *is a maximum.*

Solution: Let $f(x, y, z) = xy^2z^3$.

Given $x + y + z = 24$.

Let $g(x, y, z) = x + y + z - 24$.

Form the following function using the Lagrange's multiplier λ

$$V = f(x, y, z) + \lambda g(x, y, z) = xy^2z^3 + \lambda(x + y + z - 24)$$

For extremum values,

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0 \text{ and } x + y + z = 24$$

$$\Rightarrow y^2z^3 + \lambda = 0$$
$$2xyz^3 + \lambda = 0$$
$$3xy^2z^2 + \lambda = 0$$
$$x + y + z = 24$$

Solving the above equations we have $2x = y$ and $3x = z$. Using the above values of x, y, z in $x + y + z = 24$ we get

$$x + 2x + 3x = 24$$
$$6x = 24$$
$$x = 4$$

Therefore $y = 8, z = 12$.

Hence, the maximum value of $f(x, y, z) = xy^2z^3$ is 442368.

Example 7.1.7 *Find the minimum value of*

(i) $f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$

(ii) $f(x, y, z) = x^2 + y^2 + z^2$ *if* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Solution: (i)

$$f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$$
$$g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

Form the following function using the Lagrange's multiplier λ

$$V = f(x, y, z) + \lambda g(x, y, z)$$
$$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0 \text{ and } g(x, y, z) = 0$$

$$2a^3x - \frac{\lambda}{x^2} = 0$$
$$2b^3y - \frac{\lambda}{y^2} = 0$$
$$2c^3z - \frac{\lambda}{z^2} = 0$$
$$\text{and } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

Solving the above equations we get

$$a^3x^3 = b^3y^3 = c^3z^3$$

$$\text{or } ax = by = cz = k \text{, say}$$
$$\Rightarrow x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c}$$

Substituting these values in $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

we get $a + b + c = k.$

Therefore

$$x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$$

Hence the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ is

$$a^3\left(\frac{a+b+c}{a}\right)^2 + b^3\left(\frac{a+b+c}{b}\right)^2 + c^3\left(\frac{a+b+c}{c}\right)^2$$

$$= a(a + b + c)^2 + b(a + b + c)^2 + c(a + b + c)^2 \text{ or } (a + b + c)^3$$

The extremum is a minimum because, if we increase x, y, z satisfying $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, $\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$ increases indefinitely.

(ii) To prove this, set $a = b = c = 1$.

Then $\min(x^2 + y^2 + z^2) = (1 + 1 + 1)^3 = 27$.

Example 7.1.8 *A rectangular open box, open at the top is to have a volume of* 32 *cubic feet. Find the dimensions of the box requiring least material for its construction.*

Solution: Let the length, breath and height of the box be x, y, z.

Surface area of the box $= f(x, y, z) = xy + 2xz + 2yz$

Volume of the box $= xyz = 32$

Let $g(x, y, z) = xyz - 32$.

Form the following function using the Lagrange multiplier λ

$$V = f(x, y, z) + \lambda g(x, y, z)$$
$$= xy + 2xz + 2yz + \lambda(xyz - 32)$$

Since $xyz = 32$, we can choose $xy = k, z = \frac{32}{k}$, where k is any large number. Thus the surface area $xy + 2xz + 2yz$ which contains the term xy can be increased to any extent. So the surface area has no maximum and it has a minimum. The minimum is given by,

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0 \text{ and } g(x, y, z) = 0$$

$$\Rightarrow \quad y + 2z + \lambda yz = 0$$
$$x + 2z + \lambda zx = 0$$
$$2x + 2y + \lambda xy = 0$$
$$\text{and } xyz - 32 = 0$$

Solving the above equations we have, $x = y, y = 2z$.

Using in $xyz = 32$ we have,

$$4x^3 = 32$$

$$z^3 = 8$$

$$z = 2$$

$$\Rightarrow x = 4, y = 4, z = 2.$$

The minimum surface area $= xy + 2xz + 2yz = 48$.

Example 7.1.9 *Find the positive numbers such that their sum is a constant and their product is maximum.*

Solution: Let x, y, z be the three numbers.

We have to maximize xyz subject to $x + y + z = a$.

Let

$$f(x, y, z) = xyz$$

$$g(x, y, z) = x + y + z - a$$

Form the following function using Lagrange's multiplier λ.

$$V = f(x, y, z) + \lambda g(x, y, z)$$

$$= xyz + \lambda(x + y + z - a)$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0 \text{ and } x + y + z = a.$$

$$\Rightarrow yz + \lambda = 0$$

$$xz + \lambda = 0$$

$$xy + \lambda = 0$$

We have

$$xy = yz = zx = -\lambda$$

Solving for x, y, z,

$$x = y = z$$

Hence

$$x + y + z = a \Longrightarrow x = \frac{a}{3}, y = \frac{a}{3}, z = \frac{a}{3}.$$

The maximum value of $f(x, y, z) = \frac{a^3}{27}$.

Example 7.1.10 *Find the greatest and the least distances of the point* (3, 4, 12) *from the unit sphere whose centre is at the origin.*

Solution: The equation of the unit sphere is $x^2 + y^2 + z^2 = 1$.

Distance of (3, 4, 12) from any point of the sphere is

$$\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}.$$

Let $f = (x-3)^2 + (y-4)^2 + (z-12)^2$.

We have to find the maximum and minimum of f subject to

$$x^2 + y^2 + z^2 - 1 = 0.$$

Form the following function using Lagrange's multiplier λ.

$$\begin{aligned} V &= f(x, y, z) + \lambda g(x, y, z) \\ &= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1). \end{aligned}$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0 \text{ and } x^2 + y^2 + z^2 = 1.$$

$$\begin{aligned} \Rightarrow 2(x-3) + 2x\lambda &= 0 \\ 2(y-4) + 2y\lambda &= 0 \\ 2(z-12) + 2z\lambda &= 0 \end{aligned}$$

Solving for x, y, z we have,

$$x = \frac{3}{\lambda+1}, y = \frac{4}{\lambda+1}, z = \frac{12}{\lambda+1}$$

Substituting in $x^2 + y^2 + z^2 = 1$ we get

$$\frac{9}{(\lambda+1)^2} + \frac{16}{(\lambda+1)^2} + \frac{144}{(\lambda+1)^2} = 1$$

$$(\lambda+1)^2 = 169$$

$$(\lambda+1) = \pm 13$$

$$\Rightarrow \lambda = -14, 12$$

$$\text{At } \lambda = -14, \quad (x, y, z) = \left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$$

$$\text{At } \lambda = 12, \quad (x, y, z) = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$$

$$\text{Minimum distance} = \sqrt{\left(3 - \frac{3}{13}\right)^2 + \left(4 - \frac{4}{13}\right)^2 + \left(12 - \frac{12}{13}\right)^2} = 12.$$

$$\text{Greatest distance} = \sqrt{\left(3 + \frac{3}{13}\right)^2 + \left(4 + \frac{4}{13}\right)^2 + \left(12 + \frac{12}{13}\right)^2} = 14.$$

7.1.3 Two relations between three variables

Now we consider the following case

$$u = f(x, y, z)$$

$$g(x, y, z) = 0$$

$$h(x, y, z) = 0$$

$$\left(\frac{\partial(g,h)}{\partial(x,y)}\right)^2 + \left(\frac{\partial(g,h)}{\partial(y,z)}\right)^2 + \left(\frac{\partial(g,h)}{\partial(z,x)}\right)^2 > 0 \qquad (7.10)$$

There is now a single independent variable which must be chosen in accordance with the Jacobian which is not zero. All three cases lead to the system

$$g = h = \frac{\partial(f,g,h)}{\partial(x,y,z)} = 0. \qquad (7.11)$$

The Lagrange method introduces two parameters λ and μ and leads to the system of five equations in x, y, z, λ, μ,

$$f_1 + \lambda g_1 + \mu h_1 = 0$$

$$f_2 + \lambda g_2 + \mu h_2 = 0$$

$$f_3 + \lambda g_3 + \mu h_3 = 0$$

$$g = 0$$

$$h = 0$$

Under conditions (7.10) this system is easily seen to reduce to the system (7.11) when λ and μ are eliminated.

Example 7.1.11 *Find the maximum and minimum values of* $x^2 + y^2 + z^2$ *subject to the constraint conditions* $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ *and* $z = x + y$.

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$g(x, y, z) = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1$$

$$h(x, y, z) = x + y - z.$$

Form the following function using the Lagrange's multipliers λ and μ.

$$V = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z)$$

$$= x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \mu(x + y - z).$$

For extremum values,

$$\frac{\partial V}{\partial x} = 0, \ \frac{\partial V}{\partial y} = 0, \ \frac{\partial V}{\partial z} = 0, \ g(x, y, z) = 0, \ h(x, y, z) = 0.$$

$$\Rightarrow 2x + \frac{\lambda x}{2} + \mu = 0$$

$$2y + \frac{2\lambda y}{5} + \mu = 0$$

$$2z + \frac{2\lambda z}{25} - \mu = 0$$

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1, \quad x + y - z = 0$$

Solving these equations for x, y, z, we find

$$x = -\frac{2\mu}{\lambda + 4}, \ y = -\frac{5\mu}{2\lambda + 10}, \ z = \frac{25\mu}{2\lambda + 50}$$

Substituting all the above values in $x + y - z = 0$ we get,

$$\frac{2}{\lambda + 4} + \frac{5}{2\lambda + 10} + \frac{25}{2\lambda + 50} = 0$$

$$2(2\lambda + 10)(2\lambda + 50) + 5(\lambda + 4)(2\lambda + 50) + 25(\lambda + 4)(2\lambda + 10) = 0$$

$$(2\lambda + 10)(2\lambda + 50) + 5(\lambda + 4)(\lambda + 25) + 25(\lambda + 4)(\lambda + 5) = 0$$

$$4\lambda^2 + 120\lambda + 500 + 5\lambda^2 + 500 + 145\lambda + 25\lambda^2 + 500 + 225\lambda = 0$$

$$34\lambda^2 + 490\lambda + 1500 = 0$$

$$17\lambda^2 + 245\lambda + 750 = 0$$

$$\Rightarrow (\lambda + 10)(17\lambda + 75) = 0$$

$$\Rightarrow \lambda = -10 \text{ or } \frac{-75}{17}.$$

When $\lambda = -10$,

$$x = \frac{\mu}{3}, y = \frac{\mu}{2}, z = \frac{5\mu}{6}.$$

Substituting in the first constraint condition $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ yields

$$\mu^2 = \frac{180}{19}$$

$$\Rightarrow \mu = \pm 6\sqrt{\frac{5}{19}}.$$

This gives two points $\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right)$ and $\left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right)$.

At these points $f(x, y, z) = x^2 + y^2 + z^2 = 10$.

When $\lambda = -\frac{75}{17}$,

$$x = \frac{34\mu}{7}, y = -\frac{17\mu}{4}, z = \frac{17\mu}{28}.$$

Substituting in the first constraint condition $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ yields $\mu = \pm\frac{140}{17\sqrt{646}}$.

This gives two points $\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right)$ and $\left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$.

At these points $f(x, y, z) = x^2 + y^2 + z^2 = \frac{75}{17}$.

Thus the required maximum value is 10 and the minimum value is $\frac{75}{17}$.

Example 7.1.12 *Show that the extreme values of* $x^2 + y^2 + z^2$ *constrained as* $ax^2 + by^2 + cz^2 = 1$, $px + qy + rz = 0$ *are given by the quadratic equation in* λ

$$\frac{p^2}{1 - a\lambda} + \frac{q^2}{1 - b\lambda} + \frac{r^2}{1 - c\lambda} = 0.$$

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$g(x, y, z) = ax^2 + by^2 + cz^2 - 1,$$

$$h(x, y, z) = px + qy + rz.$$

Form the following function using the Lagrange's Multipliers λ and μ.

$$V = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z)$$

$$= x^2 + y^2 + z^2 + \lambda(ax^2 + by^2 + cz^2 - 1) + \mu(px + qy + rz)$$

For extremum values

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0, g(x, y, z) = 0 \text{ and } h(x, y, z) = 0$$

$$\Rightarrow 2x + 2\lambda ax + \mu p = 0$$

$$2y + 2\lambda by + \mu q = 0$$

$$2z + 2\lambda cz + \mu r = 0$$

$$ax^2 + by^2 + cz^2 = 1$$

$$px + qy + cz = 0$$

From the above equations we have,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 0$$

$$\Rightarrow 2(x^2 + y^2 + z^2) + 2\lambda((ax^2 + by^2 + cz^2)) + \mu(px + qy + rz) = 0$$

$$\Rightarrow 2(x^2 + y^2 + z^2) + 2\lambda = 0$$

$$\Rightarrow \lambda = -(x^2 + y^2 + z^2).$$

Hence, we have

$$x = \frac{\mu p}{2(1 - a\lambda)}, y = \frac{\mu q}{2(1 - b\lambda)}, z = \frac{\mu r}{2(1 - c\lambda)}$$

Using the values of x, y, z in $px + qy + rz = 0$ we have

$$\frac{p^2}{1 - a\lambda} + \frac{q^2}{1 - b\lambda} + \frac{r^2}{1 - c\lambda} = 0.$$

Example 7.1.13 *Maximize or minimize the function* $f(x, y, z) = 3x - y - 3z$ *subject to the constraints* $x + y - z$ *and* $x^2 + 2z^2 - 1$.

Solution: Let $f(x, y, z) = 3x - y - 3z$, $g(x, y, z) = x + y - z$, $h(x, y, z) = x^2 + 2z^2 - 1$.

Form the following function using the Lagrange's Multipliers λ and μ.

$$V = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z)$$
$$= 3x - y - 3z + \lambda(x + y - z) + \mu(x^2 + 2z^2 - 1)$$

For extremum values,

$$\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0, x + y - z = 0 \text{ and } x^2 + 2z^2 - 1 = 0$$

$$\Rightarrow 3 + \lambda + 2\mu x = 0$$
$$-1 + \lambda = 0$$
$$-3 - \lambda + 2z\mu = 0$$
$$x + y - z = 0$$
$$x^2 + 2z^2 - 1 = 0$$

Solving these equations we have,

$$\lambda = 1, x = -\frac{2}{\mu}, y = \frac{3}{\mu}, z = \frac{1}{\mu}.$$

Using the above values in $x^2 + 2z^2 = 1$ we have,

$$\left(-\frac{2}{\mu}\right)^2 + 2\left(\frac{1}{\mu}\right)^2 = 1$$
$$\frac{4}{\mu^2} + \frac{2}{\mu^2} = 1$$
$$\Rightarrow \mu = \pm\sqrt{6}$$

If $\mu = \sqrt{6}$, $x = -\frac{2}{\sqrt{6}}, y = \frac{3}{\sqrt{6}}, z = \frac{1}{\sqrt{6}}$.

If $\mu = -\sqrt{6}$, $x = \frac{2}{\sqrt{6}}$, $y = -\frac{3}{\sqrt{6}}$, $z = -\frac{1}{\sqrt{6}}$.

At $\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

$$
\begin{aligned}
f(x, y, z) &= 3x - y - 3z \\
&= \frac{6}{\sqrt{6}} + \frac{3}{\sqrt{6}} + \frac{3}{\sqrt{6}} \\
&= \frac{12}{\sqrt{6}} \\
&= 2\sqrt{6}
\end{aligned}
$$

At $\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$
\begin{aligned}
f(x, y, z) &= 3x - y - 3z \\
&= -\frac{6}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} \\
&= -\frac{12}{\sqrt{6}} \\
&= -2\sqrt{6}
\end{aligned}
$$

Therefore the maximum value is $2\sqrt{6}$ and minimum value is $-2\sqrt{6}$.

Summary

- If

  1. $f(x, y), g(x, y) \in C^1$ in a domain D
  2. $g_1^2 + g_2^2 > 0$ in D

  then the set of points (x, y) on the curve $g(x, y) = 0$, where $f(x, y)$ has maxima or minima, is included in the set of simultaneous solutions (x, y, λ) of the equations

$$
\begin{aligned}
f_1(x, y) + \lambda g_1(x, y) &= 0 \\
f_2(x, y) + \lambda g_2(x, y) &= 0 \\
g(x, y) &= 0
\end{aligned}
$$

- To solve

$$u = f(x, y, z)$$

$$g(x, y, z) = 0$$

$$g_1^2 + g_2^2 + g_3^2 > 0$$

by the method of Lagrange, we introduce the Lagrange multiplier λ, forming the function

$$V = f(x, y, z) + \lambda g(x, y, z)$$

Treating x, y, z as independent variables and set $\frac{\partial V}{\partial x} = 0$, $\frac{\partial V}{\partial y} = 0$, $\frac{\partial V}{\partial z} = 0$, we are led to the system

$$g = 0$$

$$f_1 + \lambda g_1 = 0$$

$$f_2 + \lambda g_2 = 0$$

$$f_3 + \lambda g_3 = 0$$

We can solve at least one of these for λ and thus arrive at one of the above systems.

- Consider the following case

$$u = f(x, y, z)$$

$$g(x, y, z) = 0$$

$$h(x, y, z) = 0$$

$$\left(\frac{\partial(g, h)}{\partial(x, y)}\right)^2 + \left(\frac{\partial(g, h)}{\partial(y, z)}\right)^2 + \left(\frac{\partial(g, h)}{\partial(z, x)}\right)^2 > 0 \qquad (*)$$

There is now a single independent variable which must be chosen in accordance with the Jacobian which is not zero. All three cases lead to the system

$$g = h = \frac{\partial(f, g, h)}{\partial(x, y, z)} = 0. \qquad (**)$$

The Lagrange method introduces two parameters λ and μ and leads to the system of five equations in $x, y, z, \lambda, \mu,$

$$f_1 + \lambda g_1 + \mu h_1 = 0$$

$$f_2 + \lambda g_2 + \mu h_2 = 0$$

$$f_3 + \lambda g_3 + \mu h_3 = 0$$

$$g = 0$$

$$h = 0$$

Under conditions (*) this system is easily seen to reduce to the system (* *) when λ and μ are eliminated.

Multiple Choice questions

1. In determining the minimum value of $x^2 + y^2 + z^2$ subject to the condition $x+y+z = 5$ by Lagrange's method of undetermined multiplier the value of undetermined multiplier λ is

   (a) $-\frac{10}{3}$
   (b) $\frac{3}{10}$
   (c) $-\frac{3}{10}$

2. Extreme value of xyz subject to $x + y + z = 1$ is

   (a) $\frac{1}{8}$
   (b) 8
   (c) $\frac{1}{27}$

3. In determining the minimum value of $x^2 + y^2 + z^2$ subject to condition $yz + zx + xy = 3a^2$ by Lagrange's method of undetermined multiplier the value of undetermined multiplier λ is

   a) 1 b) -1 c) $\frac{2}{3}$

4. In determining the extreme value of $f(x, y, z)$ subject to the condition $f(x, y, z) = 0$ by Lagrange's method of undetermined multiplier the value of undetermined multiplier λ satisfies the equation

(a) $F_x + \lambda f_x$

(b) $F_y + \lambda f_y$

(c) $F_z + \lambda f_z = 0$

(d) all the above

Ans: 1. a) 2. c) 3. b) 4. d)

Exercises 7

1. If

   (a) $f(x, y), g(x, y) \in C^1$ in a domain D
   (b) $g_1^2 + g_2^2 > 0$ in D

   then prove that the set of points (x, y) on the curve $g(x, y) = 0$, where $f(x, y)$ has maxima or minima, is included in the set of simultaneous solutions (x, y, λ) of the equations

$$f_1(x, y) + \lambda g_1(x, y) = 0$$
$$f_2(x, y) + \lambda g_2(x, y) = 0$$
$$g(x, y) = 0$$

2. The temperature T at any point (x, y, z) is $400xyz$. Find the highest temperature on $x^2 + y^2 + z^2 = 1$.

3. Show that the minimum and maximum distances of the origin from the ellipse $5x^2 + 6xy + 5y^2 = 8$ are 1 and 2 respectively.

4. Find the greatest and the least distances of the point $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

BLOCK IV

Line and Surface Integrals

Unit 8

Line Integral

Learning Outcomes :

After studying this unit, students will be able

- F To define and evaluate line integrals.
- F To know the applications of line integrals.

8.1 Introduction

In block IV we study two new concepts line integrals and surface integrals . These integrals have important applications to geometry and physics. If the function to be integrated is defined along an arc of a curve in two or three dimensions, we can define an integral over that region, the result is called a line integral or curvilinear integral over the arc. In the same manner if the region of integration of a double integral is taken as a region on a curved surface, the result is called a surface integral.

8.1.1 Curves

We shall be dealing with curves of various types. For easy reference let us introduce names for them.

Curve : A curve in the *xy*-plane is a set of points (x, y) for which

$$x = \varphi(t), \quad y = \psi(t), \quad a \le t \le b \tag{8.1}$$

where $\varphi(t) \in C, \psi(t) \in C$ in $a \le t \le b$.

Closed curve : If $\varphi(a) = \varphi(b), \psi(a) = \psi(b)$, the curve is called *closed* curve.

Jordon curve : A closed curve is called a *Jordan curve* if it has no double points. That is, no two distinct values *t* in $a \le t \le b$ yield the same point *(x, y)*.

Jordon curve theorem : Any continuous simple closed curve in the plane separates the plane into two disjoint regions the inside and the outside.

Definition 8.1.1 *The curve (8.1) is regular if it has no double points and if the interval (a, b) can be divided into a finite number of subintervals in each of which* $\varphi(t) \in C^1, \psi(t) \in C^1$ *and* $[\varphi'(t)^2 + \psi'(t)^2] > 0$.

It is clear that a regular curve is sectionally smooth because it consists of a finite number of arcs, each of which has a continuously turning tangent whose direction is determined by the quotient of $\varphi'(t)$ and $\psi(t)$ as they do not vanish simultaneously.

Remark :

1. A regular curve may have corners where the arcs are joined together. Example : The boundary of a rectangle.

2. A Jordan curve can fail to be regular. For example, when it contains a piece of the curve $y = x\sin(\frac{1}{x})$ near the origin.

