MASTER OF SCIENCES

(M.Sc.)

MMT-103

REAL ANALYSIS

Semester-I

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COURSE TITLE : REAL ANALYSIS

COURSE CODE : MMT-103

COURSE CREDIT : 4

COURSE OBJECTIVES

While studying the **REAL ANALYSIS**, the Learner shall be able to:

CO 1: Discuss the concepts of compactness and its properties.

CO2:Review about the concept of term by term differentiation for uniform convergence.

CO 3: Represent essential supremum of measurable functions.

CO 4: Discuss the derivation of Lebesgue's monotone convergence theorem.

CO 5: Describe the application of Radon-Nikodym theorem.

COURSE LEARNING OUTCOMES

After completion of the **REAL ANALYSIS**, the Learner will be able to:

CLO1: Interpret the difference between monotonically increasing and monotonically decreasing.

CLO2: Enable to distinguish between uniformly pointwise bounded sequence of functions and pointwise bounded sequence of functions.

CLO 3: Enable to explain the concept of measure space and its properties.

CLO 4: Demonstrate an understanding of the treatment of Integration in the sense of both Riemann and Lebesgue.

CLO5:Represent the methods of Decomposing signed measures which has applications in probability theory and Functional Analysis.

BLOCK I:CONTINUITY AND RIEMANN - STIELTJES INTEGRAL

Limit – Continuity - Connectedness and Compactness - Definition and existence of the integral - Properties of the integral - Integration and Differentiation.

BLOCK II:SEQUENCES AND SERIES OF FUNCTIONS

Pointwise convergence - Uniform convergence - Uniform convergence and continuity -Uniform convergence and Integration, Uniform Convergence and differentiation. Equi – continuous families of functions, Weierstrass and Stone-Weierstrass theorem.

BLOCK III:MEASURE AND MEASURABLE SETS

Lebesgue Outer Measure - Measurable Sets - Regularity - Measurable Functions - Abstract Measure - Outer Measure - Extension of a Measure - Measure Spaces.

BLOCK IV:LEBESGUE INTEGRAL

Integrals of simple functions - Integrals of Non Negative Functions - Fatou's Lemma, Lebesgue monotone convergence Theorem - The General Integral - Riemann and Lebesgue Integrals - Integration with respect to a general measure - Lebesgue Dominated Convergence Theorem.

BLOCK V:LEBESGUE DECOMPOSITION

Signed measures and Hahn Decomposition - Radon-Nikodym Theorem and Lebesgue Decomposition Theorem - Riez Representation Theorem for L1 and Lp .

REFERENCE BOOKS :

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.

2. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, SecondEdition, 2013.

3. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

4. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt.Ltd., Second Edition, 2007.

5. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Web Resource:

Contents

Block-I

Block-I

UNIT-1

BASIC CONCEPTS

Overview

In this unit, we will illustrate the basic concepts of countable sets, compact sets and connected sets. Also, we concentrate the concepts of convergent sequence and convergent series.

Objectives

After successful completion of this lesson, students will be able to

- understand the concept of countable sets and uncountable sets.
- classify and explain the different types of functions.
- define metric spaces with an appropriate example.
- understand the concept of limit point, closed, neighborhood, dense set.
- understand the concept of compact sets and connected sets.
- understand the concept of convergent sequence and divergent sequence.

1.1. Introduction:

In this chapter, we shall recall some basic concepts which we were studied in the lower classes.

1.2. Finite, Countable and Uncountable Sets:

Definition 1.1. Consider any two sets A and B, whose elements may be any different objects. Then a rule or correspondence, which associates each element of A to a unique element of B, is called a *function* from a set A to set B, which we denote by $f(x)$. The set A is called the *domain of f* (we can also f is defined on A), and the elements of $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

Different type of functions:

(i) *Onto function:* Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, then $f(E)$ is defined to be the set of all elements of $f(x)$ for $x \in E$. Simply, we call $f(E)$ is the image of E under f. In this way, we can say that $f(A)$ is the range of f. It is very clear that $f(A) \subset B$. If $f(A) = B$ (*i.e.*, range of $f = B$), then we say that f maps A *onto* B.

(ii) If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the inverse image of E under f.

(i) *One-One function:* If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$, then f is said to be $1-1$ (*one-to-one*) mapping of A into B.

This may also be expressed as, f is a $1-1$ mapping from set A into set B provided that $f(a) \neq f(b)$ whenever $a \neq b$, a, $b \in A$.

In other words, A function f from a set A into B. *i.e.*, $f : A \to B$ is said to be *one-to-one* (or) *injective* if and only if distinct elements of A have distinct images in B.

Definition 1.2. Two sets A and B are said to be *equivalent* (or have the same cardinal number), if there exists a $1-1$ mapping of A *onto* B and symbolically, we write $A \sim B$.

Note 1.1. The relation $\sim$ is an equivalance relation.

Notation: For any positive integer n,
let $J_n = \{1, 2, 3, \ldots, n\}$, a set containing n elements.

$J = \{1, 2, 3, \ldots\}$ the set of all positive integers.

Definition 1.3. For any set A, we say

(a) A is finite, if $A \sim J_n$ for some n (the empty set is also considered to be finite).

(b) A is infinite, if A is not finite.

(c) A is countable, if $A \sim J$.

(d) A is uncountable if A is neither countable nor finite.

(e) A is at most countable if A is finite or countable.

Countable sets are sometimes called enumerable or denumberable.

Remark 1.1. Two finite sets A and B are said to be equivalent if and only if they have the same number of elements.

For infinite sets, however the idea of **having the same number of elements** becomes quite vague, whereas the concept of $1-1$ correspondence retains its clarity.

Some important results related to countable or uncountable sets are listed below:

☞ The set of integers $\mathbb{Z}$ is countable.

☞ Any infinite subset of countable set is countable.

☞ The set of all rational numbers is countable.

☞ The set of all real numbers is uncountable.

☞ Countable Union of countable sets is countable.

1.3. Metric Spaces:

Definition 1.4. A set X whose elements are called *points*, is said to be *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the distance function from p to q, such that

a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;

b) $d(p,q) = d(q,p)$

c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $p,q,r \in X$.

Note 1.2. Any function with these three properties is called a *distance function*

Example: Euclidean space R^k (especially R^1 -real line; R^2 - complex plane) together with the distance function is defined by $d(x,y) = |x-y|$ $(x,y \in R^k)$ is a metric space.

Remark 1.2. It is very important to observe that every subset Y of a metric space X is a metric spaces in its own right with the same distance function.

Thus, every subset of a euclidean space is a metric space.

Definition 1.5.

(i) $(a, b) = \{x|\ a < x < b\}$ is called an *segment*,

(ii) $[a, b] = \{x|\ a \leq x \leq b\}$ is called the *interval*,

(iii) $[a, b) = \{x|\ a \leq x < b\}$ is called the *Half open intervals*

(iv) $(a, b] = \{x|\ a < x \leq b\}$ is also called the *Half open intervals*

Definition 1.6.

(a) k -cell:

If $a_i < b_i$, $i = 1, , 2, \ldots, k$ and $a_i, b_i \in R^1$, then the set of points
$\{x \in R^k : x = (x_1, x_2, \ldots, x_k)\ ,\ a_i \leq x_i \leq b_i,\ i = 1, 2, 3, \ldots, k\}$ is called a *k-cell.*

Note 1.3. $1-$ cell is an *interval* in R^1 ; 2 -cell is a *rectangle* in R^2 ; 3 -cell is a *cuboid* in R^3.

(b) Open ball:

If $x \in R^k$ and $r > 0$, the *open ball* B with centre at x and radius r is defined by $\{y \in R^k :\ |y - x| < r\}$.

(c) Closed ball:

If $x \in R^k$ and $r > 0$, the *open ball* B with centre at x and radius r is defined by $\{y \in R^k :\ |y - x| \leq r\}$.

Definition 1.7. A subset $E \subset R^k$ is said to be *convex* if $\lambda x + (1 - \lambda) y \in E$, whenever $x, y, \in E$ and $0 < \lambda < 1$.

Example of convex sets:

Open balls, closed balls and k -cells are convex sets in R^k.

Definition 1.8. Let X be a metric space. All points and sets mentioned below are understood to be an elements and subsets of X.

(1) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the radius of $N_r(p)$.

(2) A point p is a *limit point* of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(3) If $p \in E$ and p is not a limit point of E, then p is called an *interior point* of E.

(4) E is *closed* if every limit point of E is a point of E.

(5) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.

(6) E is *open* if every point of E is an interior point of E.

(7) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(8) E is *perfect* if E is closed and if every point of E is a limit point of E.

(9) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(10) E is dense in X if every point of X is a limit point of E or a point of E (or both).

(11) If E' denotes the set of all limit points of E in X, then the closure of E is the set $\overline{E} = E \cup E'$.

Some important results about open sets, closed sets and neighborhood are given below:

☞ Every neighborhood is an open set.

☞ If p is a limit point of a set E, then every neighborhood contains infinitely many points of E.

☞ A finite point set has not limit points.

☞ A set E is open if and only if its complement E^c is closed.

☞ For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

☞ For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

☞ For any finite collection $G_1, G_2, \ldots, G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open.

☞ For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed.

☞ A set E is closed if and only if $E = \overline{E}$.

Definition 1.9. Let X be a metric space and $E \subset Y \subset X$ is said to be *open relative* to Y if to each $p \in E$, there is an associated $r > 0$ such that $q \in E$ whenever $d(p,q) < r$ and $q \in Y$ *i.e.*, $\{q \in Y : d(p,q) \subset E\}$.

Remark 1.3. Suppose $E \subset Y \subset X$ and X be a metric space. E is open relative to Y if $E \subset Y \cap G$ for some open subset G of X.

1.4. Compact sets and Connected sets:

Definition 1.10. Let X be a metric space. A collection of open sets $\{G_\alpha\}$ of X is called an *open cover* of E, if $E \subset \bigcup_\alpha G_\alpha$.

Definition 1.11. A subset K of a metric space X is said to be *compact*, if every open cover of K contains a finite sub cover. More explicitly, if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that
$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup G_{\alpha_3} \cup \ldots \cup G_{\alpha_n}$

Remark 1.4. Every finite set is compact.

Some important results related to compact sets are given below:

- ☞ Compact subsets of metric spaces are closed.
- ☞ Closed subsets of a compact sets are compact.
- ☞ If F is closed and K is compact, then $F \cap K$ is compact.
- ☞ If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub collection of $\{K_\alpha\}$ is non-empty, then $\bigcap K_\alpha$ is non empty.
- ☞ Any infinite subset of a compact set K has a limit point.
- ☞ Every k-cell in R^k is compact.

Now, let us see two important theorem in compact spaces without proof.

Theorem 1.1 (Heine-Borel Theorem)**.**

If E is a subset of R^k, then the following one equivalent.

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Theorem 1.2 (Weierstrass Theorem)**.**

Every bounded infinite subset of R^k has a limit point in R^k.

Definition 1.12. Two subsets A and B of a metric space X are said to be separated, if both $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$

Definition 1.13. A set $E \subset X$ is said to be connected if E is not a union of two non-empty separated sets.

Note 1.4. A subset E of the real line R^1 is connected if it has the following property if $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

1.5. Sequences

1.5.1. Convergent Sequences:

Definition 1.14. A sequence $\{p_n\}$ in a metric space X is a function of f from J into X. If $f(n) = p_n$, we represent this function by its image; $p_1, p_2, p_3, \ldots,$ or simply $\{p_n\}$.

Example:

(a) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ is a sequence in R^1.

(b) $-1, \frac{1}{2}, -\frac{1}{3}, \ldots, \frac{(-1)^n}{n}, \ldots$ is also a sequence in R^1.

(c) $1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$ is also a sequence in R^1.

Definition 1.15. A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property:

For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$ (Here d denotes the distance function).

In this case, we say that $\{p_n\}$ converges to p or that p is the limit point of $\{p_n\}$.

Symbolically, we can write $\lim_{n\to\infty} p_n = p$ (or) $p_n \to p$.

If $\{p_n\}$ does not converge, it is said to be *diverge*.

Definition 1.16. If $\{p_n\}$ is a sequence, then the set of points p_n is called the *range* of the sequence $\{p_n\}$. The range may be finite or infinite.

The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

Now, we shall list out some important results on sequence below:

☞ If limit of a sequence exists, then it is unique.

☞ Let $\{p_n\}$ be a sequence in a metric space X, then $\{p_n\}$ converges to $p \in X$ if every neighborhood of p contains all but finitely many points.

☞ If $E \subset X$ and if p is a limit point of E, then there exists a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

☞ Every convergent sequence is a bounded sequence.

☞ Suppose $\{s_n\}$ and $\{t_n\}$ are complex sequence, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} cs_n = cs$

(c) $\lim_{n\to\infty} s_n t_n = st$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ provided $s_n \neq 0$ and $s \neq 0$.

1.5.2. Subsequences:

Definition 1.17. Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \dots$ Then a sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

Important Results on Subsequences:

★ If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

★ Every bounded sequence in R^k contains a convergent subsequence.

1.5.3. Cauchy Sequences:

Definition 1.18. A sequence $\{p_n\}$ in a metric space X is called a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Definition 1.19. Let $E \subset X$ be a subset of a metric space X, then the diameter of a subset E is defined by $\text{diam}E = sup\{d(p,q) : p, q \in E\}$.

Remark 1.5.

(a) If $\{p_n\}$ is a sequence in X and if E_N consists of the points $P_N, P_{N+1}, \dots$. Then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N\to\infty} \text{diam } E_N = 0$.

(b) Every Cauchy sequence in a metric space is bounded.

Definition 1.20. A metric space in which every Cauchy sequence converges is said to be *complete*.

Definition 1.21. A sequence $\{s_n\}$ of real numbers is said to be

(a) *monotonically increasing* if $s_n \leq s_{n+1}$ $(n = 1, 2, 3, \ldots)$

(b) *monotonically decreasing* if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \ldots)$

Remark 1.6. Every monotonic sequence $\{s_n\}$ converges if and only if it is bounded.

Definition 1.22. Let $\{s_n\}$ be a sequence of real numbers with the property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We write $s_n \to +\infty$.

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$. We write $s_n \to -\infty$.

Definition 1.23. Let $\{s_n\}$ be a sequence in R^1. Let E be the set of all subsequential limits of $\{s_n\}$ plus possibly the numbers $+\infty$, $-\infty$.

The numbers $s^* = \sup E$ and $s_* = \inf E$ are called the *upper limits* and the *lower limits* of $\{s_n\}$ respectively and written as

$$\limsup_{n\to\infty} s_n = s^* \quad \text{and} \quad \liminf_{n\to\infty} s_n = s_*$$

Remark 1.7.

★ Let $\{s_n\}$ be a sequence of real numbers. Let E and $*$ have the same meaning as in the above definition. Then s^* have the following two properties:

(a) $s^* \in E$

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with the above two properties.

The same result is true for s_* also.

★ If $\{s_n\} \leq \{t_n\}$ for $n \geq N$ is fixed then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

1.6. Series

Definition 1.24. Given a sequence $\{a_n\}$, we associate a sequence $\{s_n\}$ where $s_n = a_1 + a_2 + \ldots$ (or) $\displaystyle\sum_{n=1}^{\infty} a_n$.

Remark 1.8.

(i) The symbol $\sum_{n=1}^{\infty} a_n$ is called an *infinite series* or just a series.

(ii) The numbers s_n are called the partial sums of the series.

(iii) If s_n converges to s, we say that the series $\sum_{n=1}^{\infty} a_n$ converges and write

$$\sum_{n=1}^{\infty} a_n = s.$$

(iv) The number s is called the sum of the series.

(v) If s_n diverges, then the series is said to be diverge.

Cauchy Criterion of Convergence of Series:

$\sum_{n=1}^{\infty} a_n$ converges if for every $\epsilon > 0$ there is an integer N such that,

$$\text{if } m \geq n \geq N, \quad \left|\sum_{k=n}^{m} a_k\right| < \epsilon$$

Remark 1.9.

★ If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

★ If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

★ A series of non-negative terms converges if and only if its partial sum forms a bounded sequence.

Definition 1.25. Given a series $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a *power series*, the number c_n are called the *coefficients* of the series; z is a complex number.

Let us Sum up:

In this unit, the students acquired knowledge to

- countable and uncountable sets.
- metric spaces, convex sets, open balls and closed balls.
- Compact sets and Connected sets with their properties.

- basic concepts of Sequences, Convergent Sequences, Cauchy Sequences and Series.

Choose the correct or more suitable answer:

1. Let A and B be any two sets and $f(A) = B$, then we say that f maps A ...... B.
   (a) into
   (b) one-one function
   (c) onto
   (d) many to one.
2. The set of integers is
   (a) finitely countable
   (b) infinitely countable
   (c) infinitely uncountable
   (d) none of these.
3. E is ...... if every limit point of E is a point of E.
   (a) Open
   (b) Half open
   (c) closed
   (d) none of these.
4. Closed subsets of a ...... are compact.
   (a) connected sets
   (b) compact sets
   (c) closed sets
   (d) open sets.
5. A sequence of real number $\{s_n\}$ is said to be monotonically decreasing if ....
   (a) $s_n \leq s_{n+1}$ (b) $s_n < s_{n+1}$ (c) $s_n \geq s_{n+1}$ (d) $s_n > s_{n+1}$.

Answer:

(1) c (2) b (3) c (4) b (5) c

Glossaries:

1. Line Segment: It is a straight line has two endpoints, one at a beginning and other at an end.
2. Closed sets: It is a set which contains all its limit points.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.
2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-I

UNIT-2

LIMITS AND CONTINUITY-I

Structure

Objective

Overview

Overview

In this unit, we will explain the concepts of a limit of a function and continuity of a function. Further, we studied more detailed about the concepts of Continuity and Compactness.