3. A regular curve has arc length.

Definition 8.1.2 *A region is regular if it is bounded and closed and if its boundary consists of a finite number of regular Jordan curves which have no points in common with each other. We shall denote such a region by the letter* ***S****.*

Example : The set of points (x, y) for which $1 \leq x^2 + y^2 \leq 4$ is a region S. If from this region the points on the x-axis in the interval $1 < x < 2$ were removed, the region is not closed and hence not regular.

8.1.2 Definitions and Theorems

In this section we shall see some definitions and theorems needed for further study.

Definition 8.1.3 *A subdivision* Δ *of an interval* (a, b) *is a set of numbers* $\{x_k\}_{k=0}^{n}$ *or points, such that*

$$a = x_0 < x_1 < \dots < x_n = b.$$

A subdivision involving $n + 1$ *points divides the interval into* n *adjoining subintervals* $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$.

Definition 8.1.4 *The norm* $\|\Delta\|$ *of a subdivision* Δ *is*

$$\|\Delta\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$$

In otherwords, it is also the length of the largest of the subintervals.

Definition 8.1.5 *The Stieltjes integral of* $f(x)$ *with respect to* $\alpha(x)$ *from* a *to* b *is*

$$\int_a^b f(x)d\alpha(x) = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\xi_k)[\alpha(x_k) - \alpha(x_{k-1})]$$

where $x_{k-1} \leq \xi_k \leq x_k, \quad k = 1, 2, \dots, n.$

Definition 8.1.6 *Divide the interval* $a \le x \le b$ *into* n *subintervals by points*

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

and set

$$\delta = \max((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)).$$

Choose point ξ_k, $x_{k-1} \le \xi_k \le x_k$, $k = 1, 2, \ldots, n$. *Then the Riemann integral of* $f(x)$ *with respect to* x *from* a *to* b *is*

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

if this limit exists.

Theorem 8.1.1

1. $f(x) \in C, \quad a \le x \le b$

2. $\alpha(x) \in$ *class of non-decreasing functions,* $a \le x \le b$

$$\Rightarrow \int_a^b f(x)d\alpha(x) \quad \text{exists.}$$

Definition 8.1.7 *Let* $f(x) \in C$ *and* $\alpha(x) \in C^1$ *in* $a \le x \le b$. *Then*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

The integral on the right is an ordinary Riemann integral.

Theorem 8.1.2 (Dhahamel's Theorem)

1. $f(x, y), g(x, y) \in C$

2. *A subdivision* Δ *divides* R *into subregions* R_k, $k = 1, 2, \ldots, n$.

3. (x_k, y_k), (ξ_k, η_k) *are points of* R_k, $k = 1, 2, \ldots, n$.

$$\Rightarrow \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)g(\xi_k, \eta_k)\Delta S_k = \iint_R f(x, y)g(x, y)dS$$

Theorem 8.1.3 (Duhamel's theorem for Riemann integrals)

1. $f(x), g(x) \in C, \quad a \le x \le b$

2. $\alpha(x) \in$ *class of non-decreasing functions*

3. $\{x_k\}_0^n$ *is a subdivision* Δ *of* (a, b)

4. $x_{k-1} \le \xi_k \le x_k, \quad x_{k-1} \le \eta_k \le x_k, \quad k = 1, 2, \ldots, n$

$$\Rightarrow \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\xi_k) g(\eta_k)[\alpha(x_k) - \alpha(x_{k-1})] = \int_a^b f(x) g(x) d\alpha(x).$$

Definition 8.1.8 *Let* $\varphi(x)$ *and* $\psi(x) \in C$ *in* $a \le x \le b$ *and* $\varphi(x) < \psi(x)$ *in* $a < x < b$. *Then the region* R_x *or* $R[a, b, \varphi(x), \psi(x)]$ *is the region bounded by the curves* $x = a, x = b, y = \varphi(x), y = \psi(x)$.

Remark :

1. If (x_1, y_1) is a point of R_x, then $a \le x_1 \le b$ and $\varphi(x_1) \le y_1 \le \psi(x_1)$.

2. A line $x = x_1$, $a < x_1 < b$ cuts the boundary of R_x in just two points. Example: The region $R[-1, 1, -\sqrt{1 - x^2}, \sqrt{1 + x^2}]$ is the circle $x^2 + y^2 \le 1$.

3. The region R_y can be defined in an obvious way.

Theorem 8.1.4

1. $f(x, y) \in C$ *in* R_x

2. $R_x = R[a, b, \varphi(x), \psi(x)]$

$$\Rightarrow \iint_{R_x} f(x, y) dS = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x, y) dy$$

Theorem 8.1.5

1. $f(x, t) \in C^1, \quad a \le t \le b, A \le x \le B$

2. $F(x, y, z) = \int_y^z f(x, t)\, dt, \quad a \le y, z \le b, A \le x \le B$

$$\Rightarrow F_1(x, y, z) = \int_y^z f_1(x, t)dt, \quad F_2(x, y, z) = -f(x, y), \quad F_3(x, y, z) = f(x, z),$$

$a \le y, z \le b, A \le x \le B.$

Example 8.1.1 *Suppose* $G(x) = \int_{g(x)}^{h(x)} f(x, t)\, dt$ *and* $F(x, y, z)$ *is as defined in Theorem 8.1.5 then*

$$G'(x) = \int_{g(x)}^{h(x)} f_1(x, t)dt - f(x, g(x))g'(x) + f(x, h(x))h'(x)$$

8.1.3 Definition of line integrals

Let Γ be the curve given by $x = \varphi(t),\ y = \psi(t),\quad a \le t \le b$ where $\varphi(t),\ \psi(t) \in C$ in $a \le t \le b$. Let $f(x, y)$ be a function defined at every point of the curve Γ and Δ be a subdivision of the interval (a, b) by the points $t_0, t_1, \ldots, t_n$ so that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.

Then we define two types of line integrals

$$\int_\Gamma f(x, y)dx = \int_{x_0, y_0}^{x_1, y_1} f(x, y)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i'), \psi(t_i'))[\varphi(t_i) - \varphi(t_{i-1})] \tag{8.2}$$

$$\int_\Gamma f(x, y)dy = \int_{x_0, y_0}^{x_1, y_1} f(x, y)dy = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i'), \psi(t_i'))[\psi(t_i) - \psi(t_{i-1})] \tag{8.3}$$

Here $x_0 = \varphi(a), y_0 = \psi(a), x_1 = \varphi(b), y_1 = \psi(b), t_{i-1} \le t_i' \le t_i$. For the line integrals to be defined the defining limits must exist. The line integrals (8.2) and (8.3) are, in fact,

$$\int_a^b f(\varphi(t), \psi(t)d\varphi(t), \qquad \int_a^b f(\varphi(t), \psi(t)d\psi(t)$$

respectively.

Theorem 8.1.6

1. Γ *is a regular curve*

2. $f(x, y) \in C$ *on* Γ

$$\Rightarrow \int_\Gamma f(x, y)dx \text{ and } \int_\Gamma f(x, y)dy \text{ exist.}$$

Proof: Consider the curve $\Gamma : x = \varphi(t), y = \psi(t), a \le t \le b.$ where $\varphi(t), \psi(t) \in C$ in $a \le t \le b.$

Given that Γ is a regular curve. So $\varphi(t), \psi(t) \in C^1$ in $a \le t \le b.$

Let $\Delta = \{a = t_0, t_1, \ldots, t_n = b\}$ be any subdivision of $[a, b]$ and $t_i' \in (t_{i-1}, t_i).$

Consider the sum

$$\sum_{i=1}^{n} f(\varphi(t_i), \psi(t_i))[\varphi(t_i) - \varphi(t_{i-1})]$$

since $\varphi(t) \in C^1,$ applying the law of mean we have

$$\varphi(t_i) - \varphi(t_{i-1}) = \varphi'(t'_i)(t_i - t_{i-1})$$

where $t_{i-1} < t_i' < t_i.$ So, by Duhamel's theorem the limit exists and

$$\int_\Gamma f(x, y)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i), \psi(t_i))\varphi'(t_i')(t_i - t_{i-1})$$

Similarly we have for $t_{i-1} < t'_i < t_i$

$$\int_\Gamma f(x, y)dy = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i), \psi(t_i))\psi'(t_i')(t_i - t_{i-1}) \text{ exists.}$$

Note : We have the following remarks if we alter hypothesis 1 a variety of ways.

Remark 1: If the curve Γ is monotonic that is $\varphi(t)$ and $\psi(t)$ are both monotonic in (a, b) then by Theorem 8.1.1 the limits (8.2) and (8.3) both exist as Stieltjes integrals

$$\int_\Gamma f(x, y)dx = \int_a^b f(\varphi(t), \psi(t))d\varphi(t)$$

$$\int_\Gamma f(x, y)dy = \int_a^b f(\varphi(t), \psi(t))d\psi(t)$$

Remark 2: If $\varphi(t) = t$ and $\psi(t) \in C$ instead of to $C^1,$ we see that

$$\int_\Gamma f(x, y)dx = \int_a^b f(x, \psi(x))dx \qquad (8.4)$$

Thus it will be possible to extend the integral (8.2) over the boundary of a region R_x or the integral (8.3) over the boundary of a region R_y if $f(x, y) \in C$ there. Q

Example 8.1.2 *Compute* $\int_\Gamma (x+y)dx$ *if* Γ *is* $x = \cos\vartheta, y = \sin\vartheta,$ $0 \le \vartheta \le \frac{\pi}{2}$.

Solution: Given $x = \cos\vartheta, y = \sin\vartheta, 0 \le \vartheta \le \frac{\pi}{2}$

Here the integration is intended to be from (1, 0) to (0, 1) along an arc of the unit circle.

$dx = \sin\vartheta d\vartheta$

$$\begin{aligned}
\int_\Gamma (x+y)dx &= -\int_0^{\frac{\pi}{2}} (\cos\vartheta + \sin\vartheta)\sin\vartheta d\vartheta \\
&= -\int_0^{\frac{\pi}{2}} \cos\vartheta \sin\vartheta d\vartheta - \int_0^{\frac{\pi}{2}} \sin^2\vartheta \\
&= -\int_0^{\frac{\pi}{2}} \frac{\sin 2\vartheta}{2} d\vartheta - \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\vartheta}{2} d\vartheta \\
&= \left[\frac{\cos 2\vartheta}{4}\right]_0^{\frac{\pi}{2}} - \left[\frac{\vartheta}{2}\right]_0^{\frac{\pi}{2}} + \left[\frac{\sin 2\vartheta}{2}\right]_0^{\frac{\pi}{2}} \\
&= -\frac{1}{4} - \frac{1}{4} - \frac{\pi}{4} = -\frac{1}{2} - \frac{\pi}{4}.
\end{aligned}$$

Note : The above example can also be solved by using equation (8.4), $x = \cos\vartheta, y = \sin\vartheta = \sqrt{1-\cos^2\vartheta} = \sqrt{1-x^2}$ and x varies from 1 to 0. So

$$\begin{aligned}
\int_\Gamma (x+y)dx &= \int_1^0 \left(x + \sqrt{1-x^2}\right) dx \\
&= \left(\frac{x^2}{2} + \frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2}\right)_1^0 \\
&= -\frac{1}{2} - \frac{1}{2}\,\frac{\pi}{2} = -\frac{1}{2} - \frac{\pi}{4}.
\end{aligned}$$

Example 8.1.3 *Compute* $\int_\Gamma (x+y)dx$ *if* Γ *is the two line segments* $y = 0, 0 \le x \le 1; x = 0, 0 \le y \le 1.$

Solution: The integration is intended to be from $(1, 0)$ to $(0, 1)$ over the broken line.

$$\int_\Gamma (x+y)dx = \int_1^0 x dx = \left(\frac{x^2}{2}\right)_1^0 = -\frac{1}{2}.$$

Remark : From the above two examples we see that the values of the integral may depend upon the path and not merely on the end points of the path.

Example 8.1.4 *Extend the integral* $\int_\Gamma (x+y)dx + (x-y)dy$ *over the two paths* Γ *if (i)* Γ *is* $x = \cos\vartheta, y = \sin\vartheta, 0 \le \vartheta \le \frac{\pi}{2}$, *(ii)* Γ *is the two line segments* $y = 0, 0 \le x \le 1; x = 0, 0 \le y \le 1.$

Solution: (i) If Γ is $x = \cos\vartheta, y = \sin\vartheta, \quad 0 \le \vartheta \le \frac{\pi}{2}$, then

$$\begin{aligned}
&\int_\Gamma (x+y)dx + (x-y)dy \\
&= \int_0^{\frac{\pi}{2}} (\cos\vartheta + \sin\vartheta)(-\sin\vartheta) + (\cos\vartheta - \sin\vartheta)\cos\vartheta d\vartheta \\
&= \int_0^{\frac{\pi}{2}} (-\cos\vartheta\sin\vartheta - \sin^2\vartheta + \cos^2\vartheta - \sin\vartheta\cos\vartheta)d\vartheta \\
&= \int_0^{\frac{\pi}{2}} (\cos^2\vartheta - \sin^2\vartheta - 2\sin\vartheta\cos\vartheta)d\vartheta \\
&= \int_0^{\frac{\pi}{2}} (\cos 2\vartheta - \sin 2\vartheta)d\vartheta \\
&= \left[\frac{\sin 2\vartheta}{2} + \frac{\cos 2\vartheta}{2}\right]_0^{\frac{\pi}{2}} \\
&= -1
\end{aligned}$$

(ii) If Γ is the broken line

$$y = 0,\ 0 \le x \le 1, \quad x = 0,\ 0 \le y \le 1.$$

$$\begin{aligned}
\int_\Gamma (x+y)dx + (x-y)dy &= \int_1^0 x dx - \int_0^1 y dy \\
&= \left(\frac{x^2}{2}\right)_1^0 - \left(\frac{y^2}{2}\right)_0^1 = -1.
\end{aligned}$$

We will prove later that in this case the value of the integral is independent of the path.

Example 8.1.5 *Compute the following integral over the curve* Γ *if* Γ *is* $x = \cos\vartheta, y = \sin\vartheta, 0 \le \vartheta \le \frac{\pi}{2}$,

$$\int_\Gamma xydx + (x+y)dy.$$

Solution: Suppose Γ is $x = \cos\vartheta, \quad y = \sin\vartheta, \quad 0 \le \vartheta \le \frac{\pi}{2}$.
Here the integration is intended to be form $(1, 0)$ to $(0, 1)$ along an arc of the unit circle. $x = \cos\vartheta \Rightarrow dx = -\sin\vartheta d\vartheta, \quad y = \sin\vartheta \Rightarrow dy = \cos\vartheta d\vartheta$

$$\begin{aligned}
\int_\Gamma xydx + (x+y)dy &= \int_0^{\frac{\pi}{2}} (\cos\vartheta\sin\vartheta)(-\sin\vartheta)d\vartheta + (\cos\vartheta + \sin\vartheta)\cos\vartheta d\vartheta \\
&= \int_0^{\frac{\pi}{2}} (-\sin^2\vartheta\cos\vartheta + \cos^2\vartheta + \sin\vartheta\cos\vartheta)d\vartheta \\
&= -\int_0^{\frac{\pi}{2}} \sin^2\vartheta\cos\vartheta + \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\vartheta}{2} d\vartheta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\vartheta d\vartheta \\
&= -\frac{1}{3} + \frac{\pi}{4} + \frac{1}{2} \\
&= \frac{\pi}{4} + \frac{1}{6}.
\end{aligned}$$

Example 8.1.6 *Compute* $\int_\Gamma ydx + xdy$ *over the curve* Γ, *if* Γ *is* $x = \cos\vartheta$, $y = \sin\vartheta, 0 \le \vartheta \le \frac{\pi}{2}$.

Solution:

$$\begin{aligned}
\int_\Gamma ydx + xdy &= \int_0^{\frac{\pi}{2}} (-\sin^2\vartheta + \cos^2\vartheta)d\vartheta \\
&= \int_0^{\frac{\pi}{2}} \cos 2\vartheta d\vartheta \\
&= \left.\frac{\sin 2\vartheta}{2}\right|_0^{\frac{\pi}{2}} = 0.
\end{aligned}$$

Example 8.1.7 *Compute* $\int_\Gamma x^2ydx + x^3dy$, *if* Γ *is the path given by* $x = \cos t, y = \sin t, 0 \le t \le 2\pi$.

Solution:

$$\begin{aligned}
\int_\Gamma x^2ydx + x^3dy &= \int_0^{2\pi} (-\cos^2 t\sin^2 t - \cos^4 t)dt \\
&= -\int_0^{\pi} \cos^2 tdt \\
&= -\pi.
\end{aligned}$$

Example 8.1.8 *Evaluate the integral* $I = \oint_\Gamma x\,dx + y\,dy + z\,dz$ *where* Γ *is the circle* $x^2 + y^2 + z^2 = a^2,\ z = 0.$

Solution: The parametric equations of the circle are $x = a\cos t,\ y = a\sin t,\ z = 0,$ where $0 \le t \le 2\pi.$

Also $dx = -a\sin t\,dt,\quad dy = a\cos t\,dt,\quad dz = 0.$

$$\begin{aligned}
I &= \oint_\Gamma x\,dx + y\,dy + z\,dz\\
&= \int_0^{2\pi} (a\cos t)(-a\sin t\,dt) + (a\sin t)(a\cos t\,dt)\\
&= a^2 \int_0^{2\pi} (-\sin t\cos t + \sin t\cos t)\,dt\\
&= a^2 \int_0^{2\pi} 0\ dt\\
&= 0.
\end{aligned}$$

Example 8.1.9 *Evaluate* $I = \oint_\Gamma x\,dx + y\,dy$ *where* Γ *is the ellipse* $x^2 + 4y^2 = 4.$

Solution: The equation of the ellipse is $x^2 + 4y^2 = 4.$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{1^2}.$$

The parametric equations are $x = 2\cos t,\ y = \sin t$ where $0 \le t \le 2\pi.$ Therefore,

$$\begin{aligned}
I &= \oint_\Gamma x\,dx + y\,dy\\
&= \int_0^{2\pi} (2\cos t)(-2\sin t\,dt) + (\sin t)(\cos t)\,dt\\
&= -\int_0^{2\pi} 3\sin t\cos t\,dt\\
&= -\frac{3}{2}\int_0^{2\pi} \sin 2t\,dt\\
&= \frac{3}{4}(\cos 2t)_0^{2\pi}\\
&= 0.
\end{aligned}$$

Example 8.1.10 *Find the value of* $I = \oint_\Gamma (x + y^2)dx + (x^2 - y)dy$ *taken in the clockwise sense along the closed curve* Γ *formed by* $y^3 = x^2$ *and the chord joining* $(0, 0)$ *and* $(1, 1)$.

Solution: Equation of the chord joining $(0, 0)$ and $(1, 1)$ is $y = x$.

The curve Γ consist of the arc OA, $y^3 = x^2$ and the line AO, $y = x$.

Along OA, $y^3 = x^2$, $y = x^{\frac{2}{3}}$,

Along AO, $y = x$.

Therefore

$$I = \oint_\Gamma (x + y^2)dx + (x^2 - y)dy$$
$$= \int_0^1 (x + x^{\frac{4}{3}})dx + (y^3 - y)dy + \int_1^0 (x + x^2)dx + (x^2 - x)dx$$
$$= \int_0^1 x^{\frac{4}{3}}dx + y^3 dy + \int_1^0 2x^2 dx$$
$$= \frac{1}{84}.$$

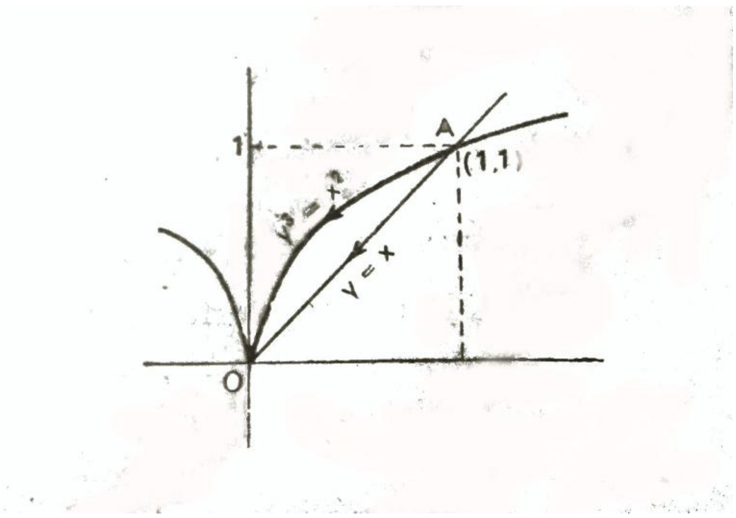

Figure 8.1

Example 8.1.11 *Find the value of* $I = \oint_C x^2ydx + xy^2dy$ *taken in the clockwise sense along the hexagon whose vertices are* $(\pm 3a, 0)$, $(\pm 2a, \pm\sqrt{3}a)$.

Solution: Equations of the line forming the curve Γ are

$$AB : y = -\sqrt{3}a$$
$$BC : y - \sqrt{3}x + 3\sqrt{3}a = 0$$
$$CD : y + \sqrt{3}x - 3\sqrt{3}a = 0$$
$$DE : y = \sqrt{3}a$$
$$EF : y - \sqrt{3}x - 3\sqrt{3}a = 0$$
$$FA : y + \sqrt{3}x + 3\sqrt{3}a = 0$$

Therefore

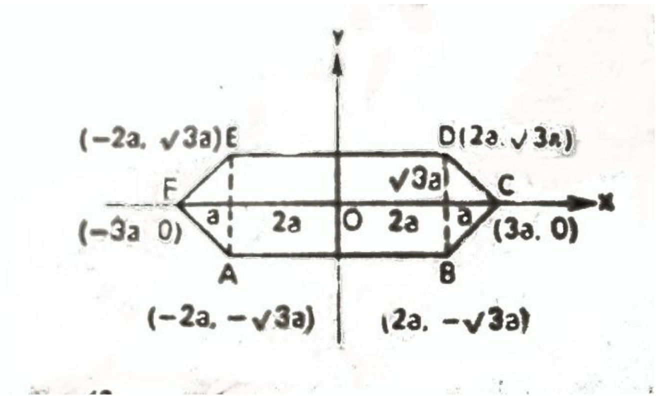

Figure 8.2

$$I_\Gamma = I_{AF} + I_{FE} + I_{ED} + I_{DC} + I_{CB} + I_{BA}$$
$$= -\int_{-2a}^{-3a} x^2(\sqrt{3}x + 3\sqrt{3}a)dx - \int_{-\sqrt{3}a}^{0} y^2\left(\frac{y}{\sqrt{3}} + 3a\right)dy$$
$$+\left(\int_{-3a}^{-2a} x^2(\sqrt{3}x + 3\sqrt{3}a)dx - \int_{0}^{\sqrt{3}a} y^2\left(\frac{y}{\sqrt{3}} - 3a\right)dy\right)$$
$$+\int_{-2a}^{2a} \sqrt{3}ax^2dx$$
$$+\int_{2a}^{3a} x^2(-\sqrt{3}x + 3\sqrt{3}a)dx + \int_{\sqrt{3}a}^{0} y^2\left(\frac{-y}{\sqrt{3}} + 3a\right)dy$$
$$+\left(\int_{3a}^{2a} x^2(\sqrt{3}x - 3\sqrt{3}a)dx + \int_{0}^{-\sqrt{3}a} y^2\left(\frac{y}{\sqrt{3}} + 3a\right)dy\right)$$
$$-\int_{2a}^{-2a} \sqrt{3}ax^2dx$$

$$= 4\int_{2a}^{3a}(-x)^2(\sqrt{3}x - 3\sqrt{3}a)dx + 4\int_{0}^{\sqrt{3}a} y^2\left(-\frac{y}{\sqrt{3}} - 3a\right)dy + 2\sqrt{3}a\int_{-2a}^{2a} x^2 dx$$

$$= 4a^4\left(\frac{65\sqrt{3}}{4} - 19\sqrt{3}\right) + 4a^4\left(\frac{9}{4\sqrt{3}} - 3\sqrt{3}\right) + \frac{32}{\sqrt{3}}a^4$$

$$= \frac{38}{\sqrt{3}}a^4.$$

Example 8.1.12 *Evaluate* $\int_\Gamma (x^2 + xy)dx + (x^2 + y^2)dy$ *where* Γ *the square formed by the lines* $x = \pm 1, y = \pm 1.$

Solution: Equations of the line following the curve Γ are $AB : y = 1$, $BC : x = -1$, $CB : y = -1$, $DA : x = 1$. Therefore

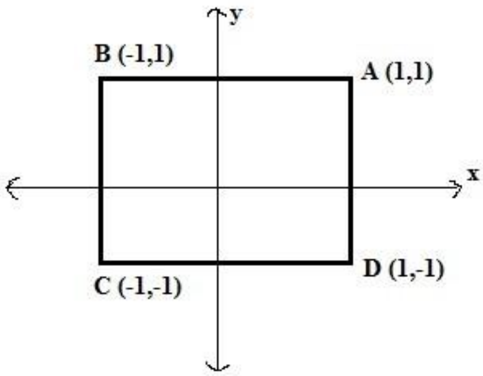

Figure 8.3

$$I_\Gamma = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

$$= \int_1^{-1}(x^2 + x)dx + \int_1^{-1}(1 + y^2)dy + \int_{-1}^{1}(x^2 - x)dx + \int_{-1}^{1}(1 + y^2)dy$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^{-1} + \left[y + \frac{y^3}{3}\right]_1^{-1} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} + \left[y + \frac{y^3}{3}\right]_{-1}^{1}$$

$$= -\frac{2}{3} - \frac{8}{3} + \frac{2}{3} + \frac{8}{3}$$

$$= 0.$$

8.1.4 Work

One of the natural application of a line integral is to the problem of defining the work done by a field of force on a particle moving along a curve through the field.

Let the field be given by two functions $X(x, y)$ and $Y(x, y)$ which are to be the x- and y-components, respectively, of a force at the point (x, y).

The magnitude of the force at the point $= \sqrt{X^2 + Y^2}$,

Direction of the force $= \tan^{-1}\dfrac{Y}{X}$.

Let us have the following assumptions:

(i) The particle describe the regular curve $x = \varphi(t)$, $y = \psi(t)$, $a \le t \le b$ where $\varphi(t), \psi(t) \in C$.

(ii) Δ be the subdivision of the interval (a, b) by the points $t_0, t_1, \ldots, t_n$ such that $a = t_0 < t_1 < \ldots < t_n = b$.

(iii) Δs_i = Arc length of the curve between the points $t = t_{i-1}$ and $t = t_i$.

(iv) ϑ_i = Angle between the direction of the force of the field at the point t_i and the direction of the tangent to the curve at t_i directed in the line of motion.

Now, work done on the particle as it traverses the whole path is

$$= \lim_{\|\Delta\|\to 0} \sum_{i=0}^{n} \sqrt{X_i^2 + Y_i^2}\cos\vartheta_i \Delta s_i \tag{8.5}$$

$$X_i = X(\varphi(t_i), \psi(t_i)), \quad Y_i = Y(\varphi(t_i), \psi(t_i))$$

The direction components of the tangent are $\varphi'(t_i), \psi'(t_i)$ and of the direc-tion of the force, X_i, Y_i, so that

$$\cos\vartheta_i = \frac{X_i\varphi'(t_i) + Y_i\psi'(t_i)}{\sqrt{X_i^2 + Y_i^2}\sqrt{[\varphi'(t_i)]^2 + [\psi'(t_i)]^2}}$$

$$\Delta s_i = \int_{t_{i-1}}^{t_i} \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}\,dt = \Delta t_i\sqrt{[\varphi'(\xi_i)]^2 + [\psi'(\xi_i)]^2}$$

where $t_{i-1} < \xi_i < t_i$. Then equation (8.5) becomes

$$\text{Workdone} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (X_i\varphi'(t_i) + Y_i\psi'(t_i))\Delta t_i$$

By Duhamel's theorem we see that the limit (8.5) is the line integral

$$\int_\Gamma X(x, y)dx + Y(x, y)dy$$

that is

$$\text{Work done on the particle} = \int_\Gamma X(x, y)dx + Y(x, y)dy.$$

Example 8.1.13 *If* $X(x, y) = x + y$, $Y(x, y) = x - y$, *find the work done by the field on a particle moving from* $(1, 0)$ *to* $(0, 1)$ *along the straight line* $x + y = 1$.

Solution: $x + y = 1 \Rightarrow y = 1 - x \Rightarrow dy = -dx$. Now,

$$\begin{aligned}
\text{Work done} &= \int_{\Gamma} X(x, y)dx + Y(x, y)dy \\
&= \int_{1}^{0} dx - (2x - 1)dx \\
&= \int_{0}^{1} (2x + 2)dx \\
&= \left(2\frac{x^2}{2} + 2x \right)_{1}^{0} = -1.
\end{aligned}$$

Example 8.1.14 *If* $X(x, y) = 2x^2y$, $Y(x, y) = 3xy$, *find the work done by the field on a particle moving from* $(0, 0)$ *to* $(1, 4)$ *along the curve* $y = 4x^2$.

Solution: Given $X(x, y) = 2x^2y$, $Y(x, y) = 3xy$

$$\begin{aligned}
\text{Work done} &= \int_{\Gamma} X(x, y)dx + Y(x, y)dy \\
&= \int_{\Gamma} 2x^2ydx + 3xydy \\
&= \int_{0}^{1} 2x^2(4x^2)dx + 3x(4x^2)8xdx \\
&= \int_{0}^{1} 8x^4dx + 96x^4dx \\
&= \int_{0}^{1} 104x^4dx \\
&= 104 \left(\frac{x^5}{5} \right)_{0}^{1} \\
&= \frac{104}{5}.
\end{aligned}$$

Example 8.1.15 *If* $X(x, y, z) = 3xy$, $Y(x, y, z) = x + y$, $Z(x, y, z) = -z$, *Find the work done by the field on a particle moving from* $(2, 0, 1)$ *to* $(4, 2, 9)$ *along the curve* $x = t + 1, y = t - 1, z = t^2$.

Solution: The parametric equation of Γ are $x = t + 1, y = t - 1, z = t^2$. Evidently $t = 1$ gives $(2, 0, 1)$ and $t = 3$ gives $(4, 2, 9)$.

Also $dx = dt, dy = dt, dz = 2t\,dt.$

$$\begin{aligned}
\text{Workdone} &= \int_\Gamma X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz \\
&= \int_1^3 3(t^2 - 1)\,dt + 2t\,dt - t^2(2t)dt \\
&= \int_1^3 (-2t^3 + 3t^2 + 2t - 3)dt \\
&= \left[\frac{-t^4}{2} + t^3 + t^2 - 3t\right]_1^3 \\
&= -12.
\end{aligned}$$

Remark 1 : When the work is independent of the path the field is called conservative.

Remark 2 : The negative sign indicates the particle has done work on the field, that is if the particle moved as a result of the forces of the field only, it would move in the opposite direction over most of the path.

Summary

- **Line integral :** Let Γ be the curve given by $x = \varphi(t), y = \psi(t), \quad a \le t \le b$ where $\varphi(t), \psi(t) \in C$ in $a \le t \le b$. Let $f(x, y)$ be a function defined at every point of the curve Γ and Δ be a subdivision of the interval (a, b) by the points $t_0, t_1, ..., t_n$ so that $a = t_0 < t_1 < t_2 < ... < t_n = b$.

  Then we define two types of line integrals

$$\int_\Gamma f(x, y)dx = \int_{x_0, y_0}^{x_1, y_1} f(x, y)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i^*), \psi(t_i^*))[\varphi(t_i) - \varphi(t_{i-1})]$$

$$\int_\Gamma f(x, y)dy = \int_{x_0, y_0}^{x_1, y_1} f(x, y)dy = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i^*), \psi(t_i^*))[\psi(t_i) - \psi(t_{i-1})]$$

  Here $x_0 = \varphi(a), y_0 = \psi(a), x_1 = \varphi(b), y_1 = \psi(b), t_{i-1} \le t_i^* \le t_i$. For the line integrals to be defined the defining limits must exist.

- If

  1. Γ is a regular curve

  2. $f(x, y) \in C$ on Γ

  $\Rightarrow \int_\Gamma f(x, y)dx$ and $\int_\Gamma f(x, y)dy$ exist.

- Work done by a field given by two functions $X(x, y)$ and $Y(x, y)$ of a force at the point (x, y) on a particle moving along a curve through the field $= \int_\Gamma X(x, y)dx + Y(x, y)dy$

- If the work is independent of the path then the field is called conservative

Multiple Choice questions:

1. When the work is independent of the path then the field is called

   a) solenoidal b) irrotational field c) conservative field

2. The value of $\int_\Gamma ydx + xdy$ where Γ is $x = \cos\vartheta, y = \sin\vartheta, 0 \le \vartheta \le \frac{\pi}{2}$ is

   a) 0 b) $\frac{\pi}{2}$ c) $\frac{\pi}{4}$.

3. The line integral is denoted by

   (a) Triple integral

   (b) Double integral

   (c) Integral along a curve

Ans: 1. c) 2. a) 3. c)

Exercises 8

1. Define line integral

2. If Γ is a regular curve and $f(x, y) \in C$ on Γ, then prove $\int_\Gamma f(x, y)dx$ and $\int_\Gamma f(x, y)dy$ exist.

3. Compute the integral $\int_{\Gamma} xydx + (x + y)dy$ over the curve Γ if Γ is the two line segments $y = 0, 0 \leq x \leq 1, x = 0, 0 \leq y \leq 1$.

Ans : $\frac{1}{2}$

4. Compute $\int_{\Gamma} xydx + (x + y)dy$ where Γ is the boundary of the triangle with vertices (0, 0), (0, 2), (1, 0) integration in the clockwise direction.

Ans: $-\frac{2}{3}$.

5. Compute $\int_{\Gamma} (x^2 + y)dx + (2x + y^2)dy$ over the boundary of the square with vertices (1, 1), (1, 2), (2, 2), (2, 1) in the clockwise sense.

Ans : -1.

6. If $X(x, y) = 3xy$, $Y(x, y) = -y^3$, find the workdone by the field on the particle moving from (0, 0) to (1, 2) along the curve $y = 2x^2$ in the $xy-$plane.

Ans : $-\frac{5}{2}$.

7. Prove that the workdone by the field given by the two function $X(x, y)$ and $Y(x, y)$ in moving a particle along a regular curve Γ is $\int_{\Gamma} X(x, y)dx + Y(x, y)dy$.

Unit 9

Green's Theorem

Learning Outcomes :

After studying this unit, students will be able

- F To acquire knowledge about Green's theorem which provides a formula connecting a line integral over its boundary with a double integral over a region.
- F To know the applications of Green's theorem.

9.1 Green's Theorem

We shall now discuss Green's theorem which provides a formula connecting a line integral over its boundary with a double integral over a region. It is sometimes refered to as Gauss's theorem.

If a region is bounded by one or more curves the positive direction over the boundary is the one that leaves the region to the left. Thus for the region between two concentric circles the positive direction is counterclockwise for the outer boundary, clockwise for the inner one.

Definition 9.1.1 *An iterated integral is an integral of the form*

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

where φ_1 or φ_2 or both are functions of x or constants.

This means that for each fixed x between a and b, the integral

$$F(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

is evaluated and then the integral $\int_a^b F(x)\,dx.$

So

$$\int_a^b F(x)dx = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

and the other repeated integral

$$\int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x, y)dx \text{ or } \int_c^d \left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x, y)dx \right) dy$$

is defined in the same way.

9.1.1 First form of Green's theorem

Theorem 9.1.1

1. *R is a region R_x and also R_y*
2. *Γ is the boundary of R*
3. *$P(x, y), Q(x, y) \in C^1$ in R*

$$\Rightarrow \int_\Gamma Pdx + Qdy = \iint_R [Q_1(x, y) - P_2(x, y)]dS \qquad (9.1)$$

the line integral being taken in the positive sense.

Proof: $R_x = R[a, b, \varphi(x), \psi(x)]$ is a region bounded by the curves $x = a, y = b, y = \varphi(x), y = \psi(x)$, where $\varphi(x), \psi(x) \in C$ in $a \le x \le b$ and $\varphi(x) < \psi(x)$ in $a < x < b$.

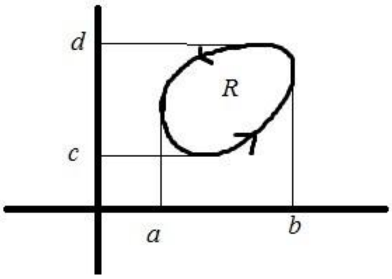

Figure 9.1

Given that the function $P(x, y)$ along with its partial derivative P_2 is continuous in the region R, and Γ is the boundary of R then by Theorem 8.1.4 we have,

$$\begin{aligned}
\iint_R P_2(x, y)dS &= \int_a^b dx \int_{\varphi(x)}^{\psi(x)} P_2(x, y)dy \\
&= \int_a^b \left(P(x, y)\right)_{\varphi(x)}^{\psi(x)} dx \\
&= \int_a^b P\left(x, \psi(x)\right) dx - \int_a^b P\left(x, \varphi(x)\right) dx \\
&= -\int_b^a P\left(x, \psi(x)\right) dx - \int_a^b P\left(x, \varphi(x)\right) dx
\end{aligned}$$

Using equation (8.4),

$$\Rightarrow \iint_R P_2(x, y)dS = -\int_\Gamma P(x, y)dx \tag{9.2}$$

Similarly considering $R_y = R[c, d, \varphi(y), \psi(y)]$, a region bounded by $y = c, y = d, x = \varphi(y), x = \psi(y)$, where $\varphi(y), \psi(y) \in C$ in $c \leq y \leq d$ and $\varphi(y) < \psi(y)$ in $c < y < d$.

$$\begin{aligned}
\iint_R Q_1(x, y)dS &= \int_c^d dy \int_{\varphi(y)}^{\psi(y)} Q_1(x, y)dx \\
&= \int_c^d \left(Q(x, y)\right)_{\varphi(y)}^{\psi(y)} dy \\
&= \int_c^d Q\left(\psi(y), y\right) dy - \int_c^d Q\left(\varphi(y), y\right) dy \\
&= \int_c^d P(\psi(y), y)dy + \int_c^d Q(\varphi(y), y)dy
\end{aligned}$$

$$\Rightarrow \iint_R Q_1(x, y)dS = \int_\Gamma Q(x, y)dy \tag{9.3}$$

Adding (9.2) and (9.3) we have

$$\int_{\Gamma} Pdx + Qdy = \iint_{R} (Q_1(x, y) - P_2(x, y))\, dS.$$

Q

9.1.2 Second form of Green's theorem

Theorem 9.1.2

1. *R is a region R_x and a regular region S*
2. *Γ is the boundary of R*
3. *$P(x, y), Q(x, y) \in C^1$ in R*

$$\Rightarrow \int_{\Gamma} Pdx + Qdy = \iint_{R} (Q_1(x, y) - P_2(x, y))\, dS$$

the line integral being taken in the positive sense.

Proof: Given that R is a region R_x and a regular region S and Γ is boundary of R. So Γ is a regular curve.

Hence we can apply previous proof as for as it concerns $P(x, y)$,

$$\Rightarrow \iint_{R} P_2(x, y)dS = -\int_{\Gamma} P(x, y)dx \qquad (9.4)$$

To find $\int_{\Gamma} Q(x, y)dy.$

The boundary Γ consists of four regular arcs. Hence

$$\int_{\Gamma} Q(x, y)dy = \int_a^b Q(x, \varphi(x))\varphi'(x)dx - \int_a^b Q(x, \psi(x))\psi'(x)dx + \int_{\varphi(b)}^{\psi(b)} Q(b, y)dy - \int_{\varphi(a)}^{\psi(a)} Q(a, y)dy \qquad (9.5)$$

and

$$\iint_{R} Q_1(x, y)dS = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} Q_1(x, y)dy \qquad (9.6)$$

Let

$$F(x) = \int_{\varphi(x)}^{\psi(x)} Q(x, y)dy$$

Then by Example 8.1.1 we have,

$$F'(x) = \int_{\varphi(x)}^{\psi(x)} Q_1(x, y)dy + Q(x, \psi(x))\psi'(x) - Q(x, \varphi(x))\varphi'(x)$$

$$\int_{\varphi(x)}^{\psi(x)} Q_1(x, y)dy = F'(x) - Q(x, \psi(x))\psi'(x) + Q(x, \varphi(x))\varphi'(x)$$

From (9.5) and (9.6), we have

$$\iint_R Q_1(x, y)dS = \int_a^b \left[F'(x) - Q(x, \psi(x))\psi'(x) + Q(x, \varphi(x))\varphi'(x)\right] dx$$

$$= \int_a^b F'(x)\,dx - \int_a^b Q(x, \psi(x))\psi'(x)\,dx + \int_a^b Q(x, \varphi(x))\varphi'(x)\,dx$$

$$= [F(x)]_a^b - \int_a^b Q(x, \psi(x))\psi'(x)\,dx + \int_a^b Q(x, \varphi(x))\varphi'(x)\,dx$$

$$= F(b) - F(a) - \int_a^b Q(x, \psi(x))\psi'(x)\,dx + \int_a^b Q(x, \varphi(x))\varphi'(x)\,dx$$

$$= \int_{\varphi(b)}^{\psi(b)} Q(b, y)dy - \int_{\varphi(a)}^{\psi(a)} Q(a, y)dy - \int_a^b Q(x, \psi(x))\psi'(x)dx + \int_a^b Q(x, \varphi(x))\varphi'(x)dx$$

$$\Rightarrow \iint_R Q_1(x, y)dS = \oint_\Gamma Q(x, y)dy \qquad (9.7)$$

From (9.4) and (9.7) we have

$$\oint_\Gamma Pdx + Qdy = \iint_R (Q_1(x, y) - P_2(x, y))dS.$$

Q

Remark 1: If a regular region S is such that it can be divided into a finite number of regions R_x (or R_y) by cross cuts, equation (9.1) still holds where Γ is the total boundary, consisting of one or more regular closed curves.

Remark 2: We can apply Green's theorem to find the area of a region defined by the equations of its boundary curves. Suppose R is a region to which Green's theorem applies and which is bounded by Γ, then the area of R is given by any of the three formulas

$$A = -\oint \qquad \oint_\Gamma xdy, \quad A = \frac{1}{2}\oint_\Gamma (-y)dx + xdy$$

ydx, *A* = $\frac{1}{}$

Γ Γ

the integration being in the positive sense.

For, In Green's theorem from equation (9.1) we have,if

(i) $P = y, \quad Q = 0$

$$-\oint_\Gamma y\,dx = \iint_R dS = A.$$

(ii) $Q = x, \quad P = 0$

$$\oint_\Gamma x\,dy = \iint_R dS = A$$

(iii) $P = -y, \quad Q = x$

$$\oint_\Gamma x\,dy - y\,dx = 2\iint_R dS = 2A \Longrightarrow A = \frac{1}{2}\oint x\,dy - y\,dx.$$

Example 9.1.1 *Find the area of the ellipse* $x = a\cos\vartheta, y = b\sin\vartheta.$

Solution: Given $x = a\cos\vartheta \Longrightarrow dx = -a\sin\vartheta d\vartheta$

$$Area\ A = -\oint_\Gamma y\,dx$$

$$= -\int_0^{2\pi} b\sin\vartheta(-a\sin\vartheta)d\vartheta$$

$$= ab\int_0^{2\pi}\sin^2\vartheta d\vartheta$$

$$= ab\int_0^{2\pi}\frac{1-\cos 2\vartheta}{2}d\vartheta$$

$$= ab\left[\frac{\vartheta}{2} - \frac{\sin\vartheta}{4}\right]_0^{2\pi}$$

$$= ab(\pi - 0)$$

$$= \pi ab.$$

Example 9.1.2 *Find the area enclosed by the folium* $x^3 + y^3 = 3axy.$

Solution: The parametric equation are got by putting $y = tx.$

We have $x = \dfrac{3at}{1+t^3}, \quad y = \dfrac{3at^2}{1+t^3}$

Since $t = \dfrac{y}{x} = \tan\vartheta,\ \vartheta$ varies from 0 to $\dfrac{\pi}{2}$, t varies from 0 to ∞.