Objectives

After completion of this unit, students will be able to

- ★ understand the concept of limits and continuity and also identity whether the given function is continuous or not at a point.
- ★ understand the concept of components of continuous vector functions are continuous.
- ★ understand the concept of uniform continuity and also they identify the difference between continuity and uniform continuity.
- ★ explain the concept of compactness and its properties.

2.1. Limits of functions:

Definition 2.1. Let X and Y be metric spaces, Suppose $E \subset X$, f maps E into Y and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, (or) $\lim_{x \to p} f(x) = q$, if there is a point $q \in Y$ with the following property:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon \tag{2.1}$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta \tag{2.2}$$

Remark 2.1.

(1) The symbols d_X and d_Y refer to the distances in metric spaces X and Y respectively.

If X and/or Y are replaced by the real line, the complex plane, or by the euclidean space R^k, the distances d_X, d_Y are replaced by absolute values or by norms of differences.

(2) In the above definition, we observed that $p \in X$, but that p need not be a point of E. Moreover, even if $p \in E$, $\lim_{x \to p} f(x) \neq f(p)$.

Theorem 2.1. *Let X and Y be a metric spaces, Suppose $E \subset X$, Suppose f maps E*

into Y and p is a limit point of E. Then

$$\lim_{x\to p} f(x) = q \tag{2.3}$$

if and only if
$$\lim_{n\to\infty} f(p_n) = q \tag{2.4}$$

for every sequence $\{p_n\}$ in E such that
$$p_n \neq p, \quad \lim_{n\to\infty} p_n = p. \tag{2.5}$$

Proof. Assume that $\lim\limits_{x\to p} f(x) = q$ holds good.

Choose $\{p_n\}$ in E such that $p_n \neq p$, $\lim\limits_{n\to\infty} p_n = p$.

Let $\epsilon > 0$ be given.

$$\begin{aligned}
\lim_{x\to p} f(x) &= q \\
&\Rightarrow \text{ there exists a } \delta > 0 \text{ such that } d_Y(f(x), q) < \epsilon, \text{ if} x \in E \text{ and} \\
&\quad 0 < d_X(x, p) < \delta \qquad (2.6)\\
p_n &\neq p, \quad \lim_{n\to\infty} p_n = p \\
&\Rightarrow \text{ for given } \delta > 0 \text{ there exist } N \text{ such that } n > N \text{ implies} \\
&\quad 0 < d_X(p_n, p) < \delta \qquad (2.7)\\
\text{For } n > N &\Rightarrow 0 < d_X(p_n, p) < \delta \quad \text{(from (2.7))} \\
&\Rightarrow d_Y(f(p_n), q) < \epsilon \quad \text{(from (2.6))} \\
&\Rightarrow \lim_{n\to\infty} f(p_n) = q \qquad (2.8)
\end{aligned}$$

Converse Part:

Given: $\lim\limits_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim\limits_{n\to\infty} p_n = p$.

To Prove: $\lim\limits_{x\to p} f(x) = q$.

Assume that $\lim\limits_{x\to p} f(x) \neq q$

Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ) for which $d_Y(f(x), q) \geq \epsilon$ but $0 < d_X(x, p) < \delta$.

Taking $\delta_n = \dfrac{1}{n}$ $(n = 1, 2, 3, \dots)$ there exists a sequence in $\{x_n\}$ in E such that $0 < d_X(x_n, p) < \dfrac{1}{n}$ and $d_Y(f(x_n), q) > \epsilon$ which is a contradiction.

Therefore $\lim\limits_{x\to p} f(x) = q$ ■

Corollary 2.1. *If f has a limit at p, then it is unique.*

Proof. Suppose that $\lim\limits_{x\to p} f(x) = q_1$; $\lim\limits_{x\to p} f(x) = q_2$. *i.e.*, f has two different limits.

$\lim\limits_{x\to p} f(x) = q_1 \Rightarrow$ there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$; $\lim\limits_{n\to\infty} f(p_n) = q_1$ (By previous theorem).

Similarly, $\lim\limits_{x\to p} f(x) = q_2 \Rightarrow$ there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$; $\lim\limits_{n\to\infty} f(p_n) = q_2$.

We know that, if the limit of a sequence exists and it is unique.

$\therefore \quad q_1 = q_2.$ ■

Definition 2.2. Let X be a metric space, E be a subset of X and f, g are two complex functions defined on E. Then $f+g, f-g, fg, \frac{f}{g}$ are also defined on E and they are defined on E as follows:

(a) $(f \pm g)(x) = f(x) \pm g(x) \quad \forall x \in E.$

(b) $(f \cdot g)(x) = f(x) \cdot g(x) \quad \forall x \in E.$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$ if $g(x) \neq 0; \quad \forall x \in E.$

(d) $(cf)(x) = cf(x); \quad \forall x \in E$ and c is a constant.

Definition 2.3. Suppose $\mathbf{f}$ and $\mathbf{g}$ map E into R^k, if $x \in E$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}-\mathbf{g}$ and $\lambda\mathbf{f}$ are functions defined on E and they are defined as follows:

(a) $(\mathbf{f}+\mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x) \quad \forall x \in E.$

(b) $(\mathbf{f}-\mathbf{g})(x) = \mathbf{f}(x) - \mathbf{g}(x) \quad \forall x \in E.$

(c) $(\lambda\mathbf{f})(x) = \lambda\mathbf{f}(x) \ \forall x \in E,$ where λ is real.

Theorem 2.2. *Suppose $E \subset X$, a metric space and p is a limit point of E, f and g are complex functions defined on E and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$. Then*

(a) $\lim_{x\to p} (f+g)(x) = A + B.$

(b) $\lim_{x\to p} (f-g)(x) = A - B.$

(c) $\lim_{x\to p} (fg)(x) = AB.$

(d) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, *if* $B \neq 0$

Proof. Assume that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B.$

$\lim_{x\to p} f(x) = A \Rightarrow$ there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$ and $\lim_{n\to\infty} f(p_n) = A.$

Similarly, $\lim_{x\to p} g(x) = B \Rightarrow$ there exists a sequence $\{p_n\}$ in E such that $p_n \neq p$ and $\lim_{n\to\infty} g(p_n) = B.$

(a) $\lim_{n\to\infty} (f+g)(p_n) = \lim_{n\to\infty} [f(p_n) + g(p_n)] = \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n) = A + B.$

(b) $\lim_{n\to\infty} (f-g)(p_n) = \lim_{n\to\infty} [f(p_n) - g(p_n)] = \lim_{n\to\infty} f(p_n) - \lim_{n\to\infty} g(p_n) = A - B.$

(c) $\lim_{n\to\infty} (fg)(p_n) = \lim_{n\to\infty} [f(p_n)g(p_n)] = \lim_{n\to\infty} f(p_n) \lim_{n\to\infty} g(p_n) = AB.$

(d) $\lim_{n\to\infty} \left(\frac{f}{g}\right)(p_n) = \lim_{n\to\infty} \left[\frac{f(p_n)}{g(p_n)}\right] = \frac{\lim_{n\to\infty} f(p_n)}{\lim_{n\to\infty} g(p_n)} = \frac{A}{B}.$ ■

Remark 2.2. If $\mathbf{f}$ and $\mathbf{g}$ maps E into R^k, then (a) and (b) remains good always, but (c) remains true if we can write $\lim_{x\to p} (\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{A} \cdot \mathbf{B}.$

2.2. Continuous functions:

The theory of continuity of a function plays a crucial role in examining the properties of a function. In this section, we are going to discuss the concept of continuity of a function.

Definition 2.4. Suppose X and Y are metric spaces; $E \subset X$, $p \in E$ and f maps E into Y. Then f is said to be *continuous* at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E then f is said to be *continuous* on E.

Theorem 2.3. *If p is a limit point of E, then f is continuous at p if and only if* $\lim_{x \to p} f(x) = f(p)$.

Proof.

$$\begin{aligned} f \text{ is continuous} \quad &\Leftrightarrow \quad \text{for every } \epsilon > 0 \text{ there exists a } \delta > 0 \\ &\quad \text{such that } d_Y(f(x), f(p)) < \epsilon \text{ whenever } x \in E,\ d_X(x,p) < \delta \\ &\Leftrightarrow \quad \lim_{x \to p} f(x) = f(p) \end{aligned}$$

■

Definition 2.5. Suppose X, Y, Z are metric space, $E \subset X$, f maps E into Y and g maps $f(E)$ into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x))$, $x \in E$. Then h is called the *composition* of f and g.

Remark 2.3. The function h is called the *composition* or *composite* of f and g and it is denoted by $h = g \circ f$.

Theorem 2.4. *Suppose X, Y, Z are metric space, $E \subset X$, f maps E into Y and g maps $f(E)$ into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x))$, $x \in E$. If f is continuous at a point $p \in E$ and if g is continuous at $f(p)$, then the composite function h is continuous at $g(f(p))$.*

Proof. Given that f is continuous at p and g is continuous at $f(a)$.

To Prove: h is continuous at p.

Let $\epsilon > 0$ be given.

$$\begin{aligned} g \text{ is continuous at } f(p) \quad &\Rightarrow \quad \text{there exists } \eta > 0 \text{ such that } d_z(g(y), g(f(p))) < \epsilon \\ &\quad \text{whenever } y \in f(E),\ d_Y(y, f(p)) < \eta \qquad (2.9) \\ f \text{ is continuous at } p \quad &\Rightarrow \quad \text{for given } \eta > 0 \text{ there exists } \delta > 0 \text{ such that} \\ &\quad d_Y(f(x), f(p)) < \eta \\ &\quad \text{whenever } x \in f(E),\ d_X(x,p) < \delta \qquad (2.10) \end{aligned}$$

From (2.9) and (2.10), we get

$$\begin{aligned} \text{For } x \in E \text{ and } d_X(x,p) < \delta \quad &\Rightarrow \quad d_Y(f(x), f(p)) < \eta \text{ and hence} \\ &\qquad d_z(g(f(x)), g(f(p))) = d_z(h(x), h(p)) < \epsilon \end{aligned}$$

which shows that $h(x)$ is continuous at p.

This completes the proof of the theorem. ■

Remark 2.4. The above theorem can also be state as *the composite of two continuous function is also continuous.*

Next, we shall discuss very useful characterization of continuity.

Theorem 2.5. *A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y. In other words, f is continuous if and only if inverse image of an open set is open.*

Proof. Given: Assume that f is continuous on X and V is open in Y.

To Prove: $f^{-1}(V)$ is open in X, *i.e.*, it is enough to prove that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

For this, Let $p \in X \Rightarrow f(p) \in f(V)$

Since V is open, then there exists a $\delta > 0$ such that $y \in V$ if $d_Y(f(p), y) < \epsilon$.

Also, given that f is continuous at p, then there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$.

$$\begin{aligned} \text{If } d_X(x,p) < \delta \quad &\Rightarrow \quad d_Y(f(x), f(p)) < \epsilon \\ &\Rightarrow \quad y \in V \\ &\Rightarrow \quad f(x) \in V \\ &\Rightarrow \quad x \in f^{-1}(V) \end{aligned}$$

That is $N_\delta(p) \subset f^{-1}(V)$ and p is an interior point. Since p is an arbitrary point, thus every point is an interior point. Therefore $f^{-1}(V)$ is open.

Converse part: Assume that $f^{-1}(V)$ is open in X for every open set V in Y.

Fix $p \in X$ and $\epsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \epsilon$.

Clearly, V is open and hence $f^{-1}(V)$ is open (by given condition).

Let $x \in f^{-1}(V)$ then there exists a $\delta > 0$ such that $d_X(p, x) < \delta$.

But if $x \in f^{-1}(V) \Rightarrow f(x) \in V$ and thus $d_Y(y, f(p)) < \epsilon$.

i.e., Given $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(y, f(p)) < \epsilon$ if $d_X(p, x) < \delta$.

Thus f is continuous at p and hence f is continuous on X (since p is an arbitrary point) ■

Corollary 2.2. *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

Proof. Assume that C is closed in Y, then C^c is open in Y.

Thus, by previous theorem

$$
\begin{aligned}
f \text{ is continuous} &\Leftrightarrow f^{-1}(C^c) \text{ is open in } X \text{ if } C^c \text{ is open in } Y \\
&\Leftrightarrow \left(f^{-1}(C)\right)^c \text{ is open in } X \text{ if } C^c \text{ is open in } Y \\
&\Leftrightarrow f^{-1}(C) \text{ is closed in } X \text{ if } C \text{ is closed in } Y
\end{aligned}
$$

■

Theorem 2.6. *Let f and g be complex continuous functions on a metric space X, then $f+g, fg$ and f/g are also continuous on X.*

Proof. Assume that f and g are continuous at p, then $\lim_{x\to p} f(x) = f(p)$; and $\lim_{x\to p} g(x) = g(p)$

(a) $\lim_{x\to p}(f+g)(x) = \lim_{x\to p}[f(x)+g(x)] = \lim_{x\to p} f(x) + \lim_{x\to p} g(x) = f(p)+g(p)$
Thus $f+g$ is continuous.

(b) $\lim_{x\to p}(fg)(x) = \lim_{x\to p}[f(x)g(x)] = \lim_{x\to p} f(x) \lim_{x\to p} g(x) = f(p)g(p)$
Thus fg is continuous.

(c) $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \lim_{x\to p}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)} = \frac{f(p)}{g(p)}$ (if $g(x) \neq 0$ for all $x \in X$)
Thus $\frac{f}{g}$ is continuous. ■

Theorem 2.7 (Components of continuous vector functions are continuous)**.**

(a) Let $f_1, f_2, \ldots, f_k$ be real functions on a metric space X and let $\mathbf{f}$ be the mapping of X into R^k defined by

$$\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x)) \quad (x \in X);$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, f_2, \ldots, f_k$ is continuous.

(b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous mapping of X into R^k, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ are continuous on X.

Proof. Assume that $\mathbf{f} = (f_1, f_2, \ldots, f_k)$ is continuous and let $x \in X$ and $\epsilon > 0$ be given.

$$\mathbf{f} \text{ is continuous at } x \Leftrightarrow \text{there exists a } \delta > 0 \text{ such that}$$

$$|\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon \text{ if } d(x,y) < \delta \tag{2.11}$$

Now, if $d(x,y) < \delta$ then for each i, by (2.11), we have

$$|f_i(x) - f_i(y)| < \left(\sum_{i=1}^{n} |f_i(x) - f_i(y)|^2\right)^{1/2} = |\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$$

Thus each f_i $(i = 1, 2, \ldots, k)$ is continuous.

Converse part: Assume that each f_i $(i = 1, 2, \ldots, k)$ is continuous. It remains to prove that **f** is continuous.

For if, let $x \in X$ and $\epsilon > 0$ be given.

since f_i is continuous at x, then there exists a $\delta_i > 0$ such that

$$|f_i(x) - f_i(y)| < \frac{\epsilon}{\sqrt{k}} \text{ if } d(x, y) < \delta_i$$

Choose $\delta = \min\{\delta_i : i = 1, 2, \ldots, k\}$. Then,

$$|f_i(x) - f_i(y)| < \frac{\epsilon}{\sqrt{k}} \text{ if } d(x, y) < \delta$$

$$\text{Thus, } |\mathbf{f}(x) - \mathbf{f}(y)| = \sum_{i=1}^{k} \left[|f_i(x) - f_i(y)|^2\right]^{1/2} < \left(\frac{k\epsilon^2}{k}\right)^{1/2} = \epsilon \text{ if } d(x, y) < \delta$$

Hence **f** is continuous. This completes the proof of the theorem. ■

2.3. Continuity and Compactness:

In this section, let us discuss the properties of continuous functions defined on compact metric space. In this regard, first we shall recall the definition of compact set of a metric space.

Definition 2.6. If X is a metric space and $E \subset X$ is a compact subset if every open cover of E has a finite sub-cover.

Definition 2.7. A mapping **f** of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

Theorem 2.8. *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact. In other words, continuous image of compact space is compact.*

Proof. Given that f is a continuous mapping of a compact metric space X into Y.

Let $\{V_\alpha : \alpha \in I\}$ is an open cover of $f(X)$.

Since f is continuous, then $f^{-1}(V_\alpha)$ is an open set in a compact set X.

Thus, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover of a compact set X. Hence, there are finitely many indices say $\alpha_i \in I$ $(i = 1, 2, 3, \ldots, n)$ such that $X \subset \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$ and hence $f(X) \subset \bigcup_{i=1}^{n} V_{\alpha_i}$

Since $f\left(f^{-1}(E)\right) \subset E$ for every $E \subset Y,$ So $f(X)$ is compact.

This completes the proof of the theorem. ∎

Remark 2.5. Here we used the relation $f\left(f^{-1}(E)\right) \subset E,$ is valid for $E \subset Y.$ If $E \subset X,$ then $f^{-1}(f(E)) \supset E;$ in both the case equality does not holds good.

Theorem 2.9. *If* **f** *is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(X)$ is closed and bounded. Thus* **f** *is bounded.*

Proof. This result proves directly from Heine-Borel theorem (1.1). ∎

The following shows that a real continuous function on a compact metric space attain the bounded.

Theorem 2.10. *Suppose f is a continuous real function on a compact metric space X and*

$$M \quad = \sup_{p \in X} f(p); \quad m = \inf_{p \in X} f(p) \tag{2.12}$$

Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m.$

Proof. Given that f is a continuous function on a compact metric space X and thus $f(X)$ is compact. (since continuous image of a compact set is compact).

Thus $f(X)$ is a compact subset of R^1 and hence by Heine-Borel theorem (1.1), $f(X)$ is closed and bounded. Hence sup and inf value of $f(X)$ exists.

Let $M = \sup_{p \in X} f(p); \quad m = \inf_{p \in X} f(p).$

Also, we know that, if $E \subset R^k$, then $\sup E$ and $\inf E$ are limits points of $E.$ Therefore M and m are the limit points of $f(X)$ and Moreover $f(X)$ is closed. Hence $m \in f(X)$ and $M \in f(X).$ Thus $M = f(p); \quad m = f(q)$ for some $p, q \in X.$ ∎

Remark 2.6. The statement of the above theorem can also be stated as follows:

Suppose f is a continuous real function on a compact metric space $X,$ then there exists points p and q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X.$ *i.e.*, f attains its maximum value at p and minimum value at $q.$

Theorem 2.11. *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space $Y.$ Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) \quad = \quad x \qquad (x \in X)$$

is a continuous mapping of Y onto $X.$

Proof. Given f is a continuous 1-1 mapping of compact metric space X onto a metric space $Y.$ Now, our aim is to prove that f^{-1} is continuous on $Y.$

The inverse mapping f^{-1} is defined by $f^{-1}(f(x)) = x \qquad (x \in X).$

Let V be an open set in X, then V^c is closed subset of a compact metric space X.

But, we know that a closed subset of a compact metric space is compact and thus V^c is compact.

Also, we know that a continuous image of a compact metric space is compact and hence, we get $f(v^c)$ is compact subset of Y.

But, a compact subset is closed and thus, we have $f(V^c)$ is closed subset of Y. Since f is 1-1 and onto, $f(V^c) = f(V)^c$ and hence $f(V)$ is open in Y. Thus f^{-1} is continuous on Y. ∎

Definition 2.8. Let f be a mapping of a metric space X into a metric space Y, We say that f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon \tag{2.13}$$

for all p and q in X for which $d_X(p,q) < \delta$.

Remark 2.7. Let us now discuss the differences between the concept of continuity and uniform continuity.

1. Uniform continuity is a property of a function on a set, where as continuity can be defined at a single point.
2. If f is continuous on X, then it is possible to find, for each given $\epsilon > 0$ and for each point p of X, there exists a δ depends on ϵ and point $p \in X$. But, in the case of uniform continuity on X then it is possible for each given $\epsilon > 0$, to find a $\delta > 0$ which depends only on ϵ and independent of the points.

Note 2.1. It is very clear that, every uniformly continuous function is continuous. Now, we shall see some examples uniformly continuous and not uniformly continuous.

Example 2.1. Consider $f : [a,b] \to R$ defined by $f(x) = \dfrac{c}{x-1}$, where a, b, c are all positive constants with $a > 1$.

Now, we shall prove that f is uniformly continuous on $[a,b]$.

For this, let $\epsilon > 0$ be given, then

$$f(x) - f(y) = \frac{c}{x-1} - \frac{c}{y-1} = \frac{c(y-x)}{(x-1)(y-1)}$$

Since $a > 1 \Rightarrow a = 1 + \eta$ for some $\eta > 0$.

If $x, y \in [a,b]$, then $|x-1| = x-1 \geq \eta$, similarly $|y-1| = y-1 \geq \eta$.

Choose $\delta < \dfrac{\eta^2 \epsilon}{c}$ and thus, if $|x-y| < \delta$ and $x, y \in [a,b]$, then

$$|f(x) - f(y)| = \frac{c|y-x|}{|x-1||y-1|} \leq \frac{c\delta}{\eta^2} < \epsilon$$

Thus, f is uniformly continuous on $[a,b]$.

Example 2.2. Let $f : (0, 1) \to R$ be defined by $f(x) = x^2 + x - 1$. Now, we shall prove that f is uniformly continuous.

For this, let $\epsilon > 0$ be given, then

$$f(x) - f(y) = (x - y)(x + y + 1)$$

If $x, y \in (0, 1)$, then $x + y + 1 < 3$

Choose $\delta < \frac{\epsilon}{3}$ and thus, if $|x - y| < \delta$ and $x, y \in (0, 1)$, then

$$|f(x) - f(y)| = |x - y||x + y + 1| < 3\delta < \epsilon$$

Thus, f is uniformly continuous on $(0, 1)$.

Example 2.3. Let $f : R \to R$ be defined by $f(x) = x^2$.

Fix $\epsilon = 1$ and $\delta > 0$ be given. Take $x_n = n$ and $y_n = n + \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

If $n > \frac{1}{\delta}$, then we have

$$\begin{aligned} |x_n - y_n| &= \frac{1}{n} < \delta, \\ \text{but } \left|x_n^2 - y_n^2\right| &= |(x_n - y_n)(x_n + y_n)| = \frac{1}{n}\left(2n + \frac{1}{n}\right) > 2. \end{aligned}$$

Thus, if $|x_n - y_n| < \delta$, but $|f(x_n) - f(y_n)| > 2 > 1 = \epsilon$.

Hence f is not uniformly continuous on R.

Example 2.4. Let $f : (0, \infty) \to R$ be defined by $f(x) = \frac{1}{x}$. Note that the function f is continuous, Now, we shall prove that the function f is not uniformly continuous.

For this, for given $\epsilon > 0$ and any $\delta > 0$.

Choose $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$ $(n = 1, 2, 3, \ldots)$ so that for $n > \max\left\{\epsilon, \frac{1}{3}\right\}$.

$$\begin{aligned} \text{If } |x_n - y_n| &= \left|\frac{1}{n} - \frac{1}{2n}\right| = \frac{1}{2n} < \delta \\ \text{but } |f(x_n) - f(y_n)| &= |n - 2n| = n > \epsilon \end{aligned}$$

Thus, f is not uniformly continuous.

Remark 2.8. The last two example shows that the function is continuous but not uniformly continuous.

The next theorem asserts that the continuity and uniformly continuity are equivalent on compact sets.

Theorem 2.12. *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Proof. Given that f is a continuous mapping of a compact metric space X into metric space Y.

Let $\epsilon > 0$ be given. Since f is continuous, to each $p \in X$, there exists $\phi(p) > 0$ such that

$$q \in X, \quad d_X(p,q) < \phi(p) \text{ implies } d_Y(f(p), f(q)) < \frac{\epsilon}{2} \tag{2.14}$$

Let $J(p)$ denote the set of all $q \in X$ such that

$$d_X(p,q) < \frac{1}{2}\phi(p) \tag{2.15}$$

For each $p \in X$, $p \in J(p)$ and hence $\{J(p)\}$ is an open cover of X and X is compact, there is a finite set of points $p_1, p_2, \ldots, p_n$ in X such that

$$X \subset J(p_1) \cup J(p_2) \cup \ldots \cup J(p_n) \tag{2.16}$$

$$\text{Choose } \delta = \frac{1}{2}\min\left[\phi(p_1), \phi(p_2), \ldots, \phi(p_n)\right] \tag{2.17}$$

Clearly $\delta > 0$. Now, let q and p be points of X such that $d_X(p,q) < \delta$. By (2.16), there is an integer m, $1 \le m \le n$ such that $p \in J(p_m)$ and hence by (2.15) $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$.

Also, we have

$$\begin{aligned} d_X(p, p_m) &\le d_X(q,p) + d_X(p, p_m) \\ &\le \delta + \frac{1}{2}\phi(p_m) < \frac{1}{2}\phi(p_m) + \frac{1}{2}\phi(p_m) < \phi(p_m). \end{aligned}$$

Thus, from (2.14), we have

$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m) + d_Y(f(q), f(p_m)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$ ■

Example 2.5. If E is not compact, then there exists a continuous function on E which is not uniformly continuous on E.

Proof. Consider the function $f(x) = \frac{1}{x}$, $(0 < x < 1)$ defined on the non-compact set $E = (0,1)$ of R^1. Since, it is not closed and bounded. Clearly, the function f is continuous on $(0,1)$. However, the function f is not uniformly continuous on $(0,1)$.

For this, Let $\epsilon = 1 > 0$ and $\delta > 0$ be any positive real. Then there exists N such that $\frac{1}{N} < \delta$. Choose $x = \frac{1}{4N}$, $y = \frac{1}{2N}$. If $x, y \in (0,1)$, then

$$\begin{aligned} |x - y| &= \left|\frac{1}{4N} - \frac{1}{2N}\right| = \frac{1}{2N} < \frac{1}{N} < \delta \\ \text{But } |f(x) - f(y)| &= |4N - 2N| = 2N \ge 1 > \epsilon. \end{aligned}$$

Thus, f is not uniformly continuous. ■

Example 2.6. If E is not compact, then there exists a continuity function on E which is not bounded.

Proof. Consider the real continuous function $f(x) = \frac{1}{x}$ $(0 < x < 1)$ defined on non-compact subset of R^1. Since, it is unbounded. Now, we shall prove that f is not bounded.

For this, if $N > 0$ there exists $x_0 = \frac{1}{N+1}$ such that $f(x_0) = \frac{1}{x_0} = N + 1 > N$. Hence f is not bounded. This completes the proof. ∎

Example 2.7. If E is not compact, then there exists a continuous function on E which is bounded but it has no maximum.

Proof. Consider the real continuous function $g(x) = \frac{1}{1+x^2}$ $(0 < x < 1)$ defined on non-compact subset of R^1. Since $0 < g(x) < 1$, g is bounded and $\sup g(x) = 1$. But there is no $x \in (0, 1)$ such that $g(x) = 1$. Hence g has no maximum on E. This completes the proof. ∎

Example 2.8. Let X be the half-open interval $[0, 2\pi)$ on the real line and **f** be the mapping of X onto the circle Y consisting of all points whose distance from the origin is 1, is given by

$$\mathbf{f}(t) = (\cos t, \sin t) \qquad (0 \leq t < 2\pi)$$

We know that the trigonometric functions sin and cos are continuous function and hence by theorem (2.7), we have **f** is continuous. Thus **f** is continuous, 1-1 mapping of X onto Y and inverse mapping exists. But, the $\mathbf{f}^{-1}$ is not continuous at the origin and also X is not compact.

Hence, we conclude that **f** is 1-1, continuous from a non-compact X into Y but its inverse mapping $\mathbf{f}^{-1}$ is not continuous.

Remark 2.9. The compactness property is essential to prove the theorems (2.10), (2.11), (2.12)

Let us Sum Up:

In this unit, the students acquired knowledge to

- limits of functions and its properties.
- continuous functions and its properties.
- concept of continuity and compactness.

Check Your Progress:

1. Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1]$.

2. Prove that

$$f(x) = \sin\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

   is not uniformly continuous on $[0, \infty)$

3. Show that $f(x) = x^3$ is uniformly continuous in $[1, 2]$.

4. Show that $f(x) = \sqrt{x}$ is uniformly continuous in $[0, 2]$.

5. If f and g are uniformly continuous on the same interval, prove that $f + g$ and $f - g$ are also uniformly continuous on the same interval.

6. A real valued function f defined in (a, b) is said to be *convex* if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ whenever $a < x < b$, $a < y < b$. Prove that every convex function is continuous.

7. Assume that f is continuous real function defined in (a, b) such that for all $x, y \in (a, b)$. Prove that f is convex.

Choose the correct or more suitable answer:

1. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then
   (a) f is continuous on X.
   (b) f is continuous on Y.
   (c) f is uniformly continuous on Y.
   (d) f is uniformly continuous on X.

2. Continuous image of a compact space is ......
   (a) closed.
   (b) open.

(c) connected space.

(d) compact space.

3. f is continuous if and only if inverse image of ......

(a) closed set is open.

(b) open set is closed.

(c) open set is open.

(d) open set is half open.

Answer:

(1) d (2) d (3) c

Glossaries:

1. Continuous function: A function that is continuous at every point of the set.
2. Compact Sets: A set E is compact if Every open cover of E admits a finite sub cover.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.
2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-I

UNIT-3

LIMITS AND CONTINUITY-II

Overview

In this unit, we discuss about the image of a connected set under a continuous map. Further we discuss in detail about discontinuites of a function at a point. Also, we explained about the limit of a function at infinity.

Objectives

After completion of this unit, students will be able to

- ★ explain the concept of connectedness and its properties.
- ★ understand the difference between monotonically increasing and monotonically decreasing.
- ★ understand the concept of infinite limits and limits at infinity.

3.1. Continuity and Connectedness:

First, we recall the definition of connectedness. In this section, we shall prove the result that image of a connected sets is connected under a continuous mapping.

Definition 3.1. A set $E \subset X$ is said to be *connected* if E is not a union of two non-empty separated sets.

Theorem 3.1. *If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected. In other words, "Continuous image of a connected subset is connected".*

Proof. Given that f is continuous mapping of X into Y and E is a connected subset of X. Now, we shall prove that $f(E)$ is connected. Let us prove this result by contradiction.

Assume that $f(E)$ is not connected, then $f(E) = A \cup B$, where A and B are non-empty separated subsets of Y; $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

$$\begin{aligned} \text{Put } G &= E \cap f^{-1}(A) \text{ and } H = E \cap f^{-1}(B). \text{ Then} \\ G \cup H &= \left(E \cap f^{-1}(A)\right) \cup \left(E \cap f^{-1}(B)\right) \\ &= E \cap \left(f^{-1}(A) \cup f^{-1}(B)\right) = E \cap f^{-1}(A \cup B) = E \cap E = E \end{aligned}$$

and neither G nor H is non-empty.

Since $A \subset \overline{A}$, $G = E \cap f^{-1}(A) \subset f^{-1}(\overline{A})$ and f is continuous.

$f^{-1}\left(\overline{A}\right)$ is closed and hence $\overline{G} \subset f^{-1}\left(\overline{A}\right)$. Since $f(H) = B$ and $\overline{A} \cap B = \emptyset$.

Thus, we have $\overline{G} \cap H \subseteq f^{-1}\left(\overline{A}\right) \cap f^{-1}(B) = f^{-1}(\overline{A} \cap B) = f^{-1}(\emptyset) = \emptyset$

$(\because f(H) = B \text{ and } \overline{A} \cap B = \emptyset)$.

Therefore $\overline{G} \cap H = \emptyset$. Similarly, we can prove that $G \cap \overline{H} = \emptyset$.

Thus, G and H are two separated subsets of E and $\overline{G} \cap H = \emptyset$, $G \cap \overline{H} = \emptyset$. *i.e.*, E is not a connected subset of X, which is a contradiction. Hence $f(E)$ is connected. This completes the proof of the theorem. ∎

Theorem 3.2 (Intermediate value theorem for continuous functions)**.**

Let f be a continuous real function on the interval $[a,b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a,b)$ such that $f(x) = c$.

Proof. Given that f is a continuous function on the interval $[a,b]$.

We know that "A subset E of the real line R^1 is connected if and only if it has the following properties: if $x \in E, y \in E$ and $x < z < y$, then $z \in E$."

Hence, by this result, $[a,b]$ is connected subset of R^1.

Also, we known that "Continuous image of a connected subset is connected" and thus $f([a,b])$ is connected subset of R^1. If c is a number such that $f(a) < c < f(b)$. Then by our first result, $c \in f([a,b])$. That is $c = f(x)$ for some $x \in [a,b]$. ∎

Remark 3.1. The above theorem holds good, if $f(a) > f(b)$

3.2. Discontinuities:

If x is a point in the domain of the function f at which f is not continuous, then we say that f is discontinuous at x or that f has a discontinuity at x.

Definition 3.2. Let f be defined on (a,b). Consider any point x such that $a \le x < b$. We write $f(x+) = q$, if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$.

Definition 3.3. Let f be defined on (a,b). Consider any point x such that $a < x \le b$. We write $f(x-) = q$, if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (a,x) such that $t_n \to x$.

Remark 3.2. If x is any point of (a,b), $\lim_{t \to x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

Definition 3.4. Let f be a function defined on (a,b).

1. A function f is said to have a discontinuity of *first kind* at x, if both $f(x+)$ and $f(x-)$ exist.

2. A function f is said to have a discontinuity of *second kind* at x, if either $f(x+)$ or $f(x-)$ or both does not exist.

Example 3.1. Examine the nature of discontinuity of $f(x)$ defined by

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

If x is rational, there exists $\epsilon = \frac{1}{2} > 0$ such that for any $\delta > 0$, $x < t < x + \delta$ and t is irrational then

$$|f(x) - f(t)| = |1 - 0| = 1 > \frac{1}{2} = \epsilon$$

So, $f(x+)$ does not exists.

If x is rational, there exists $\epsilon = \frac{1}{2} > 0$ such that for any $\delta > 0$, $x - \delta < t < x$ and t is irrational then

$$|f(x) - f(t)| = |1 - 0| = 1 > \frac{1}{2} = \epsilon$$

So, $f(x-)$ does not exists.

If x is irrational, there exists $\epsilon = \frac{1}{2} > 0$ such that for any $\delta > 0$, $x < t < x + \delta$ and t is rational then

$$|f(x) - f(t)| = |0 - 1| = 1 > \frac{1}{2} = \epsilon$$

So, $f(x+)$ does not exists.

If x is irrational, there exists $\epsilon = \frac{1}{2} > 0$ such that for any $\delta > 0$, $x - \delta < t < x$ and t is rational then

$$|f(x) - f(t)| = |0 - 1| = 1 > \frac{1}{2} = \epsilon$$

So, $f(x-)$ does not exists.

Thus, in all the cases both $f(x+)$ and $f(x-)$ does not exist and hence f has a discontinuity of the second kind at all points.

3.3. Monotonic Functions:

Definition 3.5. Let f be a real valued function defined on (a, b). Then f is said to be *monotonic* function on (a, b), if either

1. f is *monotonically increasing* *i.e.*, if $a < x < y < b \Rightarrow f(x) \leq f(y)$ (or)
2. f is *monotonically decreasing* *i.e.*, if $a < x < y < b \Rightarrow f(x) \geq f(y)$

Theorem 3.3. *Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t) \tag{3.1}$$

Further, if $a < x < y < b$, then

$$f(x+) \leq f(y-) \tag{3.2}$$

Proof. Given that f is monotonically increasing on (a, b). Our aims is to prove that $f(x+)$ and $f(x-)$ exist at every point of x.