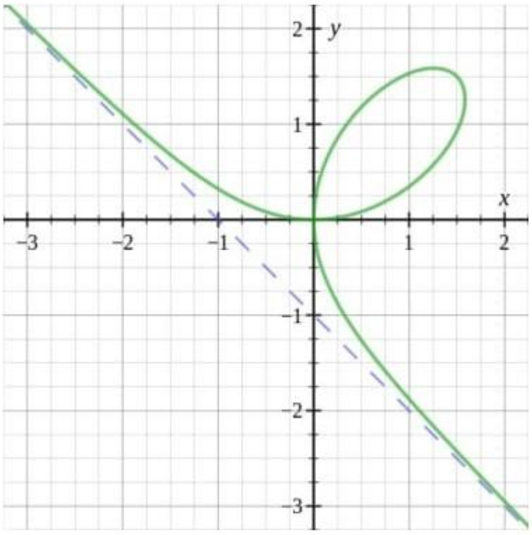

Figure 9.2

$$dx = \frac{3a(1-2t^3)}{(1+t^3)^2}dt, \quad dy = \frac{3a(2t-t^4)}{(1+t^3)^2}dt$$

Hence Area of the loop is

$$\begin{aligned}
A &= \frac{1}{2}\oint_\Gamma xdy - ydx \\
&= \frac{1}{2}\int_0^\infty \frac{3at}{1+t^3}\left(\frac{2t-t^4}{(1+t^3)^2}\right)dt - \frac{3at^2}{1+t^3}\left(\frac{3a(1-2t^3)}{(1+t^3)^2}\right)dt \\
&= \frac{9a^2}{2}\int_0^\infty \frac{t^2}{(1+t^3)^2}dt \\
&= \frac{3}{2}a^2.
\end{aligned}$$

9.2 Verification of Green's Theorem

Example 9.2.1 *Verify Green's theorem for*

$$\oint_\Gamma (2xy - x^2)dx + (x + y^2)dy$$

where Γ *is the closed curve of the region bounded by* $y = x^2$ *and* $y^2 = x$.

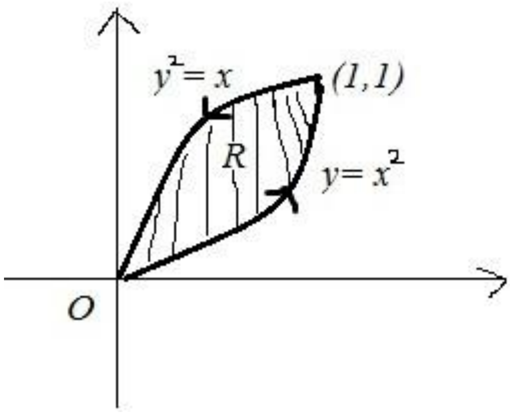

Figure 9.3

Solution: By Green's Theorem we have,

$$\oint_{\Gamma} Pdx + Qdy = \iint_{R} (Q_1(x,y) - P_2(x,y))dxdy.$$

The plane curve $y = x^2$ and $y^2 = x$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing Γ is an shown in the above figure.

Evaluation of $\oint_{\Gamma} Pdx + Qdy$

We shall take Γ in two different paths namely, (i). along $y = x^2$, (ii). along $y^2 = x$.

(i).Along $y = x^2$.

$y = x^2, \Longrightarrow dy = 2xdx.$

$$\int_0^1 (2x)(x^2) - x^2 \, dx + (x + x^4)2xdx = \int_0^1 (2x^3 + x^2 + 2x^5)dx$$
$$= \frac{7}{6}.$$

(ii).Along $y^2 = x$.

$y^2 = x, \Longrightarrow 2ydy = dx.$

The line integral

$$\int_1^0 (2y^2)y - y^4 \; 2ydy + (y^2 + y^2)dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)dy$$
$$= -\frac{17}{15}.$$

$$\oint_{\Gamma} (2xy - x^2)dx + (x + y^2)dy = \frac{7}{6} - \frac{17}{15} = \frac{1}{30}.$$

Evaluation of $\iint_R (Q_1(x,y) - P_2(x,y))\,dxdy$

$$\iint_R \big(Q_1(x,y) - P_2(x,y)\big)\,dxdy = \iint_R \left(\frac{\partial(x+y^2)}{\partial x} - \frac{\partial(2xy - x^2)}{\partial y}\right) dxdy$$

$$= \iint_R (1-2x)dxdy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (1-2x)dydx$$

$$= \int_{x=0}^{1} \big(y - 2xy\big)_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_{x=0}^{1} (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3)dx$$

$$= \frac{1}{30}.$$

Hence Green's Theorem is verified.

Example 9.2.2 *Verify Green's Theorem for* $\int_\Gamma x^2dx + xydy$ *where* Γ *is the curve given by* $x = 0, y = 0, x = a, y = a, a > 0.$

Solution: By Green's theorem we have,

$$\int_\Gamma Pdx + Qdy = \iint_R (Q_1(x,y) - P_2(x,y))dxdy.$$

Here

$$P = x^2, \quad Q = xy.$$

$$P_2 = 0, \quad Q_1 = y.$$

Evaluation of $\int_\Gamma Pdx + Qdy.$

We shall take Γ in four different segments namely, (i). along OA ($y = 0$) *(ii)*. along AB ($x = a$) (iii). along BC ($y = a$) (iv). along CO ($x = 0$).

(i). Along OA ($y = 0$), x varies from 0 to a.

$$\int_\Gamma x^2dx + xydy = \int_{OA} x^2dx + xydy$$

$$= \int_0^a x^2dx$$

$$= \frac{a^3}{3}.$$

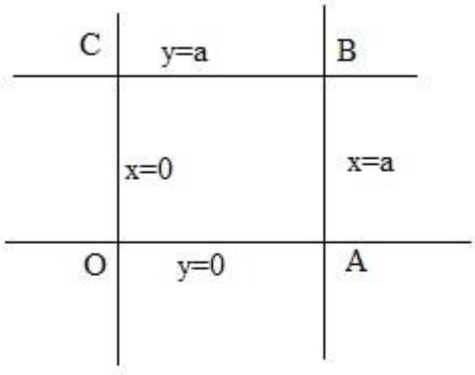

Figure 9.4

(ii). Along AB ($x = a$), y varies from 0 to a.

$$\int_{\Gamma} x^2dx + xydy = \int_{AB} x^2dx + xydy$$
$$= a\int_0^a ydy$$
$$= \frac{a^3}{2}.$$

(iii). Along BC ($y = a$), x varies from a to 0.

$$\int_{\Gamma} x^2dx + xydy = \int_{BC} x^2dx + xydy$$
$$= \int_a^0 x^2dx$$
$$= -\frac{a^3}{3}.$$

(iv). Along CO, $x = 0$

$$\int_{\Gamma} x^2dx + xydy = \int_{CO} x^2dx + xydy$$
$$= 0.$$

So

$$\int_{\Gamma} x^2dx + xydy = \int_{OA} x^2dx + xydy + \int_{AB} x^2dx + xydy + \int_{BC} x^2dx + xydy$$
$$+ \int_{CO} x^2dx + xydy$$
$$= \frac{a^3}{2}.$$

Evaluation of $\iint_{\Gamma} (Q_2(x, y) - P_1(x, y))dxdy.$

$$\iint_{\Gamma} (Q_2(x, y) - P_1(x, y))dxdy = \iint_{R} ydxdy$$
$$= \int_0^a \int_0^a ydxdy$$
$$= \int_0^a y\,(x)_0^a\,dy$$
$$= a\int_0^a ydy$$
$$= \frac{a^3}{2}.$$

Hence Green's theorem is verified.

Example 9.2.3 *Evaluate by Green's theorem* $\int_{\Gamma} e^{-x}(\sin y dx+\cos y dy)$, *where* Γ *is the rectangle with vertices* $(0, 0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$.

Solution: By Green's theorem, we have

$$\int_{\Gamma} Pdx + Qdy = \iint_{R} (Q_1(x, y) - P_2(x, y))dxdy.$$

Here

$$P = e^{-x}\sin y, \quad Q = e^{-x}\cos y.$$

$$P_2 = e^{-x}\cos y, \quad Q_1 = -e^{-x}\cos y.$$

From the figure x various from $(0, \pi)$ and y various from $(0, \frac{\pi}{2})$.

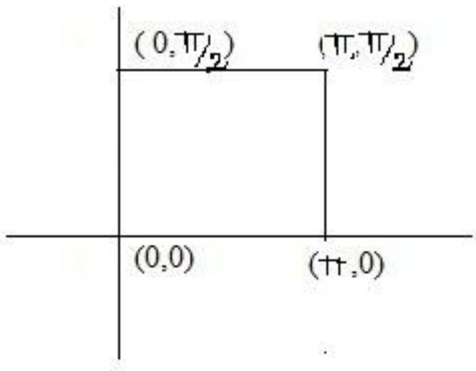

Figure 9.5

So

$$\oint_\Gamma e^{-x}\sin y\,dx + e^{-y}\cos y\,dy = \int_0^{\frac{\pi}{2}}\int_0^{\pi} (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\frac{\pi}{2}} \left[-e^{-x}\cos y\right]_0^{\pi} dy$$

$$= -2\int_0^{\frac{\pi}{2}} (e^{-\pi}\cos y + \cos y)\,dy$$

$$= -2\left[-e^{-\pi}\sin y + \sin y\right]_0^{\frac{\pi}{2}}$$

$$= 2(e^{-\pi} - 1).$$

9.3 Applications

In this section, we will study

(i) When $P(x,y)dx + Q(x,y)dy$ is the differential of a function $F(x,y)$?

(ii) Under what conditions will F exist such that $F_1 = P, F_2 = Q$? and

(iii) How can one find F if it exists ?

9.3.1 Existence of Exact Differentials

Simply connected domain: A domain D is *simply connected* if no Jordan curve in D contains in its interior a boundary point of D.We shall denote a simply connected domain by using the sign $*$ as a superscript.

Theorem 9.3.1

1. $P(x,y), Q(x,y) \in C^1$ *in* D^*

2. $Q_1(x,y) = P_2(x,y)$ *in* D^*

$\Leftrightarrow$ *there exists* $F(x,y) \in C^2$ *in* D^* *such that* $F_1 = P, \quad F_2 = Q.$

Proof: Necessary condition:

Suppose there exists $F(x, y) \in C^2$ in D^* such that $F_1 = P, F_2 = Q$.
Consider

$$F_1 = P \Longrightarrow F_{12} = P_2$$

and

$$F_2 = Q \Longrightarrow F_{21} = Q_1.$$

Since $F(x, y) \in C^2$ in D^*, $F_{12} = F_{21}$ in D^*

$$\text{So } \; P_2(x, y) = Q_1(x, y) \; in \; D^*$$

Since $F \in C^2$ and $F_1 = P$, $F_2 = Q$,

$$P(x, y), Q(x, y) \in C^1.$$

Sufficient condition :

Conversely, suppose that $P(x, y), Q(x, y) \in C^1$ in D^* and $Q_1(x, y) = P_2(x, y)$ in D^*. We define $F(x, y)$ explicitly.

Let (a, b) and (x_0, y_0) be points of D^*. Then

$$F(x_0, y_0) = \int_{a,b}^{x_0,y_0} P(x, y)dx + Q(x, y)dy \tag{9.8}$$

where the path of integration is a broken line.

Such a line exists by the definition of a domain.

In fact it is easy to see that the segments of the broken line may be taken parallel to the axes.

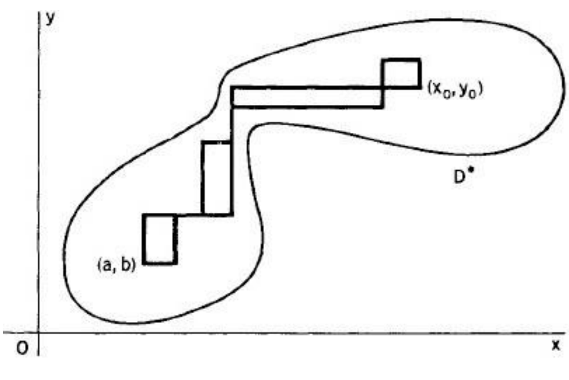

Figure 9.5

We choose the path in this way in order to simplify the type of region that can be enclosed between two such broken lines .

Claim : We first prove that F is a single-valued function and the value of the integral does not depend upon the path.

Consider two broken lines in D^* with segments parallel to the axes and joining (a, b) with (x_o, y_o).

They will form the boundaries of a finite number of rectangles.

We have assumed that $Q_1(x, y) = P_2(x, y)$ in D^*. So $Q_1(x, y) = P_2(x, y)$ in these finite number of rectangles also. By second form of Green's theorem we have the line integral (9.8) extended around the boundary of each rectangle will be zero.

Therefore,the line integral (9.8) is independent of the path.

Hence the claim.

We now prove $F_1 = P$ and $F_2 = Q$.

We shall compute F_1 and F_2 at the point (x_o, y_o) of D^*.

Since D^* is simply connected, this point is the center of a circle K which lies entirely in D^*. Choose a point $(x_o + \Delta x, y_o)$ inside K. Then

$$\begin{aligned}
\frac{\Delta F}{\Delta x} &= \frac{F(x_o + \Delta x, y_o) - F(x_o, y_o)}{\Delta x} \\
&= \frac{1}{\Delta x}\left(\int_{a,b}^{x_o+\Delta x, y_o} P(x, y)dx + Q(x, y)dy - \int_{a,b}^{x_o, y_o} P(x, y)dx + Q(x, y)dy\right) \\
&= \frac{1}{\Delta x}\left(\int_{a,b}^{x_o+\Delta x, y_o} P(x, y)dx + Q(x, y)dy + \int_{x_o, y_o}^{a,b} P(x, y)dx + Q(x, y)dy\right) \\
&= \frac{1}{\Delta x}\int_{x_o, y_o}^{x_o+\Delta x, y_o} Pdx + Qdy
\end{aligned}$$

If the path of integration is taken to be a straight line, it is evident that the integral of Q is zero. So

$$\frac{\Delta F}{\Delta x} = \frac{1}{\Delta x}\int_{x_o, y_o}^{x_o+\Delta x, y_o} P(x, y)dx \tag{9.9}$$

But by the law of the mean we have

$$\frac{F(x_o + \Delta x, y_o) - F(x_o, y_o)}{\Delta x} = F_1(x_o + \vartheta\Delta x, y_o) \tag{9.10}$$

where $0 < \vartheta < 1$. From (9.9) and (9.10)

$$\frac{\Delta F}{\Delta x} = P(x_o + \vartheta\Delta x, y_o)$$

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} P(x_o + \vartheta\Delta x, y_o)$$

$$F_1(x_o, y_o) = P(x_o, y_o).$$

As the point (x_o, y_o) is an arbitrary point of D^*, we have $F_1 = P$ in D^* Similarly, we can prove $F_2 = Q$. Q

9.3.2 Exact Differential Equations

we now integrate the exact differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

where $Q_1 = P_2$ in D^*. The primitive is $F(x, y) = c$, where c is an arbitrary constant.

Theorem 9.3.2

1. *$P(x, y), Q(x, y) \in C^1$ in D^**
2. *$Q_1(x, y) = P_2(x, y)$ in D^**
3. *Γ is a regular curve in D^* joining (a, b) with (x_o, y_o)*

$\Rightarrow$ the integral

$$F(x_o, y_o) = \int_{a,b}^{x_o,y_o} P(x, y)dx + Q(x, y)dy$$

extended over Γ is independent of Γ.

Proof: Let Γ be a regular curve in D^* joining (a, b) with (x_o, y_o). Let Γ have equations

$$x = \varphi(t), y = \psi(t) \quad 0 \le t \le 1$$

Suppose that $P(x,y), Q(x,y) \in C^1$ in D^* and $Q_1(x,y) = P_2(x,y)$ in D^*. Then by Theorem 9.3.1 , $F_1 = P$ and $F_2 = Q$ in D^*.

Also we have , $x = \varphi(t) \Rightarrow dx = \varphi'(t)dt$ and $y = \psi(t) \Rightarrow dy = \psi'(t)dt$

$$\int_{a,b}^{x_0,y_0} Pdx + Qdy = \int_0^1 [F_1(\varphi(t),\psi(t))\varphi'(t) + F_2(\varphi(t),\psi(t))\psi'(t)]dt$$
$$= \int_0^1 \frac{d}{dt} F(\varphi(t),\psi(t))$$
$$= (F(\varphi(t),\psi(t)))_0^1$$
$$= F(x_0,y_0) - F(a,b).$$

Since the final result does not depend on $\varphi(t)$ or $\psi(t)$, the integral given in the hypothesis extended over Γ is independent of Γ. Q

Example 9.3.1 *Consider the example 8.1.4. Regrouping terms we have*

$$(x+y)dx + (x-y)dy = xdx - ydy + ydx + xdy$$
$$= d(x^2/2) - d(y^2/2) + d(xy)$$

Hence

$$\int_{1,0}^{0,1} (x+y)dx + (x-y)dy = \left[\frac{x^2}{2} - \frac{y^2}{2} + xy\right]_{(1,0)}^{(0,1)} = -1.$$

Theorem 9.3.3 *If $P(x,y), Q(x,y) \in C^1$ in a domain D^*, then $Q_1 = P_2$ in $D^* \iff \int_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D^*.*

Proof: Suppose that $P(x,y), Q(x,y) \in C^1$ in a domain D^* then by theorem 9.3.2 we have, $Q_1 = P_2$ in D^*.

Let (a,b) be any point of the curve Γ. then

$$\int_\Gamma Pdx + Qdy = F(a,b) - F(a,b) = 0.$$

Conversely, suppose that $\int_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D^*

To prove $Q_1 = P_2$ in D^* $(i.e)$. $Q_1 - P_2 = 0$

Suppose that $Q_1 - P_2 > 0$ at a point (x_0, y_0) of D^*.

By continuity (x_0, y_0) is the center of a circle K of D^* with circumference C, throughout which $Q_1 - P_2 > 0$.
By Green's theorem

$$\iint_K (Q_1 - P_2)dS = \int_C Pdx + Qdy > 0$$

This contradicts the hypothesis. Similarly, if $Q_1 - P_2 < 0$ at (x_0, y_0), we obtain a contradiction. Hence, $Q_1(x_0, y_0) = P_2(x_0, y_0)$.

Since (x_0, y_0) is an arbitrary point in D^*, $Q_1 = P_2$ in D^*. Q

Remark 1: Theorem 9.3.3 remains true if the curve Γ is allowed to cut itself.

Remark 2: The results of the present section are applied to multiply connected regions by the introduction of cross cuts.

Summary

- Green's theorem provides a formula connecting a line integral over its boundary with a double integral over a region.

- **First form of Green's theorem:** If

  1. R is a region R_x and also R_y
  2. Γ is the boundary of R
  3. $P(x, y), Q(x, y) \in C^1$ in R

  $$\Rightarrow \int_\Gamma Pdx + Qdy = \iint_R [Q_1(x, y) - P_2(x, y)]dS \qquad (9.11)$$

  the line integral being taken in the positive sense.

- **Second form of Green's theorem :** If

  1. R is a region R_x and a regular region S
  2. Γ is the boundary of R
  3. $P(x, y), Q(x, y) \in C^1$ in R

$$\Rightarrow \oint_\Gamma Pdx + Qdy = \iint_R (Q_1(x, y) - P_2(x, y))\, dS$$

the line integral being taken in the positive sense.

- We can apply Green's theorem to find the area of a region defined by the equations of its boundary curves. Suppose R is a region bounded by Γ then area of R is given by any one of the three formulas

$$A = -\oint_\Gamma ydx, A = \oint_\Gamma xdy, A = \frac{1}{2}\oint_\Gamma (-y)dx + xdy$$

- Line integral is a useful tool in the investigation of exact differentials

- If

  1. $P(x, y), Q(x, y) \in C^1$ in D^*
  2. $Q_1(x, y) = P_2(x, y)$ in D^*

  $\Longleftrightarrow$ there exists $F(x, y) \in C^2$ in D^* such that $F_1 = P, \quad F_2 = Q.$

- If

  1. $P(x, y), Q(x, y) \in C^1$ in D^*
  2. $Q_1(x, y) = P_2(x, y)$ in D^*
  3. Γ is a regular curve in D^* joining (a, b) with (x_o, y_o)

  $\Rightarrow$ the integral

$$F(x_o, y_o) = \int_{a,b}^{x_o,y_o} P(x, y)dx + Q(x, y)dy$$

  extended over Γ is independent of Γ.

- If $P(x, y), Q(x, y) \in C^1$ in a domain D^*, then $Q_1 = P_2$ in $D^* \Longleftrightarrow \oint_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D^*.

Multiple Choice questions:

1. The value of the integral $\int_{(1,0)}^{(0,1)} (x+y)\,dx + (x-y)\,dy$ is

   a) 1 b) 0 c) -1

2. Suppose R is a region bounded by Γ then the area of R is given by

   a) $A = \int_\Gamma xdx$ b) $A = \frac{1}{2}\int_\Gamma xdy - ydx$ c) $A = \int_\Gamma ydy$

3. The statement of Green's theorem is

   (a) $\int_\Gamma Pdx + Qdy = \iint_R (Q_1(x,y) - P_2(x,y))dS$

   (b) $\int_\Gamma Pdx + Qdy = \iint_R Q_1(x,y)dS$

   (c) $\int_\Gamma Pdx + Qdy = \iint_R P_2(x,y)dS$

4. Green's theorem connects

   (a) line integral to surface integral

   (b) surface integral to volume integral

   (c) line integral to volume integral

Ans: 1. c) 2. b) 3. a) 4. a)

Exercises 9

1. State and prove first form of Green's Theorem .

2. Define iterated integral.

3. State and prove second form of Green's theorem.

4. Find the area of the circle $r = a\cos\vartheta$.

5. Verify Green's theorem for $\int_\Gamma (x^2 - y^2)dx + 2xydy$ where C is the boundary of the region bounded by the lines $x = 0, x = a, y = 0, y = b$.

   Ans: $2ab^2$

6. Verify Green's theorem for $\int_\Gamma (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where Γ is the region bounded by $x = 0, y = 0$ and $x + y = 1$.

7. If $P(x, y), Q(x, y) \in C^1$ in D^* and $Q_1(x, y) = P_2(x, y)$ in D^*, then prove that there exists $F(x, y) \in C^2$ in D^* such that $F_1 = P, F_2 = Q$.