Let us prove,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t) \tag{3.3}$$

Consider the set $E = \{f(t) : a < t < x\}$.

Since f is monotonically increasing and hence the set E is bounded above by $f(x)$. Therefore the set E has least upper bound say $A = \sup_{a<t<x} f(t)$. Clearly, $A \leq f(x)$. It remains to show that $A = f(x-)$.

For this, let $\epsilon > 0$ be given, then $A - \epsilon$ cannot be an upper bound. So, there exists $\delta > 0$ such that

$$a < x - \delta < x \text{ and } A - \epsilon < f(x - \delta) \leq A \tag{3.4}$$

Since f is monotonic, we have

$$f(x - \delta) \leq f(t) \leq A \qquad (x - \delta < t < x) \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$|f(t) - A| < \epsilon \quad (x - \delta < t < x)$$

Hence $f(x-) = A = \sup_{a<t<x} f(t)$.

Next, we shall prove the right hand side inequality of (3.3).

For this, consider the set $F = \{f(t) : x < t < b\}$.

Since f is monotonically increasing and hence the set F is bounded below and hence the set F is bounded below by $f(x)$. Therefore the set F has greatest lower bound say $B = \inf_{x<t<b} f(x)$. Clearly $B \geq f(x)$. It remains to prove that $B = f(x+)$.

For this, let $\epsilon > 0$ be given, then $B + \epsilon$ cannot be a lower bound, so there exists a $\delta > 0$ such that

$$x < x + \delta < b \text{ and } B < f(x + \delta) \leq B + \epsilon \tag{3.6}$$

Since f is monotonic, we have

$$B < f(t) < f(x+\delta) < B+\epsilon \qquad (x < t < x+\delta) \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$|f(t) - B| < \epsilon \quad (x < t < x+\delta)$$

Hence $f(x+) = B = \inf\limits_{x<t<b} f(t)$.

The monotonic increasing function f has right hand limit and left hand limit and hence f has discontinuities of first kind. ■

Remark 3.3. The above theorem remains holds good for monotonically decreasing function. The proof is very similar.

Corollary 3.1. *Monotonic functions have no discontinuities of the second kind.*

Proof. Proof follows directly from the above theorem and remarks. ■

Theorem 3.4. *Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.*

Proof. Given that f is monotonic on (a, b). For the sake of convenience, assume that f is monotonically increasing on (a, b). Then, by theorem (3.3), we have

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t) \tag{3.8}$$

If $a < x < y < b$, then from (3.8), we have

$$f(x+) = \inf_{x<t<b} f(t) \leq \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) \leq \sup_{a<t<y} f(y-) \tag{3.9}$$

Let E be the set of points at which f is discontinuous.

With every point x of E we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+). \tag{3.10}$$

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$ (by using (3.9)).

Hence $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$. Certainly, we have

$$f(x_1-) < r(x_1) < f(x_1+) \leq f(x_2-) < r(x_2) < f(x_2+)$$

Thus, we have established a 1-1 correspondence between the set E and the subset of the set of rational numbers which is countable.

Hence the set E is countable. This completes the proof of the theorem. ■

3.4. Infinite limits and limits at infinity:

In this section, we are going to investigate the extended real number system in which we deals with infinities.

Definition 3.6. The extended real number system consists of the real field R and two symbols $+\infty$ and $-\infty$. Maintaining the original order in R and define $-\infty < x < +\infty$ for every $x \in R$.

Remark 3.4. From the above definition, we observe that $+\infty$ is an upper bound of the extended real number system and that E a non-empty subset of real number which is not bounded above in R, then $\sup E = +\infty$. Similarly, if E is not bounded below, then we have $\inf E = -\infty$.

For any real number x, we have already defined a neighborhood of x to be any segment $(x-\delta, x+\delta)$.

Definition 3.7. For any real number c, the set of real numbers x such that $x > c$ is called a *neighborhood* of $+\infty$ and it is written as $(c, +\infty)$. Similarly the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Now, we are going to define the limit of a real function in the extended real number system.

Definition 3.8. Let f be a real function defined on $E \subset R$, we say that $f(t) \to A$ as $t \to x$, where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

Remark 3.5. When A and x are real, then it is quite interesting to observe that this definition is coincides with the definition (2.1).

The analogue of theorem (2.2) is still true and the proof is also same. For the sake of completeness, we will state the theorem for the extended real number system.

Theorem 3.5. *Let f and g be defined on $E \subset R$. Suppose $f(t) \to A$, $g(t) \to B$, as $t \to x$. Then*

(a) $f(t) \to B$ *implies* $B = A$.

(b) $(f+g)(t) \to A + B$.

(c) $(fg)(t) \to AB$.

(d) $\left(\frac{f}{g}\right)(t) \to \frac{A}{B}$.

provided the right members of $(b), (c)$, *and* (d) *are defined.*

Remark 3.6. Note that $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{A}{0}$ are not defined.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- concept of continuity and connectedness.
- understand the different types of discontinuity.
- concept of the infinite limits and limits at infinity.

Check Your Progress:

1. A function f is defined on R by

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^3 - 3x & \text{if } 1 < x \leq 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$$

   Examine f for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

2. Discuss the kind of discontinuity, if any of the function is defined as follows:

$$f(x) = \begin{cases} \frac{x - |x|}{x} & \text{when } x \neq 0 \\ 2 & \text{when } x = 0 \end{cases}$$

3. If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ for the function $f(x) = x - [x], \quad \forall x \geq 0$.

Choose the correct or more suitable answer:

1. Let f be a function defined on (a,b). Then a function f is said to have a discontinuity of first kind at x, if

   (a) $f(x+)$ exist.

   (b) $f(x-)$ exist.

   (c) both $f(x+)$ and $f(x-)$ exist.

   (d) $f(x+)$ exists and $f(x-)$ does not exist.

Answer:

(1) c

Glossaries:

Connected Set: A set is disconnected, if it can be split into two disjoint non-empty subsets such that neither contains a limit point of the other. A set is connected, if it can be split in such a way.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.

2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-I

UNIT-4

THE RIEMANN-STIELTJES INTEGRAL

Objectives

After completion of this unit, students will be able to

- ★ understand the concept of partition and refinement of partition.
- ★ explain the concept of upper sum and lower sum.
- ★ explain the basic difference between Riemann integral and Riemann Stieltjes integral.

German mathematician Riemann was the first to introduce the process of integration on purely arithmetical treatment which is broad based and free from dependence on geometrical concepts. This concept is known as *Riemann integration*, which was later on generalized by The Dutch astronomer and Mathematician Stieltjes.

Overview

In this unit, we will illustrate the condition for the existence of Riemann integral.

4.1. Definition and Existence of the integral:

Definition 4.1. Let $I = [a, b]$ be a closed and bounded interval. Then a finite set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ is called a *partition* or *division* of the interval $I = [a, b]$.

For example: Consider the interval $[0, 1]$ be a closed and bounded interval. Then $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ is a partition of $[0, 1]$.

Definition 4.2. The closed sub-interval $I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$. are called the *segment of the partition*.

Definition 4.3. The length of the sub-interval I_r is denoted by $\triangle x_r$ defined by $\triangle x_r = x_r - x_{r-1}$.

Definition 4.4. The norm of the partition P is the maximum of the length of the segments of a partition P denoted by $||P||$, defined by $||P||= max\{\triangle x_i : i = 1, 2, \ldots, n\}$.

Definition 4.5. We say that the partition P^* is a refinement of P if $P^* \supset P$ (that is every point of P is a point of P^*). Given two partition P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$.

Definition 4.6. Let f be bounded real function defined on $[a, b]$. Let P be a partition of $[a, b]$. We define $U(P, f)$ called the *upper sum* of f corresponding to P as

$$U(P, f) = \sum_{i=1}^{n} M_i \triangle x_i$$

and the *lower sum* of f corresponding to P, denoted by $L(P, f)$ is defined as

$$L(P, f) = \sum_{i=1}^{n} m_i \triangle x_i$$

Where M_i and m_i are defined by

$$M_i = \sup\{f(x) : (x_{i-1} \le x \le x_i)\}$$
$$m_i = \inf\{f(x) : (x_{i-1} \le x \le x_i)\}$$

Remark 4.1. From the definition, clearly we can see that $U(P, f) \geq L(P, f)$.

Definition 4.7. Let f bounded real function defined on $[a, b]$. We define $\overline{\int_a^b} f(x)\,\mathrm{d}x$ is called the *upper Riemann integral* of f over $[a, b]$ as

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf U(P, f)$$

where the inf are taken over all partitions P of $[a, b]$.

Definition 4.8. Let f bounded real function defined on $[a, b]$. We define $\underline{\int_a^b} f(x)\,\mathrm{d}x$ is called the *lower Riemann integral* of f over $[a, b]$ as

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup U(P, f)$$

where the sup are taken over all partitions P of $[a, b]$.

Definition 4.9. Let f bounded real function defined on $[a, b]$. We say that f is Riemann integrable on $[a, b]$ and we write $f \in \mathscr{R}$, if

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

In this case, we write $\int_a^b f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$

Remark 4.2. If f is the Riemann integral over $[a, b]$. Since f is bounded, then there exists two numbers m and M, such that

$$m \le m_i \le M_i \le M$$
$$\Rightarrow \quad m\triangle x_i \le m_i\triangle x_i \le M_i\triangle x_i \le M\triangle x_i$$

Putting $i = 1, 2, \ldots, n$ and adding all the inequalities, we get

$$m\,(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$$

Hence, the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows that the *upper* and *lower integrals* are defined for every bounded function f.

4.2. Riemann-Stieltjes Integral:

Definition 4.10. Let α be a monotonically increasing function on $[a, b]$ and P be a partition of $[a, b]$. Corresponding to each partition P of $[a, b]$, define
$\triangle\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1, 2, \ldots, n.$

Since each $\alpha(a)$ and $\alpha(b)$ are finite and α is monotonically increasing. Thus, we have α is bounded on $[a, b]$ and $\triangle\alpha_i \geq 0$.

Definition 4.11. Let f be bounded real function defined on $[a, b]$. Let P be a partition of $[a, b]$. We define $U(P, f, \alpha)$ called the *upper sum* of f corresponding to P as

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \triangle\alpha_i$$

and the *lower sum* of f corresponding to P, denoted by $L(P, f, \alpha)$ is defined as

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \triangle\alpha_i$$

Where M_i and m_i are defined by

$$\begin{aligned} M_i &= \sup f(x) \quad (x_{i-1} \le x \le x_i) \\ m_i &= \inf f(x) \quad (x_{i-1} \le x \le x_i) \end{aligned}$$

Definition 4.12. Let f bounded real function defined on $[a, b]$. We define $\overline{\int_a^b} f \, d\alpha$ is called the *upper Riemann Stieltjes integral* of f with respect to α over $[a, b]$ as

$$\overline{\int_a^b} f \, d\alpha = \inf U(P, f, \alpha)$$

where the inf are taken over all partitions P of $[a, b]$.

Definition 4.13. Let f bounded real function defined on $[a, b]$. We define $\underline{\int_a^b} f \, d\alpha$ is called the *lower Riemann Stieltjes integral* of f with respect to α over $[a, b]$ as

$$\underline{\int_a^b} f \, d\alpha = \sup U(P, f, \alpha)$$

where the sup are taken over all partitions P of $[a, b]$.

Definition 4.14. Let f bounded real function defined on $[a, b]$, We say that f is Riemann Stieltjes integrable with respect to α on $[a, b]$ and we write $f \in \mathscr{R}(\alpha)$, if

$$\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha$$

In this case, we write $\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha$ (or some times written by) $\int_a^b f(x) \, d\alpha(x)$

Remark 4.3. If we take $\alpha(x) = x$, then the Riemann-Stieltjes integral is reduced to Riemann integral. Thus, Riemann integral is a special case of Riemann-Stieltjes integral. Hence, all the theorem and properties of Riemann-Stieltjes integral are holds good for Riemann integral.

Example 4.1. Consider the function $f : [0,1] \to [0,1]$, defined by

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

Let P be a partition of $[0,1]$; a set of points $x_0, x_1, \ldots, x_n$ such that $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = 1$. Let $\triangle x_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n$.

Corresponding to each partition P of $[0,1]$,

$$\begin{aligned} M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = \sup\{0,1\} = 1 \\ m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = \inf\{0,1\} = 0, \\ & \quad \text{for each } i = 1, 2, \ldots, n \\ U(P,f) &= \sum_{i=1}^{n} M_i \triangle x_i = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1 \\ L(P,f) &= \sum_{i=1}^{n} m_i \triangle x_i = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0 \\ \overline{\int_0^1} f(x)\,\mathrm{d}x &= \inf\{U(P,f) : P \text{ is a partition of } [0,1]\} = 1 \\ \underline{\int_0^1} f(x)\,\mathrm{d}x &= \sup\{L(P,f) : P \text{ is a partition of } [0,1]\} = 0 \\ \text{Thus, } \overline{\int_0^1} f(x)\,\mathrm{d}x &\neq \underline{\int_0^1} f(x)\,\mathrm{d}x \quad \Rightarrow f \notin \mathscr{R} \text{ on } [0,1]. \end{aligned}$$

Hence f is not Riemann integrable on $[0,1]$.

Theorem 4.1. *If P^* is a refinement of P, then*

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \tag{4.1}$$

$$\text{and } U(P^*,f,\alpha) \leq U(P,f,\alpha) \tag{4.2}$$

Proof. Let P be the partition of $[a,b]$, *i.e.*, $P = \{a = x_0, x_1, \ldots, x_{i-1}, x_i, \ldots, x_n = b\}$.

Let P^* be the refinement of P by just adding one more point x^* in P,

i.e., $P^* = \{a = x_0, x_1, \ldots, x_{i-1}, x^*, x_i, \ldots, x_n = b\}$.

Let w_1 and w_2 be respectively the supremum of the functions $f(x)$ in $[x_{i-1}, x^*]$ and $[x^*, x_i]$.

Then clearly, $w_1 \leq M_i$ and $w_2 \leq M_i$ where M_i is the supremum of the function in $[x_{i-1}, x_i]$.

Let α be a non-increasing function on $[a,b]$.

Clearly $\alpha(x^*) \geq \alpha(x_{i-1})$ and $\alpha(x_i) \geq \alpha(x^*)$.

$$\begin{aligned}
U(P,f,\alpha) &= \sum_{k=1}^{n} M_k \triangle\alpha_k \\
&= \sum_{k=1}^{i-1} M_k \triangle\alpha_k + M_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] \\
&\quad + \sum_{k=i+1}^{n} M_k \triangle\alpha_k \\
U(P^*,f,\alpha) &= \sum_{k=1}^{i-1} M_k \triangle\alpha_k + w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] \\
&\quad + w_2\left[\alpha(x_i) - \alpha(x^*)\right] + \sum_{k=i+1}^{n} M_k \triangle\alpha_k \\
\text{Hence, } U(P,f,\alpha) - U(P^*,f,\alpha) &= M_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] - w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] \\
&\quad - w_2\left[\alpha(x_i) - \alpha(x^*)\right] \\
&= (M_i - w_2)\left(\alpha(x_i) - \alpha(x^*)\right] \\
&\quad + (M_i - w_1]\left[\alpha(x^*) - \alpha(x_{i-1})\right] \\
&\geq 0 \\
&\quad [\because M_i - w_1 \geq 0;. \quad M_i - w_2 \geq 0; \\
&\qquad \alpha(x_i) \geq \alpha(x^*); \quad \alpha(x^*) \geq \alpha(x_{i-1})] \\
\Rightarrow \quad U(P,f,\alpha) &\geq U(P^*,f,\alpha)
\end{aligned}$$

Hence $U(P^*,f,\alpha) \leq U(P,f,\alpha)$.

In a similar way, we can prove $L(P,f,\alpha) \leq L(P^*,f,\alpha)$. ■

Theorem 4.2.

$$\underline{\int_a^b} f\,\mathrm{d}\alpha \quad \leq \quad \overline{\int_a^b} f\,\mathrm{d}\alpha$$

Proof. Let P_1 and P_2 be the two partitions on $[a,b]$ and P^* be the common refinement of P_1 and P_2.

Then by theorem (4.1), we have

$$\begin{aligned}
L(P_1,f,\alpha) &\leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha) \\
\text{hence } L(P_1,f,\alpha) &\leq U(P_2,f,\alpha) \qquad (4.3)
\end{aligned}$$

If P_2 is fixed and the sup is taken over all P_1 in (4.3),then we have

$$\underline{\int_a^b} f\,\mathrm{d}\alpha \quad \leq \quad U(P_2,f,\alpha) \qquad (4.4)$$

Taking the inf over all P_2 in(4.4), then we have

$$\underline{\int_a^b} f\,\mathrm{d}\alpha \quad \leq \quad \overline{\int_a^b} f\,\mathrm{d}\alpha$$

This completes the proof of the theorem. ■

Theorem 4.3. *Let f be a bounded real function defined on $[a,b]$. $f \in \mathscr{R}(\alpha)$ if and only if for every $\epsilon > 0$ there exists a partition P such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Proof. Assume that $f \in \mathscr{R}(\alpha)$. Then,

$$\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha \tag{4.5}$$

$$\text{Now,} \quad \underline{\int_a^b} f \, d\alpha = \sup\{L(P,f,\alpha) : P \text{ is a partition of } [a,b]\}$$

$\Rightarrow$ $\exists$ a partition P_1 of $[a,b]$ such that

$$\underline{\int_a^b} f \, d\alpha < L(P_1,f,\alpha) + \frac{\epsilon}{2} \tag{4.6}$$

$$\text{Similarly,} \quad \overline{\int_a^b} f \, d\alpha = \inf\{U(P,f,\alpha) : P \text{ is a partition of } [a,b]\}$$

$\Rightarrow$ $\exists$ a partition P_2 of $[a,b]$ such that

$$U(P_2,f,\alpha) < \overline{\int_a^b} f \, d\alpha + \frac{\epsilon}{2} \tag{4.7}$$

If $P = P_1 \cup P_2$, then P is the common refinement of both P_1 and P_2, then by theorem (4.1), equations (4.6) and (4.7), we have

$$\begin{aligned} U(P,f,\alpha) &< U(P_2,f,\alpha) < \overline{\int_a^b} f \, d\alpha + \frac{\epsilon}{2} < L(P_1,f\alpha) + \epsilon \\ &< L(P,f,\alpha) + \epsilon \\ \Rightarrow U(P,f,\alpha) &< L(P,f,\alpha) + \epsilon \\ \Rightarrow U(P,f,\alpha) - L(P,f,\alpha) &< \epsilon \end{aligned}$$

Converse Part:

Let $\epsilon > 0$, $\exists$ a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ (4.8)

$$\begin{aligned} \underline{\int_a^b} f \, d\alpha &= \inf\{U(P,f,\alpha) : \ P \text{ is a partition of } [a,b]\} \\ &\Rightarrow \quad \overline{\int_a^b} f \, d\alpha \le U(P,f,\alpha) \\ \underline{\int_a^b} f \, d\alpha &= \sup\{L(P,f,\alpha) : \ P \text{ is a partition of } [a,b]\} \\ &\Rightarrow \quad \underline{\int_a^b} f \, d\alpha \ge L(P,f,\alpha) \\ i.e., \ L(P,f,\alpha) &< \underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha \le U(P,f,\alpha) \end{aligned} \tag{4.9}$$

Using (4.8) in (4.9), we have

$$
\begin{aligned}
\overline{\int_a^b} f\,\mathrm{d}\alpha - \underline{\int_a^b} f\,\mathrm{d}\alpha &< \epsilon \\
\Rightarrow \quad \overline{\int_a^b} f\,\mathrm{d}\alpha &< \underline{\int_a^b} f\,\mathrm{d}\alpha + \epsilon \\
\Rightarrow \quad \overline{\int_a^b} f\,\mathrm{d}\alpha &< \underline{\int_a^b} f\,\mathrm{d}\alpha \quad (\because \epsilon \text{ is arbitrary}) \qquad (4.10)\\
\text{Always} \quad \underline{\int_a^b} f\,\mathrm{d}\alpha &< \overline{\int_a^b} f\,\mathrm{d}\alpha \qquad (4.11)
\end{aligned}
$$

From(4.10) and (4.11), we get (4.12)

$$
\overline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^b} f\,\mathrm{d}\alpha
$$

$$\Rightarrow \quad f \in \mathscr{R}(\alpha)$$ ■

Theorem 4.4.

(a) If for some partition P and some $\epsilon > 0$, the inequality

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \qquad (4.13)$$

holds good then it is also holds good for partition P^* (with the same ϵ).

(b) If (4.13) *holds for $P = \{x_0, x_1, \ldots, x_n$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \triangle\alpha_i < \epsilon.$$

(c) $f \in \mathscr{R}(\alpha)$ and the hypothesis of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i) - \int_a^b f d\alpha \right| \triangle\alpha_i < \epsilon.$$

Proof.

(a) If P^* is a refinement of P, then we have

$$
\begin{aligned}
L(P, f, \alpha) &\leq L(P^*, f, \alpha) \qquad (4.14)\\
\text{and} \quad U(P^*, f, \alpha) &\leq L(P, f, \alpha) \qquad (4.15)
\end{aligned}
$$

Using (4.13), (4.14) and (4.15), we get

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

(b) Given that $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \leq M_i - m_i$.

$$\begin{aligned}
\text{Therefore, } \sum_{i=1}^{n} |f(s_i) - f(t_i)| \triangle\alpha_i &\leq \sum_{i=1}^{n} (M_i - m_i) \triangle\alpha_i \\
&= \sum_{i=1}^{n} M_i \triangle\alpha_i - \sum_{i=1}^{n} m_i \triangle\alpha_i \\
&= U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \\
\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \triangle\alpha_i &< \epsilon
\end{aligned}$$

(c) we have

$$L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\triangle\alpha_i \leq U(P, f, \alpha) \tag{4.16}$$

$$\text{and } L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha) \tag{4.17}$$

From (4.16) and (4.17), we get

$$\sum_{i=1}^{n} f(t_i)\triangle\alpha_i - \int_a^b f d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \tag{4.18}$$

$$\text{and } \int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i)\triangle\alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \tag{4.19}$$

From (4.18) and (4.19), we get, $\left| \sum_{i=1}^{n} f(t_i) - \int_a^b f d\alpha \right| \triangle\alpha_i < \epsilon.$ ■

Theorem 4.5. *If f is continuous and α is monotonically increasing on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$,*

Proof. Let $\epsilon > 0$ be given.

Since α is monotonically increasing on $[a, b]$ we can choose $\eta > 0$ such that

$$\alpha(b) - \alpha(a) < \frac{\epsilon}{\eta} \tag{4.20}$$

Since $[a, b]$ is compact and f is continuous on $[a, b]$. Thus, f is uniformly continuous on $[a, b]$.

Hence, by definition of uniform continuity, there exists $\delta > 0$ such that $x \in [a, b]$, $y \in [a, b]$.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \eta \tag{4.21}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition on $[a, b]$ such that $\triangle x_i < \delta$ $(i = 1, 2, 3, \ldots, n)$.

$$\begin{aligned}
\text{Define } M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\
\text{and } m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}
\end{aligned}$$

Then by (4.21), we have

$$M_i - m_i < \eta, \qquad i = 1, 2, \ldots, n \tag{4.22}$$

From (4.20) and (4.22), we conclude that

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} M_i \triangle\alpha_i - \sum_{i=1}^{n} m_i \triangle\alpha_i \\ &= \sum_{i=1}^{n} (M_i - m_i)\triangle\alpha_i \\ &\leq \sum_{i=1}^{n} \eta\triangle\alpha_i = \eta\,[\alpha(b) - \alpha(a)] < \eta\,\frac{\epsilon}{\eta} < \epsilon \\ \Rightarrow \quad f \in \mathscr{R}(\alpha) \end{aligned}$$

■

Theorem 4.6. *If f is monotonic and α is monotonically increasing and continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$.*

Proof. Given that α is monotonically increasing and continuous on $[a,b]$.

Let $\epsilon > 0$. Then for any positive integer n there exists a partition P of $[a,b]$ such that

$$\begin{aligned} \triangle\alpha_i &= \alpha(x_i) - \alpha(x_{i-1}) \quad (i = 1, 2, \ldots, n) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \end{aligned} \tag{4.23}$$

Assume that f is monotonically increasing on $[a,b]$. Then, we have

$$M_i = f(x_i); \qquad m_i = f(x_{i-1}) \;\; (i = 1, 2, \ldots, n) \tag{4.24}$$

From (4.23) and (4.24), we conclude that

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} (M_i - m_i)\triangle\alpha_i \\ &= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] \end{aligned}$$

Choose n sufficiently large, then

$$U(P,f,\alpha) - L(P,f,\alpha) = \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \epsilon$$

Thus, $f \in \mathscr{R}(\alpha)$ ■

Remark 4.4. The proof is similar in the case of f is monotonically decreasing.

Theorem 4.7. *Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity of $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.*

Proof. Let $E = \{c_1, c_2, \ldots, c_n\}$ be the finite set of points at which f is discontinuous.

Since E is finite and α is continuous at each point c_i $(i = 1, 2, \ldots, n)$. Hence E can be covered by n disjoint intervals $[u_j, v_j] \subset [a, b]$ such that

$$\sum_{i=1}^{n} \left[\alpha(v_j) - \alpha(u_j)\right] < \epsilon$$

We can construct these intervals in such a way that every point of $E \cap [a, b]$ lies in the interior of some $[u_j, v_j]$.

$$\text{Let } K = [a, b] - \bigcup_{j=1}^{n} (u_j, v_j)$$

Clearly, the set K is compact and also f is continuous in each sub-intervals $[a, u_1], [v_1, u_2], [v_2, u_3] \ldots [v_m, b]$, therefore f is uniformly continuous on K.

Hence, by definition of uniformly continuity, there exists a $\delta > 0$ such that for $s \in K, t \in K$.

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$$

Form a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that

(i) each u_j occurs in P.

(ii) each v_j occurs in P.

(iii) no points of any segment (u_j, v_j) occurs in P.

(iv) If x_{i-1} is not one of the u_j, then $\triangle x_j < \delta$.

$$\begin{aligned} \text{Define } M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \quad (i = 1, 2, \ldots, n) \\ m_i &= inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad (i = 1, 2, \ldots, n) \\ M &= \sup\{f(x) : x \in [a, b]\} \end{aligned}$$

$$\begin{aligned} \text{Now, } M_i - m_i &= |M_i - m_i| \\ &\leq |M_i| + |m_i| \\ &\leq 2|M_i| \\ &= 2\sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ &\leq 2\sup\{f(x) : x \in [a, b]\} \\ \Rightarrow M_i - m_i &\leq 2M \\ \text{and } M_i - m_i &\leq \epsilon, \quad \text{unless } x_{i-1} \text{ one of the } u_j \end{aligned}$$

$$\begin{aligned} \text{Therefore, } U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{x_{i-1} \neq u_j} (M_i - m_i) \triangle\alpha_i + \sum_{x_{i-1} = u_j} (M_i - m_i) \triangle\alpha_i \\ &= \epsilon \sum \triangle\alpha_i + 2M \sum \left[\alpha(v_j) - \alpha(u_j)\right] \\ &< \epsilon [\alpha(b) - \alpha(a)] + 2M\epsilon \end{aligned}$$

Since ϵ is arbitrary, thus we have
$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &< \epsilon \\ \Rightarrow \quad f &\in \mathscr{R}(\alpha) \end{aligned}$$ ■

Theorem 4.8. *Suppose* $f \in \mathscr{R}(\alpha)$ *on* $[a,b]$, $m \le f \le M$, ϕ *is continuous on* $[m,M]$ *and* $h(x) = \phi(f(x))$ *on* $[a,b]$. *Then* $h \in \mathscr{R}(\alpha)$ *on* $[a,b]$.

Proof. Since ϕ is continuous on a closed and bounded interval $[m,M]$ and hence ϕ is uniformly continuous on $[m,M]$.

Therefore, by definition of uniformly continuous, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\delta < \epsilon$.

$$|\phi(s) - \phi(t)| < \epsilon \quad \text{if} \quad |s-t| < \delta \quad \text{and} \quad s,t \in [m,M] \tag{4.25}$$

Again if $f \in \mathscr{R}(\alpha)$ if and only if $\exists$ a partion $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon = \delta^2 \tag{4.26}$$

$$\begin{aligned} \text{Let} \quad M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\}; \quad (i = 1,2,\ldots,n) \\ m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}; \quad (i = 1,2,\ldots,n) \\ M_i^* &= \sup\{\phi(t) : f(t) \in [f(x_{i-1}), f(x_i)]\}; \quad (i = 1,2,\ldots,n) \\ m_i^* &= \inf\{\phi(t) : f(t) \in [f(x_{i-1}), f(x_i)]\}; \quad (i = 1,2,\ldots,n) \end{aligned}$$

Divide the number $1,2,3,\ldots,n$ into two classes A and B such that

$$\begin{aligned} M_i - m_i &< \delta \Rightarrow i \in A \\ M_i - m_i &\ge \delta \Rightarrow i \in B \end{aligned}$$

Then if $i \in A$, then $M_i - m_i < \delta$

$$\begin{aligned} \Rightarrow \quad |\phi(M_i) - \phi(m_i)| &\le \epsilon \\ \Rightarrow \quad \left|\phi(\sup f(x)) - \phi(\inf(f(x))\right| &\le \epsilon \\ \Rightarrow \quad \left|\sup(\phi(x)) - \inf(\phi(x))\right| &\le \epsilon \\ \Rightarrow \quad \left|M_i^* - m_i^*\right| &\le \epsilon \\ \Rightarrow M_i^* - m_i^* &\le \epsilon \end{aligned}$$

And if $r \in B$, then

$$\Rightarrow \quad M_i^* - m_i^* \le \left|M_i^*\right| + \left|m_i^*\right| \le 2K \qquad \text{if} \quad K = \sup\{\phi(t) : m \le t \le M\}$$

From (4.26), we have

$$\begin{aligned} \sum_{i \in B} (M_i - m_i)\,\triangle\alpha_i &< \delta^2 \\ \Rightarrow \quad \sum_{i \in B} \delta\triangle\alpha_i &< \delta^2 \qquad (\because M_i - m_i \ge \delta, \ \text{if} \ r \in B) \\ \Rightarrow \quad \sum_{i \in B} \triangle\alpha_i &< \delta \end{aligned}$$

Therefore,

$$\begin{aligned}
U(P,h,\alpha) - L(P,h,\alpha) &= \sum_{i\in A} (M_i^* - m_i^*)\,\triangle\alpha_i + \sum_{i\in B} (M_i^* - m_i^*)\,\triangle\alpha_i \\
&\leq \epsilon \sum_{i\in A} \triangle\alpha_i + 2K \sum_{i\in B} \triangle\alpha_i \\
&\leq \epsilon\,[\alpha(b) - \alpha(a)]\,2K\delta \\
&\leq \epsilon\,[\alpha(b) - \alpha(a) + 2k] \qquad (\because\ \delta < \epsilon)
\end{aligned}$$

Since ϵ is arbitrary, thus we have

$$\begin{aligned}
U(P,h,\alpha) - L(P,h,\alpha) &< \epsilon \\
\Rightarrow \quad h \in \mathscr{R}(\alpha) &\text{ on } [a,b]
\end{aligned}$$

This completes the proof of the theorem. ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- concept about Upper Sum and Lower Sum.
- existence of Riemann-Stieltjes integral.

Check Your Progress:

1. Define Norm of the partition.
2. Define Upper Riemann Integral.
3. Define Upper sum and Lower sum.
4. Define Upper Riemann Stieltjes integral.
5. If f is monotonic and α is monotonically increasing and continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$.
6. Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity of $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Choose the correct or more suitable answer:

1. $f \in \mathscr{R}(\alpha)$ if and only if

   (a) $\overline{\int_a^b} f \, d\alpha \leq \underline{\int_a^b} f \, d\alpha$

   (b) $\overline{\int_a^b} f \, d\alpha < \underline{\int_a^b} f \, d\alpha$

   (c) $\overline{\int_a^b} f \, d\alpha \geq \underline{\int_a^b} f \, d\alpha$

   (d) $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$

Answer:

(1) *d*

Glossaries:

1. Partition: Partition of a set is a grouping of its elements into non-empty subsets.
2. Supremum: Supremum of a set is its least upper bound.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.
2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-I

UNIT-5

PROPERTIES OF INTEGRAL

Structure

Objectives

After completion of this unit, students will be able to

- ★ understand the properties of Riemann Stieljes integral.
- ★ understand the integration and differentiation are inverse process for real function.

Overview

In this unit, we will illustrate the properties of Riemann integral.

5.1. Properties of the integral:

Theorem 5.1. *If* $f_1 \in \mathscr{R}(\alpha)$ *and* $f_2 \in \mathscr{R}(\alpha)$ *then* $f_1 + f_2 \in \mathscr{R}(\alpha)$ *and* $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.$

Proof. Given that $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$.

$$f_1 \in \mathscr{R}(\alpha) \Rightarrow \exists \text{ a partition } P_1 \text{ of } [a,b] \text{ such that } U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon \quad (5.1)$$
$$f_2 \in \mathscr{R}(\alpha) \Rightarrow \exists \text{ a partition } P_2 \text{ of } [a,b] \text{ such that } U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon \quad (5.2)$$

Let $P = P_1 \cup P_2$, then P is the common refinement of both P_1 and P_2. Hence (5.1) and (5.2) can be written as

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon \quad (5.3)$$
$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon \quad (5.4)$$

Adding (5.3) and (5.4), we get

$$U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) - L(P, f_2, \alpha) < 2\epsilon \quad (5.5)$$

Put $h(x) = f_1(x) + f_2(x)$.

$$\begin{aligned}
\text{Let } M_i &= \sup\{h(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
m_i &= \inf\{h(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
M_i^* &= \sup\{f_1(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
m_i^* &= \inf\{f_1(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
M_i^{**} &= \sup\{f_2(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
m_i^{**} &= \inf\{f_2(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n)
\end{aligned}$$

In the i^{th} interval, we have

$$\begin{aligned}
f_1(x) + f_2(x) &\le M_i^* + M_i^{**} \\
\Rightarrow M_i &\le M_i^* + M_i^{**} \quad (\because \max(f_1 + f_2) \le max f_1 + max f_2) \\
\text{Similarly, } m_i &\ge m_i^* + m_i^{**}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
U(P,h,\alpha) &= \sum_{i=1}^{n} M_i \triangle\alpha_i \leq \sum_{i=1}^{n} (M_i^* + M_i^{**}) \triangle\alpha_i \\
\Rightarrow \quad U(P,h,\alpha) &\leq \sum_{i=1}^{n} M_i^* \triangle\alpha_i + \sum_{i=1}^{n} M_i^{**} \triangle\alpha_i \\
U(P,h,\alpha) &\leq U(P,f_1,\alpha) + U(P,f_2,\alpha) \qquad (5.6) \\
\text{Similarly, } L(P,h,\alpha) &\geq L(P,f_1,\alpha) + L(P,f_2,\alpha) \qquad (5.7)
\end{aligned}$$

Hence, from (5.5), we get

$$\begin{aligned}
U(P,h,\alpha) - L(P,h,\alpha) &< 2\epsilon \\
\Rightarrow \quad h \in \mathscr{R}(\alpha) \quad &i.e., \ f_1 + f_2 \in \mathscr{R}(\alpha)
\end{aligned}$$

Now, for a partition P, we have

$$\begin{aligned}
U(P,f_1,\alpha) &< \int_a^b f_1 d\alpha + \epsilon \\
U(P,f_2,\alpha) &< \int_a^b f_2 d\alpha + \epsilon
\end{aligned}$$

Adding these, we get

$$\begin{aligned}
U(P,f_1,\alpha) + U(P,f_2,\alpha) &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \\
\int_a^b h d\alpha &< U(P,f_1,\alpha) + U(P,f_2,\alpha) \\
&< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \\
\text{Since } \epsilon \text{ is arbitrary, } \int_a^b h d\alpha &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad (5.8)
\end{aligned}$$

Replace f_1 by $-f_1$ and f_2 by $-f_2$, we get

$$\int_a^b h d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad (5.9)$$

Thus, from (5.8) and (5.9), we get

$$\begin{aligned}
\int_a^b h d\alpha &= \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \\
i.e., \quad \int_a^b (f_1 + f_2)\, d\alpha &= \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha
\end{aligned}$$

■

Remark 5.1. In a similar way we can prove that, if $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ then $f_1 - f_2 \in \mathscr{R}(\alpha)$ and $\int_a^b (f_1 - f_2)\, d\alpha = \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha$.