8. If $P(x, y), Q(x, y) \in C^1$ in D^*, $Q_1(x, y) = P_2(x, y)$ in D^* and Γ is a regular curve in D^* joining (a, b) with (x_0, y_0), then prove that the integral

$$F(x_0, y_0) = \int_{a,b}^{x_0,y_0} P(x, y)dx + Q(x, y)dy$$

extended over Γ is independent of Γ.

9. If $P(x, y), Q(x, y) \in C^1$ in a domain D^*, then prove that $Q_1 = P_2$ in $D^* \iff \int_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D^*.

10. Evaluate $\int_{0,0}^{1,\pi} e^x \cos y dx - e^x \sin y dy$.

Ans: $\frac{1}{2}$

11. Evaluate $\int_{0,0}^{1,\pi} 2y \cos x dy - y^2 \sin x dx$.

Ans: $\frac{1}{2}$

Unit 10

Surface Integral and Gauss's Theorem

Learning Outcomes :

After studying this unit, students will be able

- F To learn about surface integrals.
- F To state and prove Gauss's theorem which relates the triple integral of a function on a three dimensional region of space to its double integral on the bounding surface.

10.1 Surface Integrals

In many physical problems we encounter functions defined on various surfaces. For example, density of a charge distribution over the surface of a conductor, intensity of illumination of a surface, velocity of the particles of a fluid passing through a surface etc., This section is devoted to studying integral of functions defined on surfaces, the so called surface integral.

The double integral over a plane area generalizes to a surface integral over an area of an arbitrary curved surface. We define the surface integral and we prove generalization of Green's theorem, which enable us to express a triple integral over a solid in terms of a surface integral over the surface bounding the solid.

10.1.1 Definition of Surface Integrals

Let R be a region on an arbitrary surface having area. Let Σ be a finite piece of such a surface.

Definition 10.1.1 (Subdivision) *A subdivision Δ of Σ is a set of closed curves $\{C_k\}_1^n$ lying on Σ and dividing it into a set of n subregions of areas $\Delta\Sigma_k$, $k = 1, 2, \ldots, n$.*

Definition 10.1.2 (Diameter) *The diameter of a region on Σ is the length of the largest straight line segment whose ends lie in the region.*

Remark : Since Σ may be curved, the intermediate points of the segment need not lie on Σ.

Definition 10.1.3 (Norm) *The norm of Δ, denoted by $\|\Delta\|$, is the largest of the n diameters of the subregions produced by the subdivision.*

Definition 10.1.4 (Surface Integral) *Let $P(x, y, z)$ be a function defined at every point of Σ and let (ξ_k, η_k, ζ_k) be a point on Σ inside or on the boundary of the subregion bounded by C_k. Then the surface integral of $P(x, y, z)$ over Σ is*

$$\iint_\Sigma P(x, y, z)\, d\Sigma = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} P(\xi_k, \eta_k, \zeta_k)\Delta\Sigma_k \qquad (10.1)$$

when this limit exists.

Remark: If Σ is a region R of the xy-plane, the above limit reduces to the double integral of $P(x, y, 0)$ over R. A double integral is a special case of a surface integral.

Note : Let V denote a three dimensional region.

Theorem 10.1.1

1. $P(x, y, z) \in C$ *in V*

2. Σ *is the surface* $z = f(x, y)$ *over the region R*

3. $f(x, y) \in C^1$ *in R*

4. Σ *lies in V*

$$\Rightarrow \quad A. \iint_\Sigma P(x, y, z)\,d\Sigma \quad exists$$

$$B. \iint_\Sigma P(x, y, z)\,d\Sigma = \iint_R P(x, y, f(x, y))\sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\,dS$$

Proof: Given Σ is defined by the equation $z = f(x, y)$ when (x, y) lies in a region R of the xy-plane.

That is, Σ is cut in a single point by a line parallel to the z-axis and has the projection R in the xy-plane.

Divide Σ into a set of n subregions of areas $\Delta\Sigma_k$, $k = 1, 2, ..., n$.

It induces a corresponding subdivision (which we still call Δ) of R into subregions R_k of areas ΔS_k.

The point (ξ_k, η_k, ζ_k) of Σ becomes $(\xi_k, \eta_k, f(\xi_k, \eta_k))$, where (ξ_k, η_k) is a point of R_k. Then

$$\Delta\Sigma_k = \iint_{R_k} \sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\,dS$$

where $\sec\gamma = \sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}$, γ is the acute angle between the normal to Σ and $z-$ axis.

From the law of the mean for double integrals we have a continuous function has a maximum and a minimum in a closed region R, and that it takes on each value between the two at same point of R and we have

$$\iint_R f(x,y)\,dS = f(\xi,\eta)A, \qquad (\xi,\eta)\in R$$ and A is the area of R.

So we have,

$$\Delta\Sigma_k = \sqrt{1+f_1^2(a_k,b_k)+f_2^2(a_k,b_k)}\,\Delta S_k,$$

where (a_k, b_k) is a point of R_k and ΔS_k is the area of that subregion. Substituting this value of $\Delta\Sigma_k$ in equation (4.16) we have,

$$\iint_\Sigma P(x,y,z)\,d\Sigma = \lim_{\|\Delta\|\to 0}\sum_{k=1}^{n} P(\xi_k,\eta_k,f(\xi_k,\eta_k))\Delta\Sigma_k$$

$$= \lim_{\|\Delta\|\to 0}\sum_{k=1}^{n} P(\xi_k,\eta_k,f(\xi_k,\eta_k))\sqrt{1+f_1^2(a_k,b_k)+f_2^2(a_k,b_k)}\,\Delta S_k$$

By using Duhamel's theorem, we have,

$$\iint_\Sigma P(x,y,z)\,d\Sigma = \iint_R P(x,y,f(x,y))\sqrt{1+f_1^2(x,y)+f_2^2(x,y)}\,dS$$

Q

Remark 1: In the preceding we have assumed that Σ is decomposed into a finite number of parts each of which is such that any line parallel to the $z-$ axis intersect it in only one point.

Remark 2: Suppose the surface Σ have the equation $x = f(y,z)$ or $y = f(x,z)$. If Σ can be decomposed into a finite number of parts, each of which is cut only once by a line parallel to some axis and has a continuously turning tangent plane. Also the surface integral over Σ is the sum of all the surface integral over these parts.

Then the existence of the following surface integrals are are assured.

$$\iint_\Sigma P(x,y,z)\,d\Sigma = \iint_R P(f(y,z),y,z)\sqrt{1+f_1^2(y,z)+f_2^2(y,z)}\,dS$$

and $$\iint_\Sigma P(x,y,z)\,d\Sigma = \iint_R P(x,f(x,z),z)\sqrt{1+f_1^2(x,z)+f_2^2(x,z)}\,dS$$

Remark 3: If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal, then we have $\cos\alpha dS = dydz, \quad \cos\beta dS = dzdx, \quad \cos\gamma dS = dxdy.$

Example 10.1.1 *Compute* $\iint_{\Sigma} xyz\,d\Sigma$, *where* Σ *is the surface of the terahedron bounded by the planes*

$$x = 0,\ z = 0,\ x + y = 2,\ 2z = y.$$

Solution: Consider $x + y = 2, \quad 2z = y$

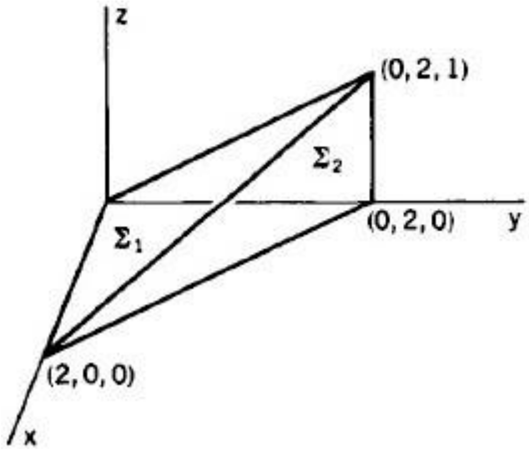

Figure 10.1

when $x = 0, \quad y = 2$ and $z = \frac{y}{2} = \frac{2}{2} = 1$. So the point is $(0, 2, 1)$.

when $z = 0, \quad x = 2$ and $y = 0$. So the point is $(2, 0, 0)$.

$$\iint_{\Sigma} xyz\,d\Sigma = \iint_{\Sigma_1} xyz\,d\Sigma + \iint_{\Sigma_2} xyz\,d\Sigma + \iint_{\Sigma_3} xyz\,d\Sigma + \iint_{\Sigma_4} xyz\,d\Sigma$$

On the coordinate faces the integrand is zero. So two of the integrals are zero.

We evaluate the other two, by projection onto the region $R_x = R[0, 2, 0, (2-x)/2]$ of the $xz-$ plane.

We have $f(x, z) = 2xz^2, f_1(x, z) = 2z^2, f_2(x, z) = 4xz$.

So $\sec\theta = \sqrt{1 + f_1^2(x, z) + f_2^2(x, z)}dS = \sqrt{5}$ on Σ_1.

$$\iint_{\Sigma_1} xyz\,d\Sigma = 2\sqrt{5}\iint_{R_x} xz^2\,dS = \frac{2\sqrt{5}}{15}.$$

Also $f(x, z) = z(2x - x^2), f_1(x, z) = z(2 - 2x), f_2(x, z) = 2x - x^2$.

So $\sec\theta = \sqrt{1 + f_1^2(x, z) + f_2^2(x, z)}dS = \sqrt{2}$ on Σ_2.

$$\iint_{\Sigma_2} xyz\,d\Sigma = \sqrt{2}\iint_{R_x} z(2x - x^2)dS = \frac{\sqrt{2}}{5}.$$

Example 10.1.2 *Evaluate* $\iint_\Sigma P(x,y,z)dS$, *where S is the surface of the paraboloid* $z = 2 - (x^2 + y^2)$ *above the xy-plane and* $P(x, y, z) = 1$.

Solution: We have

$$\iint_\Sigma P(x,y,z)d\Sigma = \iint_R P(x,y,z)\sqrt{1 + f_1^2(x,y) + f_2^2(x,y)}dS,$$

where R is the projection of S on the xy-plane given by $x^2 + y^2 = 2, z = 0$.
Here $f_1(x,y) = 2x, \quad f_2(x,y) = 2y$.

$$\iint_\Sigma P(x,y,z)d\Sigma = \iint_R \sqrt{1 + 4x^2 + 4y^2}dxdy$$

Using polar coordinates (r, ϑ), we have

$$\begin{aligned}\iint_\Sigma P(x,y,z)d\Sigma &= \int_{\vartheta=0}^{2\pi}\int_0^{\sqrt{2}}\sqrt{1 + 4r^2}\,rdrd\vartheta \\ &= \int_0^{2\pi}\frac{1}{12}\left[(1 + 4r^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}} d\vartheta \\ &= \frac{13\pi}{3}.\end{aligned}$$

Example 10.1.3 *Compute* $\iint_\Sigma x^2 z \cos\gamma d\Sigma$, *where* Σ *is the unit sphere and* γ *is the angle between the exterior normal to the sphere and the positive z-axis.*

Solution: Σ is the unit sphere $x^2 + y^2 + z^2 = 1$ and γ is the angle between the exterior normal to the sphere and the positive z-axis.
Here the z-coordinate can be expressed as single-valued function of x and y for the whole surface. Let us break it into two parts:
The upper hemisphere above the xy-plane and the lower hemisphere below it. Accordingly their equations are

$$z = \sqrt{1 - x^2 - y^2}, \quad z = -\sqrt{1 - x^2 - y^2}.$$

when $x^2 + y^2 = 1$ then both the radicals does not belong to C^1.
So theorem 10.1.1 is not directly applicable. However the decomposition

referred to above is easily performed here, so that the integral exists.

we first cut out the equator by the cylinder $x^2 + y^2 = (1 - \epsilon)^2$ and then let $\epsilon \to 0$.

On the whole sphere $\cos\gamma = z$. So $|\sec\gamma| = \frac{1}{|z|}$.

Now

$$\iint_{\Sigma} x^2 z \cos\gamma d\Sigma = \iint_{R} x^2 \sqrt{1 - x^2 - y^2} dS + \iint_{R} x^2 \sqrt{1 - x^2 - y^2} dS$$

where R is the disc $x^2 + y^2 \leq 1$.

Of the two integrals on the right hand side, the first is over the upper side of the upper hemisphere in the upward direction and the second is over the lower side of the lower hemisphere in the downward directions.

So, for lower hemisphere $|\sec\gamma|.\cos\gamma < 0$ and $z = -\sqrt{1 - x^2 - y^2} < 0$.

Hence

$$\iint_{\Sigma} x^2 z \cos\gamma d\Sigma = 2\iint_{R} x^2 \sqrt{1 - x^2 - y^2} dS.$$

Using polar coordinates

$$\iint_{\Sigma} x^2 z \cos\gamma d\Sigma = \lim_{\epsilon \to 0} 2\int_0^{2\pi} \cos^2\vartheta d\vartheta \int_0^{1-\epsilon} r^3 \sqrt{1 - r^2} dr = \frac{4\pi}{15}.$$

10.1.2 Gauss's Theorem

We use the following notations.

A surface Σ will be denoted by Σ^* if it has the following properties.

(i). It is the boundary of a three dimensional region V, which is a region V_{xy}, V_{yz}, V_{zx}.

(ii). In each case the defining function should belong to C^1.

For example if

$$V_{xy} = V(R, \varphi(x, y), \psi(x, y)). \tag{10.2}$$

This is the region bounded by the surfaces $z = \varphi(x, y)$, $z = \psi(x, y)$ and the cylinder whose rulings are perpendicular to the $xy-$ plane on the boundary of a region R of that plane.

Then $\varphi, \psi \in C^1$ in R

Clearly Σ^* will have a continuously turning tangent plane over the parts of the surface corresponding to the defining functions.

Result : If

1. $f(x, y, z) \in C$ in V_{xy}

2. $V_{xy} = V[R, \varphi(x, y), \psi(x, y)]$ then

we have

$$\iiint_{V_{xy}} f(x, y, z)\, dV = \iint_R dS \int_{\varphi(x,y)}^{\psi(x,y)} f(x, y, z)\, dz$$

Theorem 10.1.2 (Gauss's Theorem)

1. $P(x, y, z), Q(x, y, z), R(x, y, z) \in C^1$ *in V*

2. *V is bounded by* Σ^*

3. α, β, γ *are the direction angles of the exterior normal to* Σ^*

$$\Rightarrow \iiint_V [P_1(x, y, z) + Q_2(x, y, z) + R_3(x, y, z)]\, dV$$

$$= \iint_{\Sigma^*} [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma]\, d\Sigma$$

Proof: V is bounded by Σ^* given by the equation (10.2).

Since $V_{xy} = V[R, \varphi(x, y), \psi(x, y)]$ and $R(x, y, z) \in C^1$ in V, we have

$$\iiint_V R_3(x, y, z)\, dV = \iint_R dS \int_{\varphi(x,y)}^{\psi(x,y)} R_3\, dz$$

$$= \iint_R [R(x, y, z)]_{\varphi(x,y)}^{\psi(x,y)}\, dS$$

$$= \iint_R R(x, y, \psi(x, y))\, dS - \iint_R R(x, y, \varphi(x, y))\, dS$$

$$= \iint_{\Sigma_1} R(x, y, z)|\cos\gamma|\, d\Sigma - \iint_{\Sigma_2} R(x, y, z)|\cos\gamma|\, d\Sigma$$

Here Σ_1 and Σ_2 are the upper and lower nappes, respectively, of Σ^*.

we have $\cos\gamma > 0$ on Σ_1 and $\cos\gamma < 0$ on Σ_2,

Suppose **Case (i):** $\varphi = \psi$ on the boundary of R,

$$\iiint_V R_3(x, y, z)\,dV = \iint_{\Sigma_1} R(x, y, z)\cos\gamma\, d\Sigma + \iint_{\Sigma_2} R(x, y, z)\cos\gamma\, d\Sigma$$

$$= \iint_{\Sigma^*} R(x, y, z)\cos\gamma\, d\Sigma \qquad (10.3)$$

Case (ii): If $\varphi \neq \psi$ on the whole boundary of R, there is also a cylindrical surface bounding V_{xy}.

To evaluate the surface integral.

We can project the surface on the other coordinate faces.

The factor $\cos\gamma = 0$ everywhere on the cylindrical surface.

Hence the corresponding surface integral will be zero. Hence equation (10.3) holds in either case. So We have

$$\iiint_V R_3(x, y, z)\,dV = \iint_{\Sigma^*} R(x, y, z)\cos\gamma\, d\Sigma$$

Similarly by symmetry we can prove that

$$\iiint_V P_1(x, y, z)\,dV = \iint_{\Sigma^*} P(x, y, z)\cos\alpha\, d\Sigma$$

and

$$\iiint_V Q_2(x, y, z)\,dV = \iint_{\Sigma^*} Q(x, y, z)\cos\beta\, d\Sigma$$

where α, β, γ are the direction angles of the exterior normal to Σ^*.

Hence we have

$$\iiint_V [P_1(x, y, z) + Q_2(x, y, z) + R_3(x, y, z)]\,dV$$

$$= \iint_{\Sigma^*} [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma]\,d\Sigma$$

Q

Corollary 10.1.1

1. $R(x, y, z) \in C^1$ *in* $V_{xy} = V(R, \varphi(x, y), \psi(x, y))$

2. $\varphi, \psi \in C^1$ *in* R

3. γ *is the angle between the positive z-axis and the exterior normal to* Σ*, the boundary of* V_{xy}

$$\Rightarrow \iiint_{V_{xy}} R_3 dV = \iint_{\Sigma} R \cos \gamma d\Sigma$$

Proof: Here V_{xy} is neither V_{yz} nor a V_{xz}.

So in the previous theorem Σ need not be a Σ^*.

Here $V_{xy} = V(R, \varphi(x, y), \psi(x, y))$, $\varphi, \psi \in C^1$ in R, $R(x, y, z) \in C^1$ and Σ is the boundary of V_{xy}.

If γ is the angle between the positive $z-$ axis and the exterior normal to Σ, applying previous theorem we have

$$\iint_{V_{xy}} R_3 dV = \iint_{\Sigma} R \cos \gamma d\Sigma.$$

Q

Remark: The corresponding corollaries hold for regions V_{yz} or V_{xz}.

10.2 Verification of Gauss's theorem

Example 10.2.1 *Compute* $\iint_{\Sigma} x^2 z \cos \gamma d\Sigma$*, where* Σ *is the unit sphere and* γ *is the angle between the exterior normal to the sphere and the positive* $z - axis$.

Solution: Take $P = Q = 0$, and $R = x^2 z$ in hypothesis 3 of Theorem 10.1.2.

$$\iiint_{V} R_3(x, y, z) dV = \iint_{\Sigma^*} R(x, y, z) \cos \gamma d\Sigma$$

Here $R_3 = x^2$, $\Sigma^* = \Sigma$. We can evaluate $\iiint_V x^2 dV$ using spherical coordinates

$$x = r\sin\varphi\cos\vartheta, \quad 0 \le \vartheta \le 2\pi$$

$$y = r\sin\varphi\sin\vartheta, \quad 0 \le \varphi \le \pi$$

$$z = r\cos\varphi, \quad r \ge 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\vartheta)} = r^2\sin\varphi, \quad dV = r^2\sin\varphi\, dr d\varphi d\vartheta.$$

Now

$$\begin{aligned}
\iiint_V x^2 dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (r\sin\varphi\cos\vartheta)^2\, r^2\sin\varphi\, dr d\varphi d\vartheta \\
&= \int_0^{2\pi}\cos^2\vartheta\, d\vartheta \int_0^{\pi}\sin^3\varphi\, d\varphi \int_0^1 r^4 dr \\
&= \int_0^{2\pi}\cos^2\vartheta\, d\vartheta \int_0^{\pi}\sin^3\varphi\, d\varphi \left[\frac{r^5}{5}\right]_0^1 \\
&= \frac{1}{5}\int_0^{2\pi}\cos^2\vartheta\, d\vartheta \int_0^{\pi}\sin^3\varphi\, d\varphi \\
&= \frac{1}{5}\int_0^{2\pi}\cos^2\vartheta\, d\vartheta \left[-\cos\varphi + \frac{\cos^3\varphi}{3}\right]_0^{\pi} \\
&= \frac{1}{5}\int_0^{2\pi}\cos^2\vartheta\, d\vartheta \left(1 + 1 - \frac{1}{3} - \frac{1}{3}\right) \\
&= \frac{4}{5}\int_0^{2\pi}\frac{1+\cos 2\vartheta}{2}\, d\vartheta \\
&= \frac{4}{30}\left[\vartheta - \frac{\sin 2\vartheta}{2}\right]_0^{2\pi} \\
&= \frac{4}{30}(2\pi) = \frac{4\pi}{15}.
\end{aligned}$$

Example 10.2.2 *Evaluate by two methods* $\iiint_V (xy + yz + zx)\,dV$ *where V is the region bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 1$.

Solution: First we will evaluate the given integral by iteration method. V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 1$.

$$R_x = R[0, 1, \sqrt{1 - x^2}, 0].$$

We can evaluate $I = \iiint_V (xy + yz + zx)dV$ using cylindrical coordinates (r, ϑ, z).

$$x = r\cos\vartheta, \quad y = r\sin\vartheta \quad z = z$$

where $r \geq 0, 0 \leq \vartheta \leq \frac{\pi}{2}, -\infty < z < \infty.$ and

$$\frac{\partial(x, y, z)}{\partial(r, \vartheta, z)} = r \text{ and } dV = r dr d\vartheta dz.$$

Now

$$
\begin{aligned}
I &= \iint_{R_x} \left(xy + \frac{y}{2} + \frac{x}{2} \right) dS \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \left(r^2 \cos\vartheta \sin\vartheta + \frac{r\sin\vartheta}{2} + \frac{r\cos\vartheta}{2} \right) r dr d\vartheta \\
&= \int_0^1 r^3 dr \int_0^{\frac{\pi}{2}} \cos\vartheta \sin\vartheta d\vartheta + \frac{1}{2}\int_0^1 r^2 dr \int_0^{\frac{\pi}{2}} \sin\vartheta d\vartheta + \frac{1}{2}\int_0^1 r^2 dr \int_0^{\frac{\pi}{2}} \cos\vartheta d\vartheta \\
&= \int_0^1 r^3 dr \int_0^{\frac{\pi}{2}} \frac{\sin 2\vartheta}{2} d\vartheta + \frac{1}{2}\int_0^1 r^2 dr \left[-\cos\vartheta \right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^1 r^2 dr \left[\sin\vartheta \right]_0^{\frac{\pi}{2}} \\
&= \int_0^1 r^3 dr \left[\frac{-\cos 2\vartheta}{4} \right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^1 r^2 dr + \frac{1}{2}\int_0^1 r^2 dr \\
&= \frac{2}{4}\int_0^1 r^3 dr + \frac{1}{2}\left(\frac{r^3}{3} \right)_0^1 + \frac{1}{2}\left(\frac{r^3}{3} \right)_0^1 \\
&= \frac{2}{4}\left[\frac{r^4}{4} \right]_0^1 + \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{8} + \frac{1}{3} = \frac{11}{24}
\end{aligned}
$$

Now we can evaluate the given integral using Gauss's theorem

We have,

$$P_1 = xy, \quad Q_2 = yz, \quad R_3 = zx.$$

$$\text{so} \quad P = \frac{x^2 y}{2}, \quad Q = \frac{y^2 z}{2}, \quad R = \frac{z^2 x}{2}.$$

Hence

$$I = \iint_\Sigma \left(\frac{x^2 y}{2}\cos\alpha + \frac{y^2 z}{2}\cos\beta + \frac{z^2 x}{2}\cos\gamma \right) d\Sigma \qquad (10.4)$$

Here Σ consists of four plane faces and a cylindrical surface. The only plane face that contributes a value not zero is $z = 1$. If α, β, γ are the angles made by $z = 1$ with the positive $x-$ axis ,positive y axis and positive $z-$ axis respectively, then $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}, \gamma = 0$. Then we obtain

$$\iint_\Sigma \frac{z^2 x}{2}\, d\Sigma = \iint_\Sigma \frac{z^2 x}{2}\, d\Sigma = \iint_{R_x} \frac{x}{2}\, dS$$

$$= \frac{1}{2}\int_0^1 r^2\, dr \int_0^{\frac{\pi}{2}} \cos\vartheta\, d\vartheta$$

$$= \frac{1}{2}\int_0^1 r^2 dr\, [\sin\vartheta]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\int_0^1 r^2\, dr$$

$$= \frac{1}{2}\left(\frac{r^3}{3}\right)_0^1$$

$$= \frac{1}{6}.$$

Finally, for the cylindrical surface, $\cos\alpha = x$, $\cos\beta = y$, $\cos\gamma = 0$. Here we have only to consider the first two terms of the integral (10.4) in this case. The first can be expressed as a double integral over a unit square in the yz-plane, the second over a unit square in the xz-plane:

$$\iint \frac{x^2 y}{2}\cos\alpha\, d\Sigma = \frac{1}{2}\int_0^1 dz \int_0^1 (1-y^2)\, y\, dy = \frac{1}{8}$$

$$\iint \frac{y^2 z}{2}\cos\beta\, d\Sigma = \frac{1}{2}\int_0^1 z\, dz \int_0^1 (1-x^2)\, dx = \frac{1}{6}$$

Hence $I = \frac{1}{6} + \frac{1}{8} + \frac{1}{6} = \frac{11}{24}.$

Example 10.2.3 *Verify Gauss's theorem for*

$$\iint_\Sigma (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)\, d\Sigma$$

where Σ is the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$ and α, β, γ are the angle between the exterior normal to the positive $x-$ axis, $y-$ axis and $z-$ axis respectively.

Solution: Take

$$P = 4x, \quad Q = -2y^2, \quad R = z^2.$$

$$P_1 = 4, \quad Q_2 = -4y, \quad R_3 = 2z.$$

By Gauss theorem,

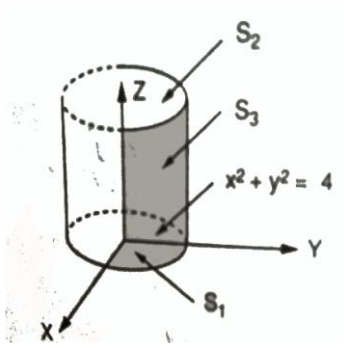

Figure 10.2

$$\iiint_V (P_1 + Q_2 + R_3)\,dV = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,d\Sigma.$$

$$\iiint_V (4 - 4y + 2z)\,dV = \int_{-2}^{2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{3} (4 - 4y + 2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3\,dy\,dx$$

$$= \int_{-2}^{2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (21 - 12y)\,dy\,dx$$

$$= \int_{-2}^{2} \left[21y - \frac{12y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 42\int_{-2}^{2} \sqrt{4 - x^2}\,dx$$

$$= 84\int_{0}^{2} \sqrt{4 - x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\pi.$$

Now

$$\iint_\Sigma (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)\,d\Sigma$$

$$= \iint_{S_1} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)\,dS_1$$

$$+ \iint_{S_2} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)\,dS_2$$

$$+ \iint_{S_3} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)\,dS_3$$

On S_1, $z = 0$, $(\alpha, \beta, \gamma) = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$,

So $4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma = 0$.

$$\iint_{S_1} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)dS_1 = 0$$

On S_2, $z = 3$, $(\alpha, \beta, \gamma) = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$,

So $z^2\cos\gamma = 9$.

$$\iint_{S_2} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)dS_2 = 9. \text{Area of } S_2 = 9 \times 4\pi = 36\pi.$$

On S_3, $x^2 + y^2 = 4$.

So $(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{2x}{\sqrt{4(x^2+y^2)}}, \frac{2y}{\sqrt{4(x^2+y^2)}}, 0\right) = \left(\frac{x}{2}, \frac{y}{2}, 0\right)$

$$\iint_{S_3} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)dS_3 = \iint_{S_3} (2x^2 - y^3)dS_3.$$

Using polar coordinates $x = 2\cos\vartheta$, $y = 2\sin\vartheta$,

$$\begin{aligned}
\iint_{S_3} (2x^2 - y^3)dS_3 &= \int_{\vartheta=0}^{2\pi}\int_{z=0}^{3} (2(2\cos\vartheta)^2 - (2\sin\vartheta)^3)2\,dz\,d\vartheta \\
&= \int_{\vartheta=0}^{2\pi} (16\cos^2\vartheta . z - 16\sin^3\vartheta . z)_0^3 d\vartheta \\
&= \int_{\vartheta=0}^{2\pi} (48\cos^2\vartheta - 48\sin^3\vartheta)d\vartheta \\
&= \int_{\vartheta}^{2\pi} 48\cos^2\vartheta d\vartheta \\
&= 48\pi.
\end{aligned}$$

So

$$\iint_{\Sigma} (4x\cos\alpha - 2y^2\cos\beta + z^2\cos\gamma)d\Sigma = 0 + 36\pi + 48\pi = 84\pi.$$

Hence the Gauss theorem is verified.

Example 10.2.4 *Evaluate* $\iint_S xy^2 dydz + (x^2y - z^3)dzdx + (2xy + y^2z)dxdy$ *where* S *is the entire surface of the hemispherical region bounded by* $z = \sqrt{a^2 - x^2 - y^2}$ *and* $z = 0$ *the divergence theorem.*

Solution: We have $dydz = \cos\alpha dS$, $dzdx = \cos\beta dS$, $dxdy = \cos\gamma dS$.

By divergence theorem

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS = \iiint_V (P_1 + Q_2 + R_3)dV$$

Here

$$P = xy^2, \quad Q = x^2y - z^3, \quad R = 2xy + y^2z; \quad P_1 = y^2, \quad Q_2 = x^2, \quad R_3 = y^2.$$

Hence

$$\iint_S xy^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dx\,dy = \iiint_V (y^2 + x^2 + y^2)\,dV$$

$$= \iiint_V a^2\,dV$$

$$= a^2.\text{Volume of the hemisphere}$$

$$= a^2 . \frac{2}{3}\pi a^3$$

$$= \frac{2}{3}\pi a^5.$$

where V is the region bounded by the hemisphere and the $xy-$plane.

10.3 Vector consideration

Let $\vec{v} = \vec{v}(x_1, x_2, x_3)$ be a vector function, defining a vector field. Suppose $(x_1, x_2, x_3) = (P, Q, R)$.

Let $\vec{\zeta} = (\cos\alpha, \cos\beta, \cos\gamma)$ be the unit vector along the exterior normal to the surface Σ^* of Theorem 10.1.2 . Then

$$Div\,\vec{v} = P_1 + Q_2 + R_3.$$

$$\vec{v}.\vec{\zeta} = P\cos\alpha + Q\cos\beta + R\cos\gamma.$$

Then the conclusion of that theorem becomes

$$\iiint_V Div\,\vec{v}\,dV = \iint_{\Sigma^*} \vec{v}.\vec{\zeta}\,d\Sigma \qquad (10.5)$$

Theorem 10.1.2 is often called the divergence theorem, since the divergence appears as the integrand of the triple integral .

Physical meaning of equation (10.5)

Suppose that $\vec{v}$ defines a velocity field for a fluid. That is, the vector $\vec{v}$ at each point gives the velocity of the fluid there both in direction and magnitude (say, in feet per second).

Suppose R is a plane region of area A and if vectors $\vec{y}$ over R are all perpendicular to R and of constant magnitude, then

$A|\vec{y}| =$ Number of cubic feet per second of the fluid flowing through R.
We know that

$$\vec{y}.\vec{\zeta} = |\vec{y}| \cos\vartheta$$

where ϑ is the angle between $\vec{y}$ and $\vec{\zeta}$, Then the integrand on the right of (10.5) is

$\vec{y}.\vec{\zeta} = |\vec{y}| \cos\vartheta =$ component of the velocity vector $\vec{y}$ in the direction of the exterior normal.

The integrand on the $\vec{y}.\vec{\zeta}$ is multiplied by the surface element $d\Sigma$,

The surface integral = the number of cubic feet per second flowing out of the whole surface Σ^* if the number is positive

= the number of cubic feet per second flowing into Σ^* if the number is negative

In particular if the fluid is incompressible the net rate of flow through $\Sigma^* = 0$ for every Σ^*, so that both integrals (10.5) must be zero.

Definition 10.3.1 *If div $\vec{y} = 0$, the fluid is incompressible .*

Summary

- A subdivision Δ of Σ is a set of closed curves $\{C_k\}_1^n$ lying on Σ and dividing it into a set of n subregions of areas $\Delta\Sigma_k$, $k = 1, 2, ..., n$.

- The diameter of a region on Σ is the length of the largest straight line segment whose ends lie in the region.

- The norm of Δ, denoted by $\|\Delta\|$, is the largest of the n diameters of the subregions produced by the subdivision.

- Let $P(x, y, ⍰)$ be a function defined at every point of Σ and let (ξ_k, η_k, ζ_k) be a point on Σ inside or on the boundary of the subregion bounded by C_k. Then the surface integral of $P(x, y, ⍰)$ over Σ

is

$$\iint_\Sigma P(x,y,z)\,d\Sigma = \lim_{\|\Delta\|\to 0}\sum_{k=1}^{n} P(\xi_k,\eta_k,\zeta_k)\Delta\Sigma_k$$

when this limit exists.

- **1.** $P(x,y,z) \in C$ in V

  2. Σ is the surface $z = f(x,y)$ over the region R

  3. $f(x,y) \in C^1$ in R

  4. Σ lies in V

$$\Rightarrow \quad A.\ \iint_\Sigma P(x,y,z)\,d\Sigma \quad exists$$

$$B.\ \iint_\Sigma P(x,y,z)\,d\Sigma = \iint_R P(x,y,f(x,y))\sqrt{1+f_1^2(x,y)+f_2^2(x,y)}\,dS$$

- **Gauss's theorem :** If

  1. $P(x,y,z), Q(x,y,z), R(x,y,z) \in C^1$ in V

  2. V is bounded by Σ^*

  3. α, β, γ are the direction angles of the exterior normal to Σ^*

$$\Rightarrow \iiint_V [P_1(x,y,z) + Q_2(x,y,z) + R_3(x,y,z)]\,dV$$

$$= \iint_{\Sigma^*} [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]\,d\Sigma$$

- **Vector consideration of Gauss's theorem**

  Let $\vec{y} = \vec{y}(x_1,x_2,x_3)$ be a vector function, defining a vector field.

  Suppose $(x_1,x_2,x_3) = (P,Q,R)$.

  Let $\vec{\zeta} = (\cos\alpha, \cos\beta, \cos\gamma)$ be the unit vector along the exterior normal to the surface Σ^* of Theorem 10.1.2 . Then

$$Div\,\vec{y} = P_1 + Q_2 + R_3.$$

$$\vec{y}.\vec{\zeta} = P\cos\alpha + Q\cos\beta + R\cos\gamma.$$

Then the conclusion of that theorem becomes

$$\iiint_V Div\,\vec{v}\,dV = \iint_{\Sigma^*} \vec{v}.\vec{\zeta}\,d\Sigma$$

- If $div\,\vec{v} = 0$, the fluid is incompressible .

Multiple Choice questions:

1. If Γ is the acute angle between the normal to the surface Σ and $z-$ axis then the value of $\sec\gamma$ is

   a) $\sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\,dS$

   b) $(1 + f_1^2(x, y) + f_2^2(x, y))dS$

   c) $\sqrt{f_1^2(x, y) + f_2^2(x, y)}\,dS$

2. The surface integral is denoted by

   (a) Triple integral

   (b) Double integral

   (c) Line integral

3. Gauss's theorem connects

   (a) line integral to surface integral

   (b) surface integral to volume integral

   (c) line integral to volume integral

Ans: 1. a) 2. b) 3. b)

Exercises 10

1. If $P(x, y, z) \in C$ in V, Σ is the surface $z = f(x, y)$ over the region R, $f(x, y) \in C^1$ in R and Σ lies in V, then prove that

   A. $\iint_\Sigma P(x, y, z)d\Sigma$ *exists.*

   B. $$\iint_\Sigma P(x, y, z)d\Sigma = \iint_R P(x, y, f(x, y))\sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\,dS$$

2. State and prove Gauss's theorem.

3. (a) $R(x, y, z) \in C^1$ in $V_{xy} = V(R, \varphi(x, y), \psi(x, y))$

   (b) $\varphi, \psi \in C^1$ in R

   (c) γ is the angle between the positive z-axis and the exterior normal to Σ, the boundary of V_{xy}, then prove that

$$\iiint_{V_{xy}} R_3 dV = \iint_{\Sigma} R \cos \gamma d\Sigma$$

4. Use divergence theorem to evaluate $\iint_{\Sigma} (x^3 \cos \alpha + y^3 \cos \beta + z^2 \cos \gamma) d\Sigma$ where Σ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

   Ans: $\frac{12}{5}\pi a^5$.

5. Verify Gauss's divergence theorem for

$$\iint_{\Sigma} ((2xy + z) \cos \alpha + y^2 \cos \beta - (x + 3y) \cos \gamma) d\Sigma$$

   taken over the region bounded by $2x + 2y + z = 6$, $x = 0$, $y = 0$, $z = 0$.

   Ans: 27.

6. Evaluate $\iint_{\Sigma} (x \cos \alpha + y \cos \beta + z \cos \gamma) d\Sigma$ where Σ is the surface of the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$ and α, β, γ are the angle between the exterior normal to the positive $x-$axis, $y-$axis and $z-$axis respectively.

BLOCK V

Transformation and Line Integrals in Space

Unit 11

Transformation and Line Integrals in Space

Learning Outcomes :

After studying this unit, students will be able

F To apply change of variable in evaluating multiple integrals.

F To transform one set of coordinates to another.

F To know about line integrals in space.

F To state and prove Stokes's theorem which relates a line integral over a closed space curve to a surface integral over a surface spanning the curve.

11.1 Introduction

In evaluating multiple integral over a region R it is often convenient to use coordinates other than rectangular such as curvilinear coordinates. In this unit we study change of variable in multiple integral and line integrals in space.

11.2 Change of Variable in Multiple Integrals

For simple integrals, suppose $x = \varphi(t)$, then

$$\int_{\varphi(a)}^{\varphi(b)} F(x)dx = \int_a^b F(\varphi(t))\varphi'(t)dt \tag{11.1}$$

Here the interval (a, b) on the t-axis is transformed into the interval $(\varphi(a), \varphi(b))$ on the x-axis. In this section we will discuss corresponding formula for a change of variable in multiple integrals.

11.2.1 Transformations

Let u, v be the coordinates of a point of a Region R_{uv} in the $uv-$ plane, bounded by a curve Γ_1 and let x, y be the coordinates of a point of a region R_{xy} bounded by a curve Γ_2 in the $xy-$ plane .
The transformation

$$\left.\begin{aligned} x &= g(u, v) \\ y &= h(u, v) \end{aligned}\right\} \tag{11.2}$$

where g and h are two single valued functions defined on the region R_{uv}, establishes a one-to-one correspondence between the points of the two regions. The equations (11.2) can be solved for u and v, the resulting functions being single valued in R_{xy}.

Example 11.2.1 *Let $g(u, v) = v$ cos u, $h(u, v) = v$ sin u. The two regions would be as indicated in Figure 11.1.*
Let the boundary of R_{xy} be the curve Γ_{xy} such that,

$$x = \varphi(t), \quad y = \psi(t) \quad 0 \le t \le \frac{\pi}{2}. \tag{11.3}$$

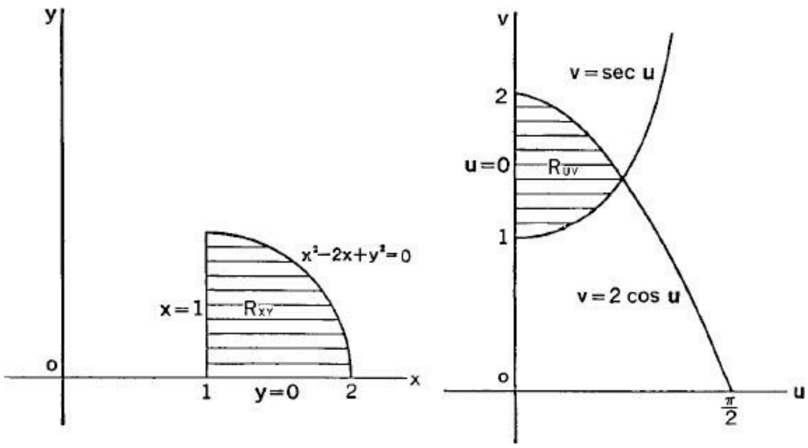

Figure 11.1

Then the boundary curve Γ_{uv} *of* R_{uv} *will have the equations*

$$\varphi(t) = g(u, v) \qquad (11.4)$$
$$\psi(t) = h(u, v)$$

Solve these equations to obtain u *and* v *as single valued functions of* t.

Solution: Consider the curve $x^2 - 2x + y^2 = 0$ whose parametric equations are

$$x = 1 + \cos t, \; y = \sin t.$$

Here $\varphi(t) = 1 + \cos t, \psi(t) = \sin t$.

From equations (11.4) we have $g(u, v) = 1 + \cos t, h(u, v) = \sin t$.

This implies

$$v \cos u = 1 + \cos t$$
$$v \sin u = \sin t$$

Solving these equations for u and v

$$\frac{\sin t}{1+\cos t} = \frac{\sin u}{\cos u}$$

$$\frac{\sin\frac{t}{2}.\cos\frac{t}{2}}{\frac{1+\cos 2\left(\frac{t}{2}\right)}{2}} = \frac{\sin u}{\cos u}$$

$$\frac{\sin\frac{t}{2}.\cos\frac{t}{2}}{\cos^2\frac{t}{2}} = \tan u$$

$$\Rightarrow \tan\frac{t}{2} = \tan u$$

$$\Rightarrow u = \frac{t}{2}$$

Also we have

$$v = \frac{\sin t}{\sin u} = \frac{\sin t}{\sin\frac{t}{2}} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{\sin\frac{t}{2}} = 2\cos\frac{t}{2}$$

We have $v = 2\cos\frac{t}{2}$ and $u = \frac{t}{2}$. This is a piece of the curve $v = 2\cos u$.

Example 11.2.2 *A region R in the xy−plane is bounded by $x + y = 6$, $x - y = 2$ and $y = 0$. Determine the region R^1 in the uv−plane into which R is mapped under the transformation $x = u + v, y = u - v$.*

Solution: The region R is a triangle bounded by the lines $x + y = 6$,

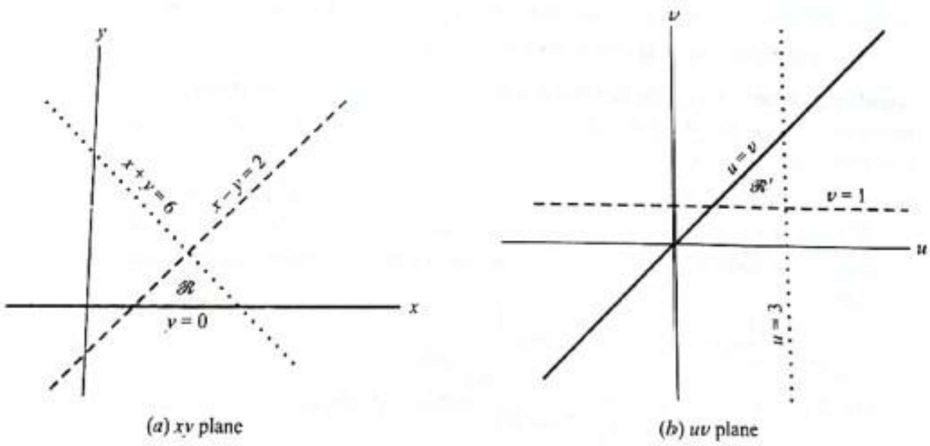

Figure 11.2

$x - y = 2$ and $y = 0$.

Given transformation is $x = u + v, y = u - v$.

Consider the line

$$x + y = 6$$

$$u + v + u - v = 6$$

$$u = 3$$

which is a line the $uv-$plane.

Similarly $x - y = 2$ becomes

$$u + v - u + v = 2$$

$$2v = 2$$

$$v = 1$$

which is a line in the $uv-$plane.

Also $y = 0 \Rightarrow u = v$, a line in the $uv-$plane. Thus the required region is bounded by $u = 3, v = 1$ and $u = v$.