Theorem 5.2. *If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then $cf \in \mathscr{R}(\alpha)$, where c is any constant.*

Also $\int_a^b cf d\alpha = c \int_a^b f d\alpha$.

Proof. Given that $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

If $c = 0$, then the theorem is quite obvious, so we may assume that $c \neq 0$.

$f \in \mathscr{R}(\alpha)$ if and only if $\exists$ a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{|c|} \tag{5.10}$$

Put $h(x) = cf(x)$.

$$\begin{aligned}
\text{Let } M_i &= \sup\{h(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
m_i &= \inf\{h(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
M_i^* &= \sup\{f_(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n) \\
m_i^* &= \inf\{f(x) : x \in [x_{i-1} x_i]\}; \quad (i = 1, 2, \ldots, n)
\end{aligned}$$

In the i^{th} interval, we have

$$\begin{aligned}
M_i &= \sup\{(cf)(x) : x_{i-1} \le x \le x_i\} \le \sup\{|c| f(x) : x_{i-1} \le x \le x_i\} = |c| M_i^* \\
\text{and } m_i &= \inf\{(cf)(x) : x_{i-1} \le x \le x_i\} \le \inf\{|c| f(x) : x_{i-1} \le x \le x_i\} = |c| m_i^*
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i=1}^{n} (M_i - m_i) \triangle\alpha_i \\
&\le \sum_{i=1}^{n} \left(|c| M_i^* - |c| m_i^*\right) \triangle\alpha_i \\
&= |c| \sum_{i=1}^{n} \left(M_i^* - m_i^*\right) \triangle\alpha_i \\
&= |c| \left[U(P, f, \alpha) - L(P, f, \alpha)\right] \\
&< |c| \frac{\epsilon}{|c|} < \epsilon \qquad \text{(using (5.10))} \\
\textit{i.e.,} \quad U(P, h, \alpha) - L(P, h, \alpha) &< \epsilon \\
\Rightarrow \quad h \in \mathscr{R}(\alpha) \quad \textit{i.e.,} \quad cf \in \mathscr{R}(\alpha)
\end{aligned}$$

Note for any constant c, we have $U(P, cf, \alpha) = cU(P, f, \alpha)$.

$$U(P, cf, \alpha) = cU(P, f, \alpha) < c \int_a^b f d\alpha + c \cdot \epsilon$$

$$\Rightarrow \int_a^b (cf) d\alpha \le c \int_a^b f d\alpha \tag{5.11}$$

Replace f by $-f$ in (5.11), we get

$$\int_a^b (cf) d\alpha \ge c \int_a^b f d\alpha \tag{5.12}$$

From (5.11) and (5.12), we conclude that

$$\int_a^b (cf) d\alpha = c \int_a^b f d\alpha$$ ■

Theorem 5.3. *If* $f \in \mathscr{R}(\alpha)$ *on* $[a, b]$ *and if* $a < c < b$. *then* $f \in \mathscr{R}(\alpha)$ *on* $[a, c]$ *and* $f \in \mathscr{R}(\alpha)$ *on* $[c, b]$.

Also, $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$

Proof. Given that $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

$f \in \mathscr{R}(\alpha)$ if and only if $\exists$ a partition P on $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon. \tag{5.13}$$

Let $P = P_1 \cup P_2$ where P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. Then

$$U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha) \tag{5.14}$$

$$\text{and} \quad L(P, f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha) \tag{5.15}$$

So by (5.13), we have

$$\begin{aligned} [U(P_1, f, \alpha) - L(P_1, f, \alpha)] &+ [U(P_2, f, \alpha) - L(P_2, f, \alpha)] \\ &= [U(P_1, f, \alpha) + U(P_2, f, \alpha)] \\ &\quad - [L(P_1, f, \alpha) + L(P_2, f, \alpha)] \\ &= U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \end{aligned}$$

$$\Rightarrow \quad U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon \quad \text{and} \quad U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon$$

$$\Rightarrow \quad f \in \mathscr{R}(\alpha) \text{ on } [a, c] \quad \text{and} \quad f \in \mathscr{R}(\alpha) \text{ on } [c, b].$$

It remains to prove that $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$

By (5.15), we have

$$L(P_1, f, \alpha) + L(P_2, f, \alpha) = L(P, f, \alpha) \leq \int_a^b f d\alpha \tag{5.16}$$

Keeping P_2 fixed and taking the supremum over all the partitions of P_1, we have

$$\int_a^c f d\alpha + L(P_2, f, \alpha) \leq \int_a^b f d\alpha$$

Now taking the supremum over all the partitions of P_2, then we have

$$\int_a^c f d\alpha + \int_c^b f d\alpha \leq \int_a^b f d\alpha \tag{5.17}$$

Similarly, from (5.15), we have

$$U(P_1, f, \alpha) + U(P_2, f, \alpha) = U(P, f, \alpha) \geq \int_a^b f d\alpha$$

Keeping P_2 fixed and taking the infimum over all the partitions of P_1, we have

$$\int_a^c f d\alpha + L(P_2, f, \alpha) \geq \int_a^b f d\alpha$$

Now taking the infimum over all the partitions of P_2, then we have

$$\int_a^c f d\alpha + \int_c^b f d\alpha \geq \int_a^b f d\alpha \tag{5.18}$$

From (5.17) and (5.18), we conclude that

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$ ■

Theorem 5.4. *If* $f_1(x) \leq f_2(x)$ *on* $[a,b]$, *then* $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha.$

Proof. Given that $f_1(x) \leq f_2(x)$ on $[a,b]$ which implies that $f_2(x) - f_1(x) \geq 0$ on $[a,b]$.

Since α is monotonically increasing on $[a,b]$ so that $\alpha(b) - \alpha(a) > 0$, then we have

$$\begin{aligned}
\int_a^b (f_2(x) - f_1(x))\, d\alpha &\geq 0 \\
\Rightarrow \int_a^b (f_2 - f_1)\, d\alpha &\geq 0 \\
\int_a^b f_2 d\alpha - \int_a^b f_1 d\alpha &\geq 0 \\
\Rightarrow \int_a^b f_2 d\alpha &\geq \int_a^b f_1 d\alpha \\
i.e., \int_a^b f_1 d\alpha &\leq \int_a^b f_2 d\alpha
\end{aligned}$$

This completes the proof of the theorem. ■

Theorem 5.5. *If* $f \in \mathscr{R}$ *on* $[a,b]$ *and if* $|f(x)| \leq M$ *on* $[a,b]$ *then* $\left|\int_a^b f d\alpha\right| \leq M\,[\alpha(b) - \alpha(a)]$.

Proof. Given that $f \in \mathscr{R}$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$; and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ $(i = 1, 2, \ldots, n)$.

$$\begin{aligned}
|U(P, f, \alpha)| &= \left|\sum_{i=1}^n M_i \triangle\alpha_i\right| \\
&\leq \sum_{i=1}^n |M_i| \triangle\alpha_i \\
&\leq \sum_{i=1}^n M \triangle\alpha_i \\
&\leq M \sum_{i=1}^n \triangle\alpha_i = M\,[\alpha(b) - \alpha(a)]
\end{aligned}$$

Taking infimum over all partitions P, then we have

$$\left|\int_a^b f d\alpha\right| \leq M\,[\alpha(b) - \alpha(a)]$$ ■

Theorem 5.6. *If* $f \in \mathscr{R}(\alpha_1)$ *on* $[a,b]$ *and* $f \in \mathscr{R}(\alpha_2)$ *on* $[a,b]$*, then* $f \in \mathscr{R}(\alpha_1+\alpha_2)$ *on* $[a,b]$.

Also, $\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$.

Proof. Given that $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$.

$$f \in \mathscr{R}(\alpha_1) \text{ if and only if } \exists \text{ a partition } P_1 \text{ on } [a,b] \text{ such that } U(P_1,f,\alpha_1) < \frac{\epsilon}{2} \tag{5.19}$$

$$f \in \mathscr{R}(\alpha_2) \text{ if and only if } \exists \text{ a partition } P_2 \text{ on } [a,b] \text{ such that } U(P_2,f,\alpha_2) < \frac{\epsilon}{2} \tag{5.20}$$

Let P be the common refinement of P_1 and P_2, then (5.19) and (5.20) also holds good.

$$\therefore \quad U(P,f,\alpha_1) - L(P,f,\alpha_1) < \frac{\epsilon}{2} \tag{5.21}$$

$$U(P,f,\alpha_2) - L(P,f,\alpha_2) < \frac{\epsilon}{2} \tag{5.22}$$

Since α_1 and α_2 is a monotonic increasing function and hence $\alpha_1+\alpha_2$ is also a monotonic increasing function. Take $\alpha = \alpha_1+\alpha_2$.

Now, consider

$$\begin{aligned}
\sum_{i=1}^{n} M_i\left[\alpha(x_i) - \alpha(x_{i-1})\right] &= \sum_{i=1}^{n} M_i\left[(\alpha_1+\alpha_2)(x_i) - (\alpha_1+\alpha_2)(x_{i-1})\right] \\
&= \sum_{i=1}^{n} M_i\left[(\alpha_1(x_i) - \alpha_1(x_{i-1})) + (\alpha_2(x_i) - \alpha_2(x_{i-1}))\right] \\
&= \sum_{i=1}^{n} M_i\left(\alpha_1(x_i) - \alpha_1(x_{i-1})\right) + \sum_{i=1}^{n} M_i\left(\alpha_2(x_i) - \alpha_2(x_{i-1})\right)
\end{aligned}$$

$$U(P,f,\alpha) = U(P,f,\alpha_1) + U(P,f,\alpha_2) \tag{5.23}$$

$$\text{Similarly,} \quad L(P,f,\alpha) = L(P,f,\alpha_1) + L(P,f,\alpha_2) \tag{5.24}$$

From (5.21) , (5.22), (5.23) and (5.24) we get

$$\begin{aligned}
U(P,f,\alpha) - L(P,f,\alpha) &= \left[U(P,f,\alpha_1) - L(P,f,\alpha_1)\right] + \left[U(P,f,\alpha_2) - L(P,f,\alpha_2)\right] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
\Rightarrow \quad U(P,f,\alpha) - L(P,f,\alpha) &< \epsilon \\
\Rightarrow \quad f \in \mathscr{R}(\alpha) \qquad & i.e., \ f \in \mathscr{R}(\alpha_1+\alpha_2)
\end{aligned}$$

Now, from (5.23), we have

$$\begin{aligned}
\inf U(P,f,\alpha) &= \inf\left[U(P,f,\alpha_1) + U(P,f,\alpha_2)\right] \\
\inf U(P,f,\alpha) &\geq \inf U(P,f,\alpha_1) + \inf U(P,f,\alpha_2) \\
i.e., \quad \int_a^b f d\alpha &\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2
\end{aligned}$$

Similarly from (5.24), we can easily find that

$$\int_a^b f d\alpha \leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

Thus, we have $\displaystyle\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$ ■

Theorem 5.7. *If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a,b]$ then $fg \in \mathscr{R}(\alpha)$ on $[a,b]$.*

Proof. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$, then by theorem (4.8), we have $\phi(f(x)) \in \mathscr{R}(\alpha)$ on $[a,b]$.

Let $\phi(t) = t^2$ *i.e.*, $\phi(f(x)) = [f(x)]^2$

Then $f \in \mathscr{R}(\alpha)$ on $[a,b] \Rightarrow [f(x)]^2 \in \mathscr{R}(\alpha)$ on $[a,b]$.

Also, we know that if $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $g \in \mathscr{R}(\alpha)$ on $[a,b]$ then $f+g \in \mathscr{R}(\alpha)$ on $[a,b]$ and $f-g \in \mathscr{R}(\alpha)$ on $[a,b]$.

Thus, we have $(f+g)^2 \in \mathscr{R}(\alpha)$ on $[a,b]$ and $(f-g)^2 \in \mathscr{R}(\alpha)$ on $[a,b]$.

From the identity

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

We can easily see that $fg \in \mathscr{R}(\alpha)$ on $[a,b]$. ■

Theorem 5.8. *If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then $|f| \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f|\, d\alpha$.*

Proof. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$, then by theorem (4.8), we have $\phi(f(x)) \in \mathscr{R}(\alpha)$ on $[a,b]$.

Let $\phi(t) = |t|$ *i.e.*, $\phi(f(x)) = |f(x)|$.

Then $f \in \mathscr{R}(\alpha)$ on $[a,b] \Rightarrow |f(x)| \in \mathscr{R}(\alpha)$ on $[a,b]$.

Choose $c = \pm 1$, so that $c\int_a^b f d\alpha \geq 0$. Then

$$\left|\int_a^b f d\alpha\right| = c\int_a^b f d\alpha = \int_a^b (cf) d\alpha \leq \int_a^b |f| d\alpha \qquad (\because cf \leq |f|)$$

This completes the proof. ■

Remark 5.2. Converse of the above theorem is not true. *i.e.*, $|f| \in \mathscr{R}(\alpha)$ does not imply $f \in \mathscr{R}(\alpha)$.

For example: Let $f(x) = \begin{cases} -1 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$

Here, we can easily see that $\displaystyle\int_a^b |f| dx$ exists, but $\displaystyle\int_a^b f(x) dx$ does not exist.

Definition 5.1. The unit step function I is defined by $I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$

Theorem 5.9. *If* $a < s < b$, f *is bounded on* $[a,b]$, f *is continuous at* s *and* $\alpha(x) = I(x-s)$, *then* $\int_a^b f d\alpha = f(s)$.

Proof. Let $a < s < b$ and f is bounded on $[a,b]$. Given that f is continuous at s.

Consider the partition $P = \{x_0, x_1, x_2, x_3\}$ of $[a,b]$

where $a = x_0 < x_1 = s < x_2 < x_3 = b$.

$$\begin{aligned}
M_i &= \sup\{f(x) : x_{i-1} \le x \le x_i\}; \quad i = 1,2,3 \\
m_i &= \inf\{f(x) : x_{i-1} \le x \le x_i\}; \quad i = 1,2,3 \\
\text{Then } U(P,f,\alpha) &= M_1\,[\alpha(x_1) - \alpha(x_0)] + M_2\,[\alpha(x_2) - \alpha(x_1)] + M_3\,[\alpha(x_3) - \alpha(x_2)] \\
&= M_1(0-0) + M_2(1-0) + M_3(1-1) = M_2 \\
\text{and } L(P,f,\alpha) &= m_1\,[\alpha(x_1) - \alpha(x_0)] + m_2\,[\alpha(x_2) - \alpha(x_1)] + m_3\,[\alpha(x_3) - \alpha(x_2)] \\
&= m_1(0-0) + m_2(1-0) + m_3(1-1) = m_2
\end{aligned}$$

Since f is continuous at s, $\lim\limits_{x_2 \to s} M_2 = f(s) = \lim\limits_{x_2 \to s} m_2$.

Thus $\sup\limits_P L(P,f,\alpha) = \lim\limits_{x_2 \to s} m_2 = f(s)$ and $\inf\limits_P U(P,f,\alpha) = \lim\limits_{x_2 \to s} M_2 = f(s)$.

i.e., $\overline{\int_a^b} f\,\mathrm{d}\alpha = f(s) = \underline{\int_a^b} f\,\mathrm{d}\alpha$

Hence $\int_a^b f d\alpha = f(s)$. ■

Theorem 5.10. *Suppose* $c_n \ge 0$ *for* $n = 1,2,3,\ldots,$ $\sum c_n$ *converges,* $\{s_n\}$ *is a sequence of distinct points in* (a,b) *and*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \tag{5.25}$$

Let f *be continuous on* $[a,b]$. *Then* $\int_a^b f d\alpha = \sum\limits_{n=1}^{\infty} c_n f(s_n)$.

Proof. Suppose $c_n \ge 0$ for $n = 1,2,3,\ldots,$ $\sum c_n$ converges. Let $\{s_n\}$ be a sequence of distinct points in (a,b) so that $\alpha(x) = \sum\limits_{n=1}^{\infty} c_n I(x-s_n)$ and f is continuous on $[a,b]$.