Remark : Let us now see how the transformation (11.2) affects the line integral. We show that

$$\int_{\Gamma_{xy}} Q(x, y)dy = \int_{\Gamma_{uv}} Q(g(u, v), h(u, v))[h_1(u, v)du + h_2(u, v)dv] \qquad (11.5)$$

We can fix the direction of integration in one of these integrals is arbitrary and the direction of the other integration is determined by the transformation (11.2).

In the above example discussed the clockwise description of Γ_{xy} corresponds to the counterclockwise description of Γ_{uv}.

As $x = \varphi(t)$ and $y = \psi(t)$,

Left hand side of equation (11.5)

$$= \int_0^1 Q(\varphi(t), \psi(t))\psi'(t)\, dt \qquad (11.6)$$

To evaluate the right-hand side to equation (11.5) we use the equations (11.4) of the curve Γ_{uv}. Consider $\psi(t) = h(u, v)$

$$\psi'(t) = h_1(u, v)\frac{du}{dt} + h_2(u, v)\frac{dv}{dt}$$

So right hand side of equation (11.5) = $\int_0^1 Q(\varphi(t), \psi(t))\psi'(t)dt.$

11.2.2 Double Integrals

We shall now obtain a formula connecting the areas of the two regions R_{uv} and R_{xy}.

Theorem 11.2.1

1. $F(x, y) \in C$ *in* R_{xy}
2. $g(u, v), h(u, v) \in C^2$ *in* R_{uv}
3. $\frac{\partial(g, h)}{\partial(u, v)} \neq 0$ *in* R_{uv}
4. R_{xy} *and* R_{uv} *correspond in a one -to- one fashion under the transformation* $x = g(u, v)$, $y = h(u, v)$ *and both are regular, simply connected region S*

$$\Rightarrow \iint_{R_{xy}} F(x, y)dS_{xy} = \iint_{R_{uv}} F(g(u, v), h(u, v)) \left|\frac{\partial(g, h)}{\partial(u, v)}\right| dS_{uv} \qquad (11.7)$$

Proof: Let us first prove the theorem for the special case $F = 1$. Then

$$\iint_{R_{xy}} F(x, y)dS_{xy} = \iint_{R_{xy}} dS_{xy}$$

$$= Area\ of\ R_{xy}$$

$$= A\ (say)$$

By Green's theorem we have $A = \oint_{\Gamma_{xy}} xdy.$

Applying equation (11.5) when $Q = x$ we have

$$\oint_{\Gamma_{xy}} xdy = \oint_{\Gamma_{uv}} g(u, v)\,[h_1(u, v)du + h_2(u, v)dv]$$

So

$$A = \oint_{\Gamma_{xy}} xdy = \oint_{\Gamma_{uv}} g(u, v)\,[h_1(u, v)du + h_2(u, v)dv]$$

Here the integration is counterclockwise. Applying Green's theorem to $\oint_{\Gamma_{uv}} g(u,v)\,[h_1(u,v)du + h_2(u,v)dv]$ we obtain

$$\begin{aligned}
\oint_{\Gamma_{uv}} g(u,v)\,[h_1(u,v)du + h_2(u,v)dv] &= \oint_{\Gamma_{uv}} g(u,v)h_1(u,v)du + g(u,v)h_2(u,v)dv \\
&= \pm \iint_{R_{uv}} \left[\frac{\partial}{\partial u}(gh_2) - \frac{\partial}{\partial v}(gh_1)\right] dS_{uv} \\
&= \pm \iint_{R_{uv}} [g_1h_2 + gh_{21} - g_2h_1 - gh_{12}]\, dS_{uv} \\
&= \pm \iint_{R_{uv}} J(u,v)dS_{uv}
\end{aligned}$$

where $J(u,v) = \dfrac{\partial(g,h)}{\partial(u,v)}$ is the Jacobian of the transformation.

If the sense of description of Γ_{uv} is counterclockwise and clock wise we have plus and minus sign respectively. But we are given $g(u,v), h(u,v) \in C^2$ in R_{uv} and $\dfrac{\partial(g,h)}{\partial(u,v)} \neq 0$ in R_{uv}, so the Jacobian never changes sign.

Also the area is always positive. So we must choose plus sign when J is positive and minus sign when J is negative.

We have

$$A = \iint_{R_{uv}} |J(u,v)|dS_{uv}$$

Hence the theorem is true when $F = 1$.

We observe that when $J > 0$ clockwise description of Γ_{xy} corresponds to clockwise descriptions of Γ_{uv}.

Let Δ be a subdivision of R_{xy} into subregions R_k of area ΔS_k, $k = 1, 2, ..., n$. By the transformation (11.2) there will correspond in the uv-plane a sub-division Δ' of R_{uv} into subregions R_k' of area $\Delta S_k'$.

Now by the above proof the areas of these subregions are related as follows:

$$\Delta S_k = \iint_{R_k'} |J(u,v)|dS_{uv} = |J(u_k,v_k)|\Delta S_k'$$

where the point (u_k, v_k) is in R_k'.

By the hypothesis R_{xy} and R_{uv} corresponds in a one-one fashion.

Let (x_k, y_k) be the point of R_k which corresponds to (u_k, v_k) under the transformation (11.2).

Since $F(x, y) \in C$ in R_{xy}, $\iint_{R_{xy}} F(x, y)\,dS_{xy}$ exists, we have

$$\iint_{R_{xy}} F(x, y)\,dS_{xy} = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} F(x_k, y_k)\Delta S_k$$

$$= \lim_{\|\Delta'\| \to 0} \sum_{k=1}^{n} F(g(u_k, v_k), h(u_k, v_k))|J(u_k, v_k)|\Delta S'_k$$

Since $g(u, v)$ and $h(u, v)$ are uniformly continuous in R_{uv} the norm of Δ approaches zero when the norm of Δ' approaches zero follows.

Hence

$$\iint_{R_{xy}} F(x, y)\,dS_{xy} = \iint_{R_{uv}} F(g(u, v), h(u, v)) \left|\frac{\partial (g, h)}{\partial(u, v)}\right| dS_{uv}$$

This concludes the proof of the theorem. Q

Example 11.2.3 *Make the transformation* $x = v\cos u$, $y = v\sin u$ *to* $\iint_{R_{xy}} y\,dS_{xy}$ *where* R_{xy} *is the region shown in Figure 11.1.*

Solution: Given $x = v\cos u$ $y = v\sin u$ (i.e). $g(u, v) = v\cos u$ $h(u, v) = v\sin u$. By theorem 11.2.1

$$\iint_{R_{xy}} y\,dS_{xy} = \iint_{R_{uv}} v\sin u \left|\frac{\partial (g, h)}{\partial(u, v)}\right| dS_{uv}$$

Now

$$\frac{\partial(g, h)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{vmatrix}$$

$$= -v.$$

So

$$\iint_{R_{x,y}} y\,dS_{xy} = \iint_{R_{uv}}$$

$v \sin u| - v| dS_{uv} =$ $v^2 \sin$

R_{uv} $u dS_{uv}$

Hence,

$$\int_1^2 dx \int_0^{\sqrt{2x-x^2}} y\,dy = \int_0^{\frac{\pi}{4}} \sin u\,du \int_{\sec u}^{2\cos u} v^2 dv$$

$$= \int_0^{\frac{\pi}{4}} \sin u\,du \left(\frac{v^3}{3}\right)_{\sec u}^{2\cos u}$$

$$= \int_0^{\frac{\pi}{4}} \sin u \left[\frac{8\cos^3 u}{3} - \frac{\sec^3 u}{3}\right] du$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \left[8\cos^3 u \sin u - \sec^3 u \sin u\right] du$$

put $t = \cos u \Rightarrow dt = -\sin u du,$

$$= \frac{1}{3}\int_1^{\frac{1}{\sqrt{2}}} \left(-8t^3 dt + \frac{1}{t^3} dt\right),$$

$$= -\frac{1}{3}\left[\frac{8t^4}{4} + \frac{1}{2t^2}\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= -\frac{1}{3}[-1]$$

$$= \frac{1}{3}.$$

Example 11.2.4 *Evaluate* $\iint (y-x)dxdy$ *over the region* R_{xy} *in the* $xy-$ *plane bounded by the straight lines*

$y = x - 3,\ y = x + 1,\ 3y + x = 5,\ 3y + x = 7.$

Solution: It is difficult to evaluate the double integral directly;however a simple change of coordinates reduces the domain of integration into a rectangle with sides parallel to the axes.

Set $y - x = u, \quad 3y + x = v.$
so that $x = \frac{1}{4}(v - 3u), \quad y = \frac{1}{4}(v + u)$
and

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{16}\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = -\frac{1}{4}.$$

So $|J| = \frac{1}{4}.$

The new domain is the rectangle R_{uv} in the $uv-$ plane bounded by the lines $u = -3, u = 1, v = 5, v = 7.$

So

$$\iint_{R_{xy}} (y - x)\,dxdy = \iint_{R_{uv}} u.\frac{1}{4}\,dudv$$
$$= \frac{1}{4}\int_{-3}^{1} u\,du \int_{5}^{7} dv$$
$$= -2.$$

Example 11.2.5 *Integrate* $x^2 + y^2$ *over the circle* $x^2 + y^2 = a^2$.

Solution: Using polar coordinates, $x = r\cos\vartheta$, $y = r\sin\vartheta$

so that

$$J = \frac{\partial(x,y)}{\partial(r,\vartheta)} = \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix} = r$$

$$\iint_{x^2+y^2\le a^2} (x^2 + y^2)\,dxdy = \int_0^{2\pi}\int_0^{a} r^2 .r\,drd\vartheta = \frac{\pi a^4}{2}.$$

11.2.3 Applications

It is frequently required to evaluate a surface integral over a surface Σ which is given parametrically:

$$x = g(u,v), \quad y = h(u,v), \quad z = k(u,v)$$

Let

$$j_1 = \frac{\partial(h,k)}{\partial(u,v)}, \quad j_2 = \frac{\partial(k,g)}{\partial(u,v)}, \quad j_3 = \frac{\partial(g,h)}{\partial(u,v)}, \quad D = \sqrt{j_1^2 + j_2^2 + j_3^2}$$

Let Σ correspond to the region R_{uv} of the uv-plane. Suppose that $D \neq 0$ in R_{uv}. Then j_1, j_2, j_3 do not vanish simultaneously.

Case (i) Suppose first that j_3 does not vanish. If γ is the acute angle between the normal to Σ and the z-axis, then

$$\sec\gamma = \frac{D}{|j_3|}.$$

If R_{xy} is the projection of Σ on the xy-plane, then by Theorem 10.1.1 and Theorem 11.2.1 we have

$$\iint_{\Sigma} P(x, y, z)d\Sigma = \iint_{R_{xy}} P(x, y, f(x, y)) \frac{D}{|j_3|} dS_{xy}$$
$$= \iint_{R_{uv}} P(g(u, v), h(u, v), k(u, v))DdS_{uv}$$

Case (ii) Suppose j_1 or j_2 which does not vanish, we may project Σ on yz or xz plane and obtain precisely the same formula.

Case (iii) Suppose no one of the Jacobians is zero throughout R_{uv}, we may divide this region into subregion in each of which some Jacobian does not vanish. Hence in all cases we obtain

$$\iint_{\Sigma} P(x, y, z)d\Sigma = \iint_{R_{uv}} P(g(u, v), h(u, v), k(u, v))DdS_{uv} \tag{11.8}$$

Example 11.2.6 *Find the area of the sphere* $x = a\sin\varphi\cos\vartheta$, $y = a\sin\varphi\sin\vartheta$, $z = a\cos\varphi$, *where* $0 \le \vartheta \le 2\pi$, $0 \le \varphi \le \pi$.

Solution: If $x = g(\varphi, \vartheta)$, $y = h(\varphi, \vartheta)$, $z = k(\varphi, \vartheta)$, then $D = \sqrt{j_1^2 + j_2^2 + j_3^2}$ where $j_1 = \frac{\partial(h, k)}{\partial(\varphi, \vartheta)}$, $j_2 = \frac{\partial(k, g)}{\partial(\varphi, \vartheta)}$, $j_3 = \frac{\partial(g, h)}{\partial(\varphi, \vartheta)}$.

$$j_3 = \frac{\partial(x, y)}{\partial(\varphi, \vartheta)}$$
$$= \begin{vmatrix} \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\vartheta} \\ \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\vartheta} \end{vmatrix}$$
$$= \begin{vmatrix} a\cos\vartheta\cos\varphi & -a\sin\varphi\sin\vartheta \\ a\cos\varphi\sin\vartheta & a\sin\varphi\cos\vartheta \end{vmatrix}$$
$$= a^2\cos^2\vartheta\cos\varphi\sin\varphi + a^2\sin\varphi\cos\varphi\sin^2\vartheta$$
$$= a^2\sin\varphi\cos\varphi.$$

$$j_2 = \frac{\partial(z, x)}{\partial(\varphi, \vartheta)}$$

$$= \begin{vmatrix} \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \vartheta} \end{vmatrix}$$

$$= \begin{vmatrix} -a \sin \varphi & 0 \\ a \cos \varphi \cos \vartheta & -a \sin \varphi \sin \vartheta \end{vmatrix}$$

$$= a^2 \sin^2 \varphi \sin \vartheta.$$

$$j_1 = \frac{\partial(y, z)}{\partial(\varphi, \vartheta)}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{vmatrix}$$

$$= \begin{vmatrix} a \cos \varphi \sin \vartheta & a \sin \varphi \cos \vartheta \\ -a \sin \varphi & 0 \end{vmatrix}$$

$$= a^2 \sin^2 \varphi \cos \vartheta.$$

$$D = \sqrt{a^4 \sin^2 \varphi \cos^2 \varphi + a^4 \sin^4 \sin^2 \vartheta + a^4 \sin^4 \varphi \cos^2 \vartheta}$$

$$= \sqrt{a^4 \sin^4 \varphi + a^4 \sin^2 \varphi \cos^2 \varphi}$$

$$= \sqrt{a^4 \sin^2 \varphi (\sin^2 \varphi + \cos^2 \varphi)}$$

$$= a^2 \sin \varphi.$$

Hence, the area is

$$A = \iint_{R_{\vartheta\varphi}} a^2 \sin \varphi dS_{\vartheta\varphi}$$

$$= a^2 \int_0^{2\pi} d\vartheta \int_0^{\pi} \sin \varphi d\varphi$$

$$= a^2 \int_0^{2\pi} d\vartheta \left[-\cos \varphi \right]_0^{\pi}$$

$$= 2a^2 \int_0^{2\pi} d\vartheta$$

$$= 2a^2 [\vartheta]_0^{2\pi}$$

$$= 4\pi a^2.$$

Example 11.2.7 *Show that the area of the surface of revolution* $x = u\cos\vartheta, y = u\sin\vartheta, z = f(u), \quad a \le u \le b, 0 \le \vartheta \le 2\pi$ *is* $2\pi\int_a^b u\sqrt{1+[f'(u)]^2}\,du.$

Solution: Given $x = u\cos\vartheta, y = u\sin\vartheta, z = f(u)$.

$$j_1 = \frac{\partial(y,z)}{\partial(u,\vartheta)} = \begin{vmatrix} \sin\vartheta & u\cos\vartheta \\ f'(u) & 0 \end{vmatrix} = f'(u).u\cos\vartheta.$$

$$j_2 = \frac{\partial(z,x)}{\partial(u,\vartheta)} = \begin{vmatrix} f'(u) & 0 \\ \cos\vartheta & -u\sin\vartheta \end{vmatrix} = -f'(u).u\sin\vartheta.$$

$$j_3 = \frac{\partial(x,y)}{\partial(u,\vartheta)} = \begin{vmatrix} \cos\vartheta & -u\sin\vartheta \\ \sin\vartheta & u\cos\vartheta \end{vmatrix} = u\cos^2\vartheta + u\sin^2\vartheta = u.$$

$$D = \sqrt{j_1^2 + j_2^2 + j_3^2} = \sqrt{[f'(u)]^2u^2\cos^2\vartheta + [f'(u)]^2u^2\sin^2\vartheta + u^2}$$
$$= u\sqrt{[f'(u)]^2 + 1}.$$

Hence the area is $A = \iint_{R_{uv}} u\sqrt{1+[f'(u)]^2}\,dS_{uv}$.

$$A = \int_a^b\int_0^{2\pi} u\sqrt{1+[f'(u)]^2}\,du\,d\vartheta$$
$$= \int_a^b u\sqrt{1+[f'(u)]^2}\,[\vartheta]_0^{2\pi}du$$
$$= 2\pi\int_a^b u\sqrt{1+[f'(u)]^2}\,du.$$

Remark : The transformation (11.2) has another useful interpretation. It may be regarded as a change of coordinates. Thus (x, y) and (u, v), connected by equations (11.2), may be thought of as different coordinates of the same point.

Example

(i) In cylindrical coordinates (r, φ, z), transformation equations

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

where $r \ge 0, 0 \le \varphi \le 2\pi, -\infty < z < \infty$.

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,z)} = r.$$

Element of volume $dV = rdrd\varphi dz$.

(ii) In spherical coordinates (r, ϑ, φ), transformation equations

$$x = r\sin\vartheta\cos\varphi, \quad y = r\sin\vartheta\sin\varphi, \quad z = r\cos\vartheta$$

where $r \geq 0, 0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2\pi$.

$$\frac{\partial(x, y, z)}{\partial(r, \vartheta, \varphi)} = r^2 \sin\vartheta.$$

Element of volume $dV = r^2 \sin\vartheta dr d\vartheta d\varphi$.

(iii) In polar coordinates (r, ϑ), transformation equations

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

where $r \geq 0, 0 \leq \vartheta \leq 2\pi$

$$\frac{\partial(x, y)}{\partial(r, \vartheta)} = r^2.$$

11.3 Line Integrals in Space

In this section we study Stokes's theorem which relates a line integral over a closed space curve to a surface integral over a surface spanning the curve. The relation reduces to Green's theorem for the plane when the curve lies in the $xy-$ plane and the spanning surface is the plane itself.

11.3.1 Definition of the Line Integral

Consider a curve Γ with parametric equations

$$x = \varphi(t), \quad y = \psi(t) \quad z = \omega(t), \quad a \leq t \leq b \tag{11.9}$$

(i) It is regular if

* it has no double points and
* if the interval (a, b) can he divided into a finite number of subintervals in each of which $\varphi(t) \in C^1, \psi(t) \in C^1, \omega(t) \in C^1$.

(ii) If $f(x, y, z)$ is defined on Γ, then with obvious notations we define the line integral

$$\int_\Gamma f(x, y, z)dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\varphi(t_i^*), \psi(t_i^*), \omega(t_i^*))\,[\varphi(t_i) - \varphi(t_{i-1})], \tag{11.10}$$

$t_{i-1} \le t_i^* \le t_i, \; i = 1, 2, \dots, n$ whenever the limit exists.

In the similar manner we can define $\int_\Gamma f(x, y, z)dy$ and $\int_\Gamma f(x, y, z)dz$.

(iii) When $f \in C$ on the regular curve Γ then we show that

$$\int_\Gamma f(x, y, z)dx = \int_a^b f(\varphi(t), \psi(t), \omega(t))\varphi'(t)dt$$

In the similar manner we can define $\int_\Gamma f(x, y, z)dy$ and $\int_\Gamma f(x, y, z)dz$.

(iv) The direction of integration in (11.10) is that direction on Γ which corresponds to the motion of a point whose parametric value t moves from a to b.

Example 11.3.1 *Compute* $\int_\Gamma xdx + xydy + xyzdz$ *where* Γ *is the piece of the twisted cubic* $x = t, y = t^2, z = t^3$ *corresponding to the interval* $0 \le t \le 1$.

Solution: Given $x = t \Rightarrow dx = dt, y = t^2 \Rightarrow dy = 2tdt,$

$z = t^3 \Rightarrow dz = 3t^2dt$

$$\begin{aligned}
\int_\Gamma xdx + xydy + xyzdz &= \int_0^1 tdt + 2\int_0^1 t^4dt + 3\int_0^1 t^8dt \\
&= \left(\frac{t^2}{2}\right)_0^1 + 2\left(\frac{t^5}{5}\right)_0^1 + 3\left(\frac{t^9}{9}\right)_0^1 \\
&= \frac{1}{2} + \frac{2}{5} + \frac{3}{9} \\
&= \frac{37}{30}.
\end{aligned}$$

11.3.2 Stokes's Theorem

Theorem 11.3.1

1. $f(x, y) \in C^2$

2. *Σ is the surface $z = f(x, y)$ bounded by the regular closed curve Γ*

3. $P(x, y, z), Q(x, y, z), R(x, y, z) \in C^1$ *on* Σ

4. α, β, γ *are direction angles of a directed normal to* Σ

$$\Rightarrow \int_\Gamma P dx + Q dy + R dz$$

$$= \iint_\Sigma [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]\, d\Sigma$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

Proof: For definiteness we choose the direction of the normal to Σ so as to make an acute angle with the positive direction on the z-axis. Then we have

$$f_1(x, y) = -\frac{\cos\alpha}{\cos\gamma}, \quad f_2(x, y) = -\frac{\cos\beta}{\cos\gamma} \qquad (11.11)$$

Let the projection of Σ on the $xy-$ plane be R_{xy} and the projection of Γ on the $xy-$ plane be Γ_{xy}.

The sense of description of Γ will give rise to a counterclockwise direction on Γ_{xy}.

Let the parametric representation of Γ_{xy} is $x = \varphi(t), y = \psi(t)$.