Since $c_n I(x - s_n) \le c_n \quad n = 1,2,\ldots$ and hence by comparison test, $\alpha(x) = \sum\limits_{n=1}^{\infty} c_n I(x-s_n)$ converges for every x.

Moreover, $\alpha(x)$ is monotonic and $\alpha(a) = 0$ and $\alpha(b) = \sum c_n$.

Since $I(a - s_n) = 0$ and $I(b - s_n) = 1$ for $n = 1,2,3,\ldots.$

Since $\sum c_n$ converges. Let $\epsilon > 0$ be given. Then there exists N such that $\sum\limits_{n=N+1}^{\infty} c_n < \epsilon$.

Now $\alpha(x) = \alpha_1(x) + \alpha_2(x)$. By theorem (5.6) and (5.9), we have

$$\int_a^b f d\alpha_1 = \sum_{n=N+1}^{\infty} c_n f(s_n),$$

and $\alpha(b) - \alpha(a) = \sum_{n=N+1}^{\infty} c_n - 0 = \sum_{n=N+1}^{\infty} c_n < \epsilon.$

Hence by theorem (5.5), we have

$$\left|\int_a^b f d\alpha_2\right| \leq M[\alpha_2(b) - \alpha_2(a)] < M\epsilon \qquad \text{where } M = \sup\{f(x) : x \in [a,b]\}$$

Since $\alpha_1 + \alpha_2 = \alpha$, then

$$\left|\int_a^b f d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right| = \left|\int_a^b f d\alpha - \int_a^b f d\alpha_1\right| = \left|\int_a^b f d\alpha_2\right| \leq M\epsilon$$

When $N \to \infty$, $\displaystyle\int_a^b f d\alpha = \sum_{n=1}^{\infty} f_n f(s_n).$ ■

Theorem 5.11. *Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case*

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx \tag{5.26}$$

Proof. Let α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$.

Let $\epsilon > 0$ be given and by theorem (4.1), there exists a partition $P = \{x_0, x_1, x_2, \ldots, x_n$ of $[a,b]$ such that

$$U(P, f, \alpha') - L(P, f, \alpha') < \epsilon \tag{5.27}$$

If s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then $\sum_{i=1}^{n} \left|\alpha'(s_i) - \alpha'(t_i)\right| \triangle\alpha_i < \epsilon$

By the mean value theorem, there are points $t_i \in [x_{i-1}x_i]$ such that $\triangle\alpha_i = \alpha'(t_i)\triangle x_i$.

Put $M = \sup|f(x)|$. Then we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} f(s_i)\triangle\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\triangle x_i\right| &= \left|\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\triangle x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\triangle x_i\right| \\
&\leq M\sum_{i=1}^{n} \left|\alpha'(s_i) - \alpha'(t_i)\right| \triangle\alpha_i \\
&< M\epsilon
\end{aligned}$$

In particular, $\sum_{i=1}^{n} f(s_i)\triangle\alpha_i \leq U(P, f, \alpha') + M\epsilon$ for all choices of $s_i \in [x_{i-1}, x_i]$.

So that, $U(P, f, \alpha) \leq U(P, f, \alpha') + M\epsilon.$

On the other hand, we have $U(P, f, \alpha') \leq U(P, f, \alpha) + M\epsilon.$

Thus, $|U(P, f, \alpha) - U(P, f\alpha')| \leq M\epsilon$ and for any partition P and so its refinement.

Hence $$\left|\overline{\int_a^b} f\,\mathrm{d}\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x\right| \leq M\epsilon$$

Since ϵ is arbitrary, it follows that
$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x.$$

Similarly, we get $$\underline{\int_a^b} f\,\mathrm{d}\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x.$$

Thus,
$$\underline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}\alpha = \int_a^b f d\alpha = \overline{\int_a^b} f\,\mathrm{d}\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,\mathrm{d}x$$

Hence $f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if $f\alpha' \in \mathscr{R}$ on $[a,b]$. *i.e.*, Riemann stieltjes integral of f with respect to α on $[a,b]$ is equal to Riemann integral of $f\alpha'$ on $[a,b]$. ∎

Theorem 5.12 (change of variable)**.** *Suppose ϕ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by*
$$\beta(y) = \alpha(\phi(y)), \qquad g(y) = f(\phi(y)) \tag{5.28}$$

Then $g \in \mathscr{R}(\beta)$ and
$$\int_A^B g d\beta = \int_a^b f d\alpha \tag{5.29}$$

Proof. Given that $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

Let $\epsilon > 0$ be given.

$$f \in \mathscr{R}(\alpha) \text{ on } [a,b] \text{ such that } U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \tag{5.30}$$

Hence $\left|U(P,f,\alpha) - \int_a^b f d\alpha\right| < \epsilon$ and $\left|L(P,f,\alpha) - \int_a^b f d\alpha\right| < \epsilon.$

To each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$, corresponds a partition $Q = \{y_0, y_1, \ldots, y_n\}$ of $[A,B]$, such that $x_i = \phi(y_i)$ for $i = 1,2,\ldots,n.$ All partitions of $[A,B]$ are obtained in this way.

For $i = 1,2,\ldots,n,$ let
$$\begin{aligned} M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \\ m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\} \\ M_i^* &= \sup\{g(y) : y_{i-1} \leq y \leq y_i\} \\ m_i^* &= \inf\{g(y) : y_{i-1} \leq y \leq y_i\} \end{aligned}$$

Since $M_i^* = g(y_i^*) = f\left(\phi(y_i^*)\right) = f(x_i^*) = M_i$ and

$\triangle\beta_i = \beta(y_i) - \beta(y_{i-1}) = \alpha(\phi(y_i)) - \alpha(\phi(y_{i-1})) = \alpha(x_i) - \alpha(x_{i-1}) = \triangle\alpha_i.$

Then $U(Q,g,\beta) = \sum_{i=1}^{n} M_i^* \triangle\beta_i = \sum_{i=1}^{n} M_i \triangle\alpha_i = U(P,f,\alpha)$.

Similarly, we can get $L(Q,g,\beta) = L(P,f,\alpha)$.

Hence by (5.30), we have

$$U(Q,g,\beta) - L(Q,g,\beta) = U(P,f\alpha) - L(P,f,\alpha) < \epsilon$$

Thus, $f \in \mathscr{R}(\beta)$ and further,

$$\begin{aligned}
\overline{\int_A^B} g\,\mathrm{d}\beta &= \inf_Q U(Q,g,\beta) = \inf_P U(P,f,\alpha) = \overline{\int_a^b} f\,\mathrm{d}\alpha = \int_a^b f d\alpha \\
\underline{\int_A^B} g\,\mathrm{d}\beta &= \sup_Q L(Q,g,\beta) = \sup_P L(P,f,\alpha) = \underline{\int_a^b} f\,\mathrm{d}\alpha = \int_a^b f d\alpha \\
\text{Hene, } \overline{\int_A^B} g\,\mathrm{d}\beta &= \underline{\int_A^B} g\,\mathrm{d}\beta = \int_A^B g d\beta = \int_a^b f d\alpha
\end{aligned}$$

So, $g \in \mathscr{R}(\beta)$ on $[A,B]$ and $\int_A^B g d\beta = \int_a^b f d\alpha$. ■

5.2. Integration and Differentiation:

For real functions integration and differentiation are in a certain sense, inverse operations. In this section, let us establish this result.

Theorem 5.13. *Let $f \in \mathscr{R}$ on $[a,b]$. For $a \le x \le b$, put*

$$F(x) = \int_a^x f(t)dt \tag{5.31}$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0) \tag{5.32}$$

Proof. Since $f \in \mathscr{R}$ on $[a,b]$ and hence it is bounded.

Therefore, $\exists$ a real number M such that

$$|f(t)| \le M \quad \text{for } a \le t \le b$$

If $a \le x \le y \le b$, then

$$\begin{aligned}
|F(y) - F(x)| &= \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right| \\
&= \left| \int_a^x f(t)dt + \int_x^y f(t)dt - \int_a^x f(t)dt \right| \\
&= \left| \int_x^y f(t)dt \right| \le \int_x^y |f(t)|\, dt \\
&\le M(y-x)
\end{aligned}$$

For given $\epsilon > 0$, we have

$$\begin{aligned}
|y - x| < \frac{\epsilon}{M} &\Rightarrow |F(y) - F(x)| < \epsilon \\
&\Rightarrow F \text{ is uniformly continuous on } [a,b] \\
&\Rightarrow F \text{ is continuous.}
\end{aligned}$$

Suppose, f is continuous at a point x_0 of $[a,b]$. Given $\epsilon > 0$, we can choose a $\delta > 0$ such that $a \le t \le b$ with

$$|t - x| < \delta \Rightarrow |f(t) - f(x_0)| < \epsilon$$

Therefore, if $x_0 - \delta \le s \le x_0 + \delta$ and $a \le s \le t \le b$, we have

$$\begin{aligned}
\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| &= \left| \frac{1}{t-s} \int_s^t f(u)du - f(x_0) \right| \\
&= \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)]\, du \right| \\
&\le \frac{1}{t-s} \int_s^t |f(u) - f(x_0)|\, du \\
&< \epsilon \frac{1}{t-s} \int_s^t du = \epsilon
\end{aligned}$$

Hence $F'(x_0) = f(x_0)$. ■

Theorem 5.14 (The fundamental theorem of calculus). *If $f \in \mathscr{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$ then*

$$\int_a^b f(x)dx = F(b) - F(a) \tag{5.33}$$

Proof. Let f be continuous on $[a,b]$ and $F'(x) = f(x)$ $\forall x \in [a,b]$.

Since $f \in \mathscr{R}$ on $[a,b]$ and hence $F' \in \mathscr{R}$ on $[a,b]$.

Let $\epsilon > 0$ be given.

Hence, by theorem (4.4), $\exists$ a partition P of $[a,b]$ such that

$$\left| \sum_{i=1}^{n} F'(t_i)(x_i - x_{i-1}) - \int_a^b F'(x)dx \right| < \epsilon \quad \text{where} \quad t_i \in [x_{i-1}, x_i] \tag{5.34}$$

By Lagrange's mean value theorem, we can say that there exists $t_i \in [x_{i-1}, x_i]$ such that

$$
\begin{aligned}
F(x_i) - F(x_{i-1}) &= (x_i - x_{i-1})F'(t_i) \\
\Rightarrow \sum_{i=1}^{n} [(x_i - x_{i-1})F'(t_i)] &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = F(b) - F(a)
\end{aligned}
$$

Using the above equations in (5.34), then we have

$$
\begin{aligned}
\left| F(b) - F(a) - \int_a^b F'(x)dx \right| &< \epsilon \\
\Rightarrow F(b) - F(a) &= \int_a^b F'(x)dx \\
&= \int_a^b f(x)dx \qquad (\because F'(x) = f(x)) \\
\text{Hence, } \int_a^b f(x)dx &= F(b) - F(a)
\end{aligned}
$$

■

Theorem 5.15 (integration by parts)**.** *Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$ and $G' = g \in \mathscr{R}$. Then*

$$
\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx. \tag{5.35}
$$

Proof. Let F and G be differentiable functions on $[a, b]$ so that $F' = f \in \mathscr{R}$ on $[a, b]$ and $G' = g \in \mathscr{R}$ on $[a, b]$.

$$
\begin{aligned}
\text{Let } H(x) &= F(x)G(x) \\
\text{then } H'(x) &= F(x)G'(x) + F'(x)G(x) \\
&= F(x)g(x) + f(x)G(x)
\end{aligned}
$$

Hence, by fundamental theorem of calculus, we have

$$
\begin{aligned}
\int_a^b H'(x)dx &= \int_a^b [F(x)g(x) + f(x)G(x)]\, dx \\
&= H(b) - H(a) \\
\text{i.e., } \int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx &= F(b)G(b) - F(a)G(a) \\
\text{i.e., } \int_a^b F(x)g(x)dx &= F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx
\end{aligned}
$$

■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- properties of Riemann Stieltjes integral.
- change of variables.
- integration by parts.

- fundamental theorem of calculus.

Check Your Progress:

1. Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$, if $x \ne x_0$. Prove that $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

2. Suppose $f \ge 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$, prove that $f(x) = 0$ for all $x \in [a,b]$.

3. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

4. Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follows that $f \in \mathscr{R}$?

Choose the correct or more suitable answer:

1. The value of $\int_0^1 x^2 dx^2$ is
   (a) 1
   (b) -1
   (c) $\frac{1}{3}$
   (d) $\frac{1}{2}$

2. The value of $\int_0^2 [x] dx^2$ is
   (a) 1
   (b) 2
   (c) 3
   (d) 4

Answer:

(1) d (2) c

Glossaries:

Change of variables: It is a basic technique used to simplifying problems in which original variables are replaced with functions of other variables.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.

2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-II

Unit-6: Uniform Convergent of Sequence and Series of functions.

Unit-7: Equicontinuous families of functions.

Block-II

UNIT-6

UNIFORM CONVERGENT OF SEQUENCE AND SERIES OF FUNCTIONS

Structure

Objectives

After completion of this unit, students will be able to

- ★ classify pointwise convergent and uniform convergent sequence of functions.
- ★ understand the concept of term by term differentiation for uniform convergent series.
- ★ construct a continuous function which is nowhere differentiable on real line.

Overview

In this unit we focus our attention to complex-valued functions (including the real-valued functions), although many of the theorems and proofs which follow extend without difficulty to vector valued functions and even to mappings into general metric spaces.

6.1. Discussion of Main problem:

Definition 6.1. Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots,$ is a sequence of functions defined on a set E and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$, we can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E) \tag{6.1}$$

We say that $\{f_n\}$ converges on E and that f is the *limit* or the *limit function*, of $\{f_n\}$. We say that $\{f_n\}$ converges to f *pointwise* on E, if (6.1) holds.

Definition 6.2. Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in E$ and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E) \tag{6.2}$$

the function f is called the *sum* of the series $\sum f_n$

Remark 6.1. If the point-wise limit of a sequence of functions $\{f_n\}$ defined on $[a, b]$, then to each $\epsilon > 0$ and to each $x \in [a, b]$, there corresponds an integer N such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$$

The main problem which arises is to determine whether important properties of functions (such as boundedness, continuity, integration, differentiation, etc.,) are preserved under the limit operations (6.1) and (6.2).

Now, we shall discuss by means of several examples that limit process cannot in general be interchanged without affecting the result.

Example 6.1. Consider the double sequence:

For $m = 1, 2, 3, \ldots,\ n = 1, 2, 3, \ldots.$ Let

$$s_{m,n} = \frac{m}{m+n}$$

Then for any fixed n, we have $\lim\limits_{m\to\infty} s_{m,n} = 1.$

So that $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} s_{m,n} = 1.$

But, on the other hand, for every fixed m, we have

$$\lim_{n\to\infty} s_{m,n} = 0$$

$$\text{so that } \lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0$$

Thus, $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} s_{m,n} \neq \lim\limits_{m\to\infty} \lim\limits_{n\to\infty} s_{m,n}.$

Hence, we conclude that *limit process cannot be interchanged in general without affecting the results in double sequence*

Example 6.2. Consider the series $\sum\limits_{n=0}^{\infty} f_n$, where $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$ (x real).

At $x = 0$, each $f_n(x) = 0$, so that the sum of the series $f(0) = 0$.

For $x \neq 0$, it forms a geometric series with common ratio $\dfrac{1}{1+x^2}$, so that its sum is $f(x) = 1 + x^2$.

Hence $f(x) = \begin{cases} 1 + x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$

Each term of the series is continuous but the sum f is not continuous.

Thus, we conclude that *a convergent series of continuous function may have discontinuous sum.*

Example 6.3. Consider the sequence of functions $\{f_m(x)\}$. For $m = 1, 2, 3, \ldots,$

$$f_m(x) = \lim_{n\to\infty} (\cos m!\,\pi x)^{2n} = \begin{cases} 1 & m!\,x \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

Let $f(x) = \lim\limits_{n\to\infty} f_m(x)$.

For irrational x, we have $m!\,x$ is not an integer, so $f_m(x) = 0 \ \forall m$ and hence $f(x) = 0$.

For rational x, put $x = \dfrac{p}{q}$, and $m!\,x$ is an integer, when $m \geq q$, so $f_m(x) = 1$ and hence $f(x) = 1$.

Thus, $f(x) = \lim_{m\to\infty}\left[\lim_{n\to\infty}(\cos m!\,\pi x)^{2n}\right] = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$

which is not Riemann integrable[Refer Page No.62], but for each m, $f_m(x)$ is Riemann integrable.

Hence, limit of a sequence of Riemann integrable function is need not be Riemann integrable.

Thus, we conclude that *limits and integration cannot be changed*.

Example 6.4. The sequence f_n where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ (x real) has the limit

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) = 0 \\ \therefore \quad f'(x) &= 0 \text{ and so } f'(0) = 0 \\ \text{But} \quad f_n'(x) &= \sqrt{n}\cos nx \\ \text{so that} \quad f_n'(0) &= \sqrt{n} \to \infty \quad \text{as } n \to \infty \end{aligned}$$

Hence at $x = 0$, the sequence $\{f_n'(x)\}$ diverges whereas the limit function $f'(x) = 0$.

Thus, we conclude that *the limit of differentials is not equal the differential of the limit*.

Example 6.5. Consider the sequence $\{f_n\}$ where

$$f_n(x) = nx\left(1 - x^2\right)^n \quad 0 \leq x \leq 1;\ n = 1, 2, 3, \ldots \tag{6.3}$$

For $0 < x \leq 1$, $\lim_{n\to\infty} f_n(x) = 0$.