Then the parametric representation for Γ is

$$x = \varphi(t), \quad y = \psi(t), \quad z = f(\varphi(t), \psi(t)) \quad a \le t \le b$$

$$dx = \varphi'(t)dt, \qquad dy = \psi'(t)dt, \qquad dz = f_1\varphi'(t) + f_2\psi'(t)$$

Then

$$\int_\Gamma P(x, y, z)dx =$$

$$\int_a^b P(\varphi(t), \psi(t), f(\varphi(t), \psi(t)))\varphi'(t)dt$$

Also

$$\int_{\Gamma_{xy}} P(x, y, f(x, y))dx = \int_a^b P(\varphi(t), \psi(t), f(\varphi(t), \psi(t)))\varphi'(t)dt$$

Hence we have,

$$\int_{\Gamma} P(x, y, z)dx = \int_{\Gamma_{xy}} P(x, y, f(x, y))dx$$

where the sense of description over Γ_{xy} is counterclockwise. By Green's theorem for the plane and using theorem 10.1.1 we have,

$$\int_{\Gamma_{xy}} P(x, y, f(x, y))dx = -\iint_{R_{xy}} \frac{\partial}{\partial y} P(x, y, f(x, y))dS_{xy}$$

$$= -\iint_{R_{xy}} [P_2 + P_3 f_2]\, dS_{xy}$$

$$= -\iint_{\Sigma} [P_2(x, y, z) + P_3(x, y, z)f_2(x, y)] \cos\gamma d\Sigma$$

$$= -\iint_{\Sigma} P_2(x, y, z) + P_3(x, y, z) \frac{-\cos\beta}{\cos\gamma} \cos\gamma d\Sigma$$

$$= -\iint_{\Sigma} [P_2(x, y, z) \cos\gamma - P_3(x, y, z) \cos\beta]\, d\Sigma$$

Hence the theorem is true if it concerns $P(x, y, z)$.

Similarly we have

$$\int_{\Gamma} Q(x, y, z)dy = \int_{\Gamma_{xy}} Q(x, y, f(x, y))dy,$$

where the sense of description of Γ_{xy} is counter clockwise.

By using Green's theorem in plane and theorem 10.1.1 we have

$$\int_{\Gamma_{xy}} Q(x, y, f(x, y))dy = \iint_{R_{xy}} \frac{\partial}{\partial x} Q(x, y, f(x, y))dS_{xy}$$

$$= \iint_{R_{xy}} [Q_1 + Q_3 f_1]\, dS_{xy}$$

$$= \iint_{\Sigma} [Q_1(x, y, z) + Q_3(x, y, z)f_1(x, y)] \cos\gamma d\Sigma$$

$$= \iint_{\Sigma} Q_1(x, y, z) + Q_3(x, y, z) \frac{-\cos\alpha}{\cos\gamma} \cos\gamma d\Sigma$$

$$= \iint_{\Sigma} [Q_1(x, y, z) \cos\gamma - Q_3(x, y, z) \cos\alpha]\, d\Sigma$$

This proves the theorem in so far as it concerns $Q(x, y, z)$.

Consider

$$\oint_{\Gamma} R(x, y, z)dz = \int_a^b R(\varphi(t), \psi(t), f(\varphi(t), \psi(t))) \left[f_1\varphi' + f_2\psi'\right] dt$$

$$= \int_a^b \left[R(\varphi(t), \psi(t), f(\varphi(t), \psi(t))) f_1\varphi' + R(\varphi(t), \psi(t), f(\varphi(t), \psi(t))) f_2\psi'\right] dt$$

$$= \oint_{\Gamma_{xy}} Rf_1dx + Rf_2dy$$

$$= \iint_{R_{xy}} [R_1f_2 + R_3f_1f_2 + Rf_{12} - R_2f_1 - R_3f_1f_2 - Rf_{21}]\, dS_{xy}$$

$$= \iint_{\Sigma} [R_1f_2 - R_2f_1] \cos\gamma d\Sigma$$

$$= \iint_{\Sigma} \left[R_1 \frac{-\cos\beta}{\cos\gamma} - R_2 \frac{-\cos\alpha}{\cos\gamma}\right] \cos\gamma d\Sigma$$

$$= \iint_{\Sigma} [R_2\cos\alpha - R_1\cos\beta]\, d\Sigma$$

Thus we have,

$$\oint_{\Gamma} Pdx + Qdy + Rdz$$

$$= \iint_{\Sigma} (-P_2\cos\gamma + P_3\cos\beta + Q_1\cos\gamma - Q_3\cos\alpha + R_2\cos\alpha - R_1\cos\beta)\, d\Sigma$$

$$= \iint_{\Sigma} [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]\, d\Sigma$$

Q

Corollary 11.3.1

1. Σ *is a surface bounded by the regular closed curve* Γ

2. Σ *has the three equations* $z = f(x, y), x = g(y, z), y = h(z, x)$ *with* $f, g, h \in C^1$

3. $P(x, y, z), Q(x, y, z), R(x, y, z) \in C^1$ *on* Σ

4. α, β, γ *are direction angles of a directed normal to* Σ

$$\Rightarrow \oint_\Gamma Pdx + Qdy + Rdz$$

$$= \iint_\Sigma [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]\, d\Sigma$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

Proof: For Stokes's Theorem we required that Σ should be cut only once by lines parallel to a single axis and that the single-valued defining function should belong to C^2 .

But in the corollary we require Σ should be cut only once by lines parallel to all three axes and that the single-valued defining functions should belong only to C^1.

From the proof of the Stokes's theorem we have

$$\oint_\Gamma P(x, y, z)dx = \iint_\Sigma (P_3 \cos\beta - P_2 \cos\gamma)d\Sigma$$

We permute symbols:

$P \to Q \to R, f \to g \to h, x \to y \to z, 1 \to 2 \to 3, \alpha \to \beta \to \gamma.$

So

$$\oint_\Gamma Q(x, y, z)dy = \iint_\Sigma (Q_1 \cos\gamma - Q_3 \cos\alpha)d\Sigma.$$

$$\oint_\Gamma R(x, y, z)dz = \iint_\Sigma (R_2 \cos\alpha - R_1 \cos\beta)d\Sigma.$$

Hence

$$\oint_\Gamma Pdx+Qdy+Rdz = \iint_\Sigma [(R_2-Q_3)\cos\alpha+(P_3-R_1)\cos\beta+(Q_1-P_2)\cos\gamma]d\Sigma.$$

Q

11.4 Verification of Stokes's theorem

Example 11.4.1 *Compute in two ways the line integral*

$$I = \oint_\Gamma xyz\,dz$$

over the circle $x = \cos t$, $y = \frac{\sin t}{\sqrt{2}}$, $z = \frac{\sin t}{\sqrt{2}}$ $0 \le t \le 2\pi$ *in the direction of increasing t.*

Solution:

Method I: Substitution gives

$$I = \frac{1}{2\sqrt{2}} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt = \frac{\pi}{8\sqrt{2}}.$$

Method II : The direction cosines of the directed normal to Γ, the plane of the circle, are $0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$. By Stokes's theorem

$$\oint_\Gamma Pdx+Qdy+Rdz = \iint_\Sigma ((R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma)\, d\Sigma$$

Here $R = xyz$. So

$$I = \oint_\Gamma xyz\,dz = \iint_\Sigma (-yz\cos\beta) d\Sigma$$

To evaluate this integral, project on the xz-plane. We have then to compute

$$I = \iint_S z^2 dS,$$

where S is the ellipse $x^2 + 2z^2 = 1$. Hence,

$$I = 4\int_0^{\frac{1}{\sqrt{2}}} z^2 dz \int_0^{\sqrt{1-2z^2}} dx = \frac{\pi}{8\sqrt{2}}.$$

Example 11.4.2 *Verify Stokes's theorem for the integral* $\oint_\Gamma ydx + zdy + xdz$, *where* Γ *is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and* Γ *is the boundary.*

Solution: From Stokes's theorem,

$$\oint_\Gamma Pdx + Qdy + Rdz = \iint_\Sigma ((R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma) d\Sigma$$

Here

$P = y; Q = z; R = x$ and $P_2 = 1; P_3 = 0; Q_1 = 0; Q_3 = 1; R_1 = 1; R_2 = 0.$

Evaluation of L.H.S:

Γ is the boundary of the upper half of the given sphere which is clearly a circle $x^2 + y^2 = 1$.

Since Γ lies on $xy-$ plane, $z = 0 \Rightarrow dz = 0$.
We use the parametric representation of the circle $x^2 + y^2 = 1$

$x = \cos\vartheta, t = \sin\vartheta.$ $dx = -\sin\vartheta d\vartheta, dy = \cos\vartheta d\vartheta$ where $0 \le \vartheta \le 2\pi$.

Now

$$\oint_\Gamma y dx + z dy + x dz = \int_0^{2\pi} \sin\vartheta(-\sin\vartheta)d\vartheta$$
$$= -\int_0^{2\pi} \sin^2\vartheta d\vartheta$$
$$= -\pi.$$

Therefore

$$\oint_\Gamma y dx + z dy + x dz = -\pi.$$

Evaluation of RHS:

$$\iint_\Sigma ((R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma) d\Sigma$$
$$= -\iint_\Sigma (\cos\alpha + \cos\beta + \cos\gamma) d\Sigma$$
$$= -\iint_{R_{xy}} (dy\, dz + dz\, dx + dx\, dy)$$
$$= -\int_{R_{xy}} dy dz$$

since the projection is the circle of radius one,

$$= -\text{Area of the circle}$$
$$= -\pi.$$

where R_{xy} is the region which is the projection of the surface on the xy plane.

$$\iint_\Sigma ((R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma) d\Sigma = -\pi$$

Hence Stokes's theorem is verified.

Example 11.4.3 *Evaluate by Stokes's theorem the integral* $\oint_\Gamma x^2z\,dx + xy\,dy$, *where* Γ *is the rectangle in the plane* $z = 0$, *where the sides are along the lines* $x = 0, y = 0, x = a, y = b$.

Solution: Here

$$P = x^2z, \quad Q = xy, \quad R = 0$$

$$P_2 = 0, P_3 = x^3, Q_1 = y, Q_3 = 0, R_1 = 0, R_2 = 0.$$

By Stokes's theorem,

$$\oint_\Gamma x^2z\,dx + xy\,dy = \iint_\Sigma (x^2\cos\beta + y\cos\gamma)\,d\Sigma$$

$$= \iint (x^2\,dz\,dx + y\,dx\,dy)$$

$$= \int_0^b\int_0^a y\,dx\,dy, \quad (\text{since } z = 0)$$

$$= \int_0^a dx \int_0^b y\,dy$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= \frac{b^2}{2}(x)_0^a$$

$$= \frac{ab^2}{2}.$$

Remark 1: If Σ is divisible into a finite number of parts, each of which satisfies the conditions of the theorem or its corollary then even if Σ is more complicated in nature Stokes's theorem remains true.

For example,

Suppose Σ is the part of the unit sphere lying in the first octant, as in the Figure 11.3. It is not possible to apply theorem 11.3.1 or its corollary directly.

But if we take the equation of Σ as $z = (1 - x^2 - y^2)^{\frac{1}{2}}$, $f(x, y)$ is single-valued but $f \notin C'$ along the unit circle of the xy-plane.

The figure shows how Σ can be decomposed into three parts.

To apply Theorem 11.3.1. we may project $\Sigma_1, \Sigma_2, \Sigma_3$ on the $xy-$ plane,

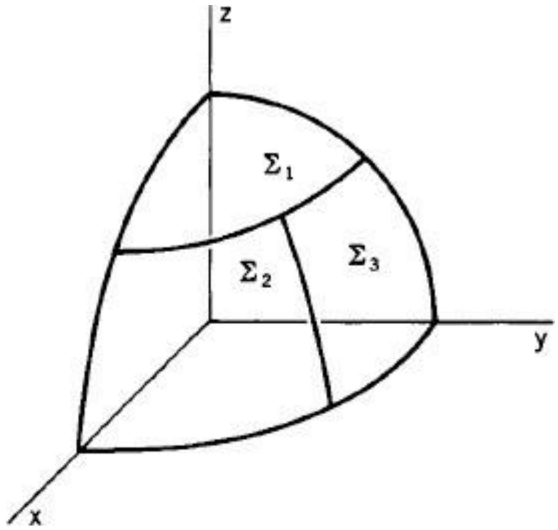

Figure 11.3

y⍰$-$ plane, ⍰$x-$ plane respectively.

It is noted that the line integrals over the auxiliary division lines cancel each other, being executed in opposite direction.

Thus Stokes's theorem is valid for the original surface Σ bounded by the three circular arcs.

Remark 2: There are surfaces for which Stokes's theorem is not applicable to the original uncut surface even if the theorem is applicable to its subdivisions.

This is the ”one-sided” surface. A sample of such a surface can be made by joining together the opposite far edges of a long strip of paper after a half twist has been made in the paper.

In the figure 11.4 the surface has been decomposed into two parts by introducing two cuts. We can apply Stokes's theorem to each cuts. But now the two line integrals over one of the cuts do not cancel each other. Moreover the single boundary of the surface is not described in the same sense over all its parts.

Hence Stokes's theorem is not applicable to the original uncut surface.

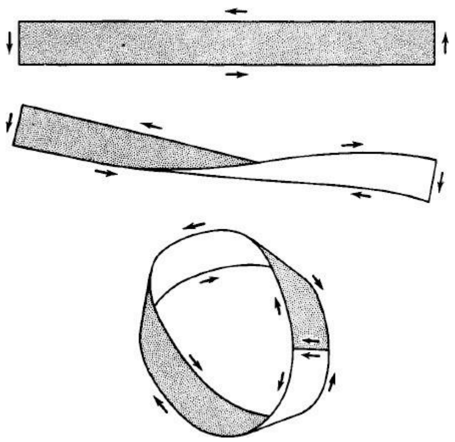

Figure 11.4

11.4.1 Vector Considerations

Both Green's theorem and Stokes's theorem take a particularly elegant form if vector notation is used.

Stokes's Theorem

Let $\vec{v} = \vec{v}(x_1, x_2, x_3)$ be a vector function, defining a vector field.

Suppose $(x_1, x_2, x_3) = (P, Q, R)$.

Let $\vec{\zeta} = (\cos\alpha, \cos\beta, \cos\gamma)$ be the unit vector along the exterior normal to the surface Σ^* of Stokes's theorem .

$$\vec{v}.\vec{\nabla} \times \vec{v} = (R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma$$

$$\vec{v}.d\vec{x} = P\,dx + Q\,dy + R\,dz$$

Then we have,

$$\int_\Gamma \vec{v}.d\vec{x} = \iint_\Sigma \vec{\zeta}.Curl\,\vec{v}\; d\Sigma \qquad (11.12)$$

where $\vec{x}$ is the unit tangent vector to Γ in the sense of the direction of integration.

Definition 11.4.1 *If $Curl\,\vec{v} \equiv 0$, the field is called irrotational.*

Work

As another interpretation we may think of $\vec{v}$ as defining a force field. Then the line integral (11.12) is the work performed by the field on a unit particle as it describes Γ in the sense of integration. If $Curl\ \vec{v} \equiv 0$ this work is zero for every closed curve and the field is called *conservative*. The integral over part of the path may be positive (when the field has done work on the particle) and negative over the rest (when the particle has done an equal amount of work on the field); thus total energy is conserved.

Summary

- The transformation

$$x = g(u, v)$$
$$y = h(u, v)$$

  where g and h are two single valued functions defined on the region R_{uv}, establishes a one-to-one correspondence between the points of the two regions R_{uv} and R_{xy}.

- Formula connecting the areas of the two regions R_{uv} and R_{xy}

  1. $F(x, y) \in C$ in R_{xy}

  2. $g(u, v), h(u, v) \in C^2$ in R_{uv}

  3. $\dfrac{\partial(g, h)}{\partial(u, v)} \neq 0$ in R_{uv}

  4. R_{xy} and R_{uv} correspond in a one -to- one fashion under transformation $x = g(u, v)$, $y = h(u, v)$ and both are regular, simply connected region S

$$\Rightarrow \iint_{R_{xy}} F(x, y)dS_{xy} = \iint_{R_{uv}} F(g(u, v), h(u, v)) \left|\frac{\partial (g, h)}{\partial(u, v)}\right| dS_{uv}$$

- The area of the surface of revolution $x = u\cos\vartheta, y = u\sin\vartheta,$ $z = f(u), a \le u \le b, 0 \le \vartheta \le 2\pi$ is $2\pi \int_a^b u \sqrt{1 + |f'(u)|^2}\, du$

- **Stokes's theorem :** If

  1. $f(x, y) \in C^2$

  2. Σ is the surface $z = f(x, y)$ bounded by the regular closed curve Γ

3. $P(x, y, z), Q(x, y, z), R(x, y, z) \in C^1$ on Σ

4. α, β, γ are direction angles of a directed normal to Σ

$$\Rightarrow \oint_\Gamma P dx + Q dy + R dz$$

$$= \iint_\Sigma [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]\, d\Sigma$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

- If Σ is divisible into a finite number of parts, each of which satisfies the conditions of the theorem or its corollary then even if Σ is more complicated in nature Stokes's theorem remains true.

- There are surfaces for which Stokes's theorem is not applicable to the original uncut surface even if the theorem is applicable to its subdivisions.

- **Vector consideration of Stokes's theorem**

  Let $\vec{v} = \vec{v}(x_1, x_2, x_3)$ be a vector function, defining a vector field.

  Suppose $(x_1, x_2, x_3) = (P, Q, R)$.

  Let $\vec{\zeta} = (\cos\alpha, \cos\beta, \cos\gamma)$ be the unit vector along the exterior normal to the surface Σ^* of Stokes's theorem .

$$\vec{v}.\vec{\nabla} \times \vec{v} = (R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma$$

$$\vec{v}.d\vec{x} = P dx + Q dy + R dz$$

  Then we have,

$$\oint_\Gamma \vec{v}.d\vec{x} = \iint_\Sigma \vec{\zeta}.Curl\,\vec{v}\, d\Sigma$$

  where $\vec{x}$ is the unit tangent vector to Γ in the sense of the direction of integration.

- If $Curl\,\vec{v} \equiv 0$, the field is called irrotational.

Multiple Choice questions

1. If $x = r\cos\vartheta$, $y = r\sin\vartheta$, then find $\dfrac{\partial(x, y)}{\partial(r, \vartheta)}$.

a) r^2 b) r c) 0

2. Choose the correct statement

a) Stokes's theorem is applicable to all surfaces.

b) There are surfaces for which Stokes's theorem is not applicable.

c) We cannot apply Stokes's theorem to surfaces if Σ is divisible into a finite number of parts, each of which satisfies the conditions of the theorem.

3. Choose the wrong answer:

A regular curve Γ with parametric equations $x = \varphi(t)$, $y = \psi(t)$, $z = \omega(t)$, $a \le t \le b$,

(a) has double points

(b) if the interval (a, b) can be divided into a finite number of sub-interval in each of which $\varphi(t) \in C^1$, $\psi(t) \in C^1$, $\omega(t) \in C^1$

4. The area of surface of revolution $x = u\cos\vartheta$, $y = u\sin\vartheta$, $z = f(u)$, $a \le u \le b$, $0 \le \vartheta \le 2\pi$

(a) $2\pi \int_a^b \sqrt{1 + (f'(u))^2}\, du$

(b) $2\pi \int_a^b u \sqrt{(f'(u))^2}\, du$

(c) $2\pi \int_a^b u \sqrt{1 + (f'(u))^2}\, du$

5. In spherical coordinates what is the value of $\dfrac{\partial(x, y, z)}{\partial(r, \vartheta, \varphi)}$

a) $r^2 \sin\vartheta$ b) r^2 c) $r^2\cos\vartheta$

Ans: 1. a) 2. b) 3. a) 4. c) 5. a)

Exercises 11

1. If $F(x, y) \in C$ in R_{xy}, $g(u, v), h(u, v) \in C^2$ in R_{uv}, $\dfrac{\partial(g, h)}{\partial(u, v)} \neq 0$ in R_{uv}, R_{xy} and R_{uv} correspond in a one -to- one fashion under transformation $x = g(u, v)$, $y = h(u, v)$, and both are regular, simply connected region S, then prove that

$$\iint_{R_{xy}} F(x, y)\,dS_{xy} = \iint_{R_{uv}} F(g(u, v), h(u, v)) \left|\frac{\partial (g, h)}{\partial(u, v)}\right| dS_{uv}$$

2. Evaluate $\iint_{R_{xy}} (x + y)^3 dS_{xy}$ where R_{xy} is the parallelogram shown in the figure 11.5. The sides of R_{xy} are straight lines having equations

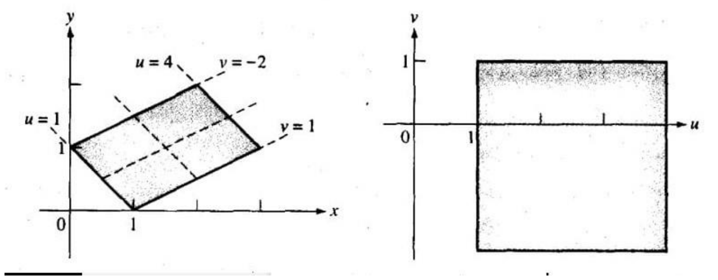

Figure 11.5

of form $x + y = c_1$, $x - 2y = c_2$ for appropriate choices of c_1, c_2.

3. Use the result of example 11.2.7 to find the area of sphere.

4. Compute $\oint_{\Gamma} xdx + xydy + xydz$, where Γ is the curve $x = \cos t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq 2\pi$.

5. State and prove Stokes's theorem.

6. If

   (a) Σ is a surface bounded by the regular closed curve Γ

   (b) Σ has the three equations $z = f(x, y)$, $x = g(y, z)$, $y = h(z, x)$ with $f, g, h \in C^1$

   (c) $P(x, y, z), Q(x, y, z), R(x, y, z) \in C^1$ on Σ

(d) α, β, γ are direction angles of a directed normal to Σ

then prove that

$$\oint_{\Gamma} Pdx + Qdy + Rdz$$
$$= \iint_{\Sigma} [(R_2 - Q_3)\cos\alpha + (P_3 - R_1)\cos\beta + (Q_1 - P_2)\cos\gamma]\, d\Sigma$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

7. Verify Stokes's theorem for the integral $\oint_{\Gamma} (2x - y)dx - yz^2dy - y^2zdz$, where Γ is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

8. Compute $\oint_{\Gamma} xdx + 2dy - ydz$ over the curve $\Gamma : x = 1 + \cos\vartheta, y = \sin\vartheta, z = 14 - 2\cos\vartheta$ where ϑ increasing from 0 to 2π. Then by the Stokes's theorem express the integral as a surface integral over $\Sigma : x^2 + y^2 + z = 16$. Evaluate the surface integral by projection on the xy- plane.

9. Evaluate by Stokes's theorem $\oint_{\Gamma} e^x dx + 2ydy - dz$ where Γ is the curve $x^2 + y^2 = 4; z = 2$.