At $x = 0$, each $f_n(0) = 0$, so that $\lim_{n\to\infty} f_n(0) = 0$.

$$\begin{aligned} \text{Thus, the limit function } f(x) &= \lim_{n\to\infty} f_n(x) = 0, \quad \text{for } 0 < x <\leq 1 \\ \therefore \quad \int_0^1 f(x)dx &= 0 \\ \text{Again,} \quad \int_0^1 f_n(x)dx &= \int_0^1 nx\left(1 - x^2\right)^n dx = \frac{n}{2n+2} \\ \text{so that} \quad \lim_{n\to\infty}\left\{\int_0^1 f_n(x)dx\right\} &= \frac{1}{2} \\ ]\text{Thus,} \quad \lim_{n\to\infty}\left\{\int_0^1 f_n(x)dx\right\} &\neq \int_0^1 f(x)dx = \int_0^1 \left[\lim_{n\to\infty}\{f_n\}(x)dx\right] \end{aligned}$$

Thus, the limit of integral is not equal to the integral of the limit.

Hence, we conclude that *the sequence of integrals may not converge to the integral of the limit of the sequence*.

These examples, which show what can go wrong if limit process are interchanged carelessly. We have to investigate under what conditions these or other properties of the terms f_n are transferred to the limit function. A concept of great importance in this regard is known as *Uniform convergence of a sequence (series)*.

6.2. Uniform Convergence:

Definition 6.3. A sequence of functions $\{f_n\}$, $n = 1, 2, 3, \ldots$ converges *uniformly* on E to a function f if for $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon \tag{6.4}$$

for all $x \in E$.

From the above definition, it is very clear that every uniformly convergent sequence is pointwise convergent.

Remark 6.2. The main difference between the pointwise convergent and uniform convergent is as follows:

If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\epsilon > 0$ and for every $x \in E$, there is an integer N, depending on ϵ and on x, such that (6.4) holds if $n \geq N$.

If $\{f_n\}$ converges uniformly on E, it is possible to for each $\epsilon > 0$ to find one integer N, which do for all $x \in E$.

Definition 6.4. The series $\sum f_n(x)$ *converges uniformly* on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

Theorem 6.1 (Cauchy criterion for uniform convergence)**.** *The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

$$|f_n(x) - f_m(x)| < \epsilon$$

Proof. Assume that $\{f_n\}$ converges uniformly on E and $f(x)$ be the limit function.

Let $\epsilon > 0$ be given.

Then by definition, there exists an integer N such that

$$n \geq N, x \in E \quad \Rightarrow \quad |f_n(x) - f(x)| < \epsilon/2 \tag{6.5}$$

If $n \geq N, m \geq N, x \in E$, then by (6.5), we have

$$\begin{aligned}|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

Converse Part: Assume that for every $\epsilon > 0$, $\exists$ an integer N such that $m \ge N$, $n \ge N, x \in E$ implies

$$|f_n(x) - f_m(x)| < \epsilon \tag{6.6}$$

Thus, for every $x \in E$, $\{f_n\}$ is a Cauchy sequence and hence converges to a limit function $f(x)$. Hence the sequence of functions $\{f_n\}$ converges uniformly to f on E.

Certainly, let $\epsilon > 0$ be given and choose N such that (6.6) holds.

Fix n and letting $m \to \infty$ in (6.6), as $f_m(x) \to f(x)$, it follows that

$$x \in E \text{ and } n \ge N \Rightarrow \quad |f_n(x) - f(x)| < \epsilon$$

This completes the proof. ■

Theorem 6.2. *Suppose* $\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E)$.

Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$.

Then $f_n \to f$ *converges uniformly on* E *if and only if* $M_n \to 0$ *as* $n \to \infty$.

Proof. Suppose the sequence $\{f_n\}$ of functions converges uniformly to f on X. Then by definition, for a given $\epsilon > 0$, $\exists$ a positive integer N such that

$$\begin{aligned} n \ge N &\Rightarrow |f_n(x) - f(x)| < \epsilon, \quad \forall x \in X \\ \text{Also, } M_n &= \sup_{x\in E} |f_n(x) - f(x)| \\ \therefore\ |f_n(x) - f(x)| &< \epsilon \ \forall\ n \ge N, \ \forall x \in X \\ \Rightarrow \quad M_n &= \sup_{x\in E} |f_n(x) - f(x)| < \epsilon \quad \forall n \ge N \\ \Rightarrow \quad M_n \to 0 &\quad \text{as} \quad n \to \infty \end{aligned}$$

Converse Part: Suppose $M_n \to 0$ as $n \to \infty$.

Let $\epsilon > 0$ be given. Then there exists N such that

$$n \ge N \Rightarrow \quad M_n = \sup_{x\in E} |f_n(x) - f(x)| \quad < \quad \epsilon.$$

$$\text{Hence } n \ge N, x \in E \quad \Rightarrow |f_n(x) - f(x)| \quad < \quad \epsilon.$$

Thus, $f_n \to f$ uniformly on E. Hence the proof. ■

Theorem 6.3 (Weierstrass M-test for uniform convergence)**.** *Suppose* $\{f_n\}$ *is a sequence of functions defined on* E *and suppose*

$$|f_n(x)| \le M_n \qquad (x \in E, n = 1, 2, 3, \ldots) \tag{6.7}$$

Then $\sum f_n$ *converges uniformly on* E *if* $\sum M_n$ *converges.*

Proof. Assume that $\sum M_n$ converges.

Let $\epsilon > 0$ be given and $S_n(x) = \sum_{i=1}^{n} f_i(x)$.

Since $\sum M_n$ converges, by definition there exists an integer N such that

$$m \geq n \geq N \Rightarrow \sum_{i=n+1}^{m} M_i < \epsilon$$

But, if $m \geq n \geq N$ and $x \in E$, then $\left|\sum_{i=n+1}^{m} f_i(x)\right| \leq \sum_{i=n+1}^{m} |f_i(x)| \leq \sum_{i=n+1}^{m} M_i < \epsilon$

$$i.e., \quad m \geq n \geq N \text{ and } x \in E \Rightarrow |s_m(x) - s_n(x)| < \epsilon \tag{6.8}$$

This implies that, if $x \in E$, $\{s_n(x)\}$ is a Cauchy sequence and hence converges to a limit function say $f(x)$.

Keeping m fixed and let $n \to \infty$ in (6.6), we get

$$\text{if } m \geq N, x \in E, \ |s_m(x) - f(x)| < \epsilon.$$

Thus, $\{s_n(x)\}$ converges uniformly on E and hence $\sum f_n(x)$ converges uniformly on E. ∎

Example 6.6. Let the function f_n defined by $f_n : R \to R$ such that $f_n = \frac{x}{n}$ $\forall x \in R$, $n = 1, 2, 3, \ldots$. Show that the sequence $\{f_n\}$ converges pointwise to the zero function.

Solution: We want to show that the sequence $\{f_n\}$ converges pointwise to the function
$f(x) = 0\ \forall x \in R$.

Let $\epsilon > 0$ be given, we can find m such that

$$\forall n \geq m \quad \Rightarrow \quad \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} \tag{6.9}$$

Let us choose $m > \frac{|x|}{\epsilon}$. Then (6.9) gives

$$\forall n \geq m \quad \Rightarrow \quad \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} < \epsilon$$

Hence, the sequence $\{f_n\}$ converges pointwise to the zero function.

Example 6.7. Show that the sequence $\{f_n\} = \{nx(1-x)^n\}$ does not converge uniformly on $[0, 1]$.

Solution:

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \\ &= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} \\ &= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)} = 0 \\ \Rightarrow f(x) &= 0 \quad \forall x \in [0,1] \\ \text{Now, } M_n &= \sup\{|f_n(x) - f(x)| : x \in [0,1]\} \\ &= \sup\{nx(1-x)^n : x \in [0,1]\} \end{aligned}$$

Taking $x = \frac{1}{n} \in [0,1]$, we have

$$\begin{aligned} M_n &\geq n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^n \\ &\to \frac{1}{e} \quad \text{as } n \to \infty. \end{aligned}$$

Hence, Weierstrass M test, $\{f_n\}$ does not converge uniformly on $[0,1]$.

Example 6.8. Show that the sequence of functions $f_n(x) = \frac{x}{1+nx^2}$ $\forall x \in R$ converges uniformly on R.

Solution:

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \quad \forall x \in R \\ M_n &= \sup_{x\in R}\{|f_n(x) - f(x)|\} \\ &= \sup\left\{\left|\frac{x}{1+nx^2} - 0\right|\right\} \\ &= \max\left\{\frac{x}{1+nx^2}\right\} = \frac{1}{2\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Hence, Weierstrass M test, the sequence $f_n(x) = \frac{x}{1+nx^2}$ $\forall x \in R$ converges uniformly on R.

6.3. Uniform convergence and Continuity:

Theorem 6.4. *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that*

$$\lim_{t\to x} f_n(t) = A_n, \quad (n = 1, 2, 3, \ldots) \tag{6.10}$$

$$\textit{Then } \{A_n\} \textit{ converges, and } \lim_{t\to x} f(t) = \lim_{n\to\infty} A_n \tag{6.11}$$

In other words, $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$

Proof. Let $\epsilon > 0$ be given.

Since $f_n \to f$ converges uniformly on E. By Cauchy's condition for uniform convergence, there exists N such that

$$n \geq N, \quad m \geq N, t \in E \quad \Rightarrow \quad |f_n(t) - f_m(t)| < \epsilon \tag{6.12}$$

Suppose that $\lim_{t\to x} f_n(t) = A_n$, $n = 1, 2, 3, \ldots$ and x is a limit point of E.

Letting $t \to x$ in (6.12), we get $|A_n - A_m| < \epsilon \quad n \geq N, \ m \geq N.$

Thus, $\{A_n\}$ is a Cauchy sequence and therefore converges to A (say).

$$\text{Next} \quad |f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \tag{6.13}$$

Since $f_n \to f$ uniformly on E, then by definition $\exists\ N_1$ such that

$$n \geq N_1, \quad t \in E \quad \Rightarrow |f_n(t) - f(t)| < \epsilon/3 \tag{6.14}$$

Since $A_n \to A$, then by definition, $\exists\ N_2$ such that

$$n \geq N_2 \quad \Rightarrow |A_n - A| < \epsilon/3 \tag{6.15}$$

Choose $N_0 = \max\{N_0, N_1\}$. Then (6.14) and (6.15) are holds good for N_0.

Then for this N_0, we choose a neighborhood V of x such that

$$|f_{N_0}(t) - A_{N_0}| \leq \epsilon/3 \quad \text{if } t \in V \cup E, \quad t \neq x \tag{6.16}$$

Substituting (6.14), (6.15) and (6.16) in (6.13), we get

$$|f(t) - A| < \epsilon \qquad \text{provided } t \in V \cap E, \quad t \neq x$$

Hence, $\lim_{t\to x} f(t) = A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$

Thus, $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$. This completes the proof. ■

Theorem 6.5. *If* $\{f_n\}$ *is a sequence of continuous functions on* E *and if* $f_n \to f$ *uniformly on* E*, then* f *is continuous on* E.

Proof. For each $n = 1, 2, 3, \ldots$ since f_n is continuous on E. Then

$$\text{for each } x \in E, \quad \lim_{t\to x} f_n(t) = f_n(x)$$

By theorem (6.4), we have

$$\begin{aligned}
\lim_{t\to x} f(t) &= \lim_{t\to x}\lim_{n\to\infty} f_n(t) \\
&= \lim_{n\to\infty}\lim_{t\to x} f_n(t) \\
&= \lim_{n\to\infty} f_n(x) = f(x) \\
\text{i.e.,}\quad \lim_{t\to x} f(t) &= f(x)
\end{aligned}$$

Thus, f is continuous at x and hence f is continuous on E. This completes the proof. ■

Remark 6.3. The converse of the above theorem is not true. *i.e.*, a limit of a continuous function is continuous, but not uniformly convergent. This will be explained in the following example.

Example 6.9. Consider the function $f_n(x) = n^2x(1-x)^n$, $0 \le x \le 1$ for each $n = 1, 2, 3, \ldots$. Show that the limit function is continuous, but $\{f_n\}$ is not uniformly convergent to f.

Solution: We have $\lim_{n\to\infty} n^2x(1-x)^n = 0 \quad (0 \le x \le 1)$.

Thus, $\{f_n\}$ converges to 0 on $[0,1]$ which is also a continuous function.

$$\begin{aligned}
Mn &= \sup_{0\le x\le 1} |f_n(x) - f(x)| = \sup_{0\le x\le 1} n^2x(1-x)^n \\
&= \max\{n^2x(1-x)^n\} = n\left(\frac{n}{n+1}\right)^{n+1} \\
\text{and}\quad \lim_{n\to\infty} M_n &= \lim_{n\to\infty} n\left(1 - \frac{1}{n+1}\right)^{n+1} = \lim_{n\to\infty} \frac{n}{e} = \infty \neq 0
\end{aligned}$$

Therefore, by Weierstrass's M-test, the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Hence, $\{f_n\}$ converges to a continuous limit function on $[0,1]$, but $\{f_n\}$ does not converges uniformly on $[0,1]$.

Theorem 6.6. *Suppose K is compact and*

(a) $\{f_n\}$ is a sequence of continuous function on K.

(b) $\{f_n\}$ converges pointwise to a continuous function f on K.

(c) $f_n \ge f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly on K.

Proof. Let $\epsilon > 0$ be given.

Put $g_n = f_n - f$. Then by the given conditions, g_n is continuous and $g_n \to 0$ pointwise on compact K.

Now, it remains to prove that $g_n \to 0$ uniformly on K.

Let $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.

Since each g_n is continuous and also K_n is a closed subset of K.

Since, closed subset of a compact set is compact and hence K_n is compact.

Since $g_n \geq g_{n+1}$, it follows that $K_n \supset K_{n+1}$ (n=1,2,3,...).

Fix $x \in K$, since $g_n(x) \to 0$, $x \notin K_n$, if n is sufficiently large.

Thus $x \notin \bigcap_{n=1}^{\infty} K_n$.

In other words, $\bigcap_{n=1}^{\infty} K_n$ is empty.

We know that if $\{K_n\}$ is a sequence of non-empty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Thus, it follows that K_N is empty for some N.

i.e., $0 \leq g_n(x) < \epsilon$ for all $x \in K$ and for all $n \geq N$.

This shows that $g_n \to 0$ uniformly. *i.e.*, f_n converges uniformly to f on K. This completes the proof of the theorem. ■

Remark 6.4. The following example explains how the importance of compactness is needed in the hypothesis of the above theorem.

Example 6.10. Consider the function $f_n(x) = \dfrac{1}{nx+1}$ $\quad (0 < x < 1;\ n = 1, 2, 3, \ldots)$.

Then $f_n(x) \to 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

Choose $\epsilon = \dfrac{1}{2}$ and given n, choose $x = \dfrac{1}{2n}$ in $(0, 1)$. Then

$$|f_n(x) - f(x)| = \left| \frac{1}{1 + \frac{n}{2n}} - 0 \right| = \frac{2}{3} > \frac{1}{2} = \epsilon$$

Thus, for given $= \dfrac{1}{2} > 0$ there exist n such that for every $x \in E$, $|f_n(x) - f(x)| > \epsilon$.

i.e., The sequence $\{f_n\}$ does not converge uniformly in $(0, 1)$, which is not compact.

Definition 6.5. Let X be a metric space. Then $\mathscr{C}(X)$ will denote the set of all complex valued, continuous, bounded functions with domain X.

Definition 6.6. If X is a compact, then $\mathscr{C}(X)$ consists of all continuous functions on X.

Supremum norm on $\mathscr{C}(X)$ is defined by

$$\|f\| = \sup_{x \in X} |f(x)| \quad \text{if } f \in \mathscr{C}(X)$$

It is well defined, since X is a compact.

Example 6.11. If X is a compact metric space, sup norm defines a metric space on $\mathscr{C}(X)$ by $d(f, g) = \|f - g\|$, where $\|f\| = \sup_{x \in X} |f(x)|$. Then prove that $\mathscr{C}(X)$ is a metric space.

Proof.

(a) $d(f,g) = \|f - g\| \geq 0$, $\quad [\because |f(x)| \geq 0 \text{ for every} x \in X.]$

(b) $d(f,g) = 0 \Leftrightarrow \sup_{x\in X}|f(x) - g(x)| = 0 \Leftrightarrow f(x) = g(x) \text{ for every} x \in X.$

(c) $d(f,g) = \|f - g\| = \sup_{x\in X}|f(x) - g(x)| = \sup_{x\in X}|g(x) - f(x)| = d(g,f).$

(d) $d(f,g) = \sup_{x\in X}|f(x) - g(x)| = \sup_{x\in X}|f(x) - h(x) + h(x) - g(x)|$

$\leq \sup_{x\in X}|f(x) - h(x)| + \sup_{x\in X}|h(x) - g(x)| = d(f,h) + d(h,g).$ ■

Theorem 6.7. *A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Proof. Let $\{f_n\}$ be a sequence of functions in $\mathscr{C}(X)$.

We know that By Weierstrass's M test for uniform convergence of function, $f_n \to f$ uniformly on X if and only if $M_n \to 0$ as $n \to \infty$.

Then $\{f_n\}$ converges uniformly to f on E
$$\begin{aligned} &\Leftrightarrow M_n \to 0 \text{ as } n \to \infty \\ &\Leftrightarrow \sup_{x\in X}|f_n(x) - f(x)| \to 0 \text{ as } n \to \infty \\ &\Leftrightarrow \|f_n - f\| \to 0 \text{ as } n \to \infty \\ &\Leftrightarrow f_n \to f \text{ in } \mathscr{C}(X) \end{aligned}$$ ■

Theorem 6.8. *The metric space $\mathscr{C}(X)$ is a complete metric space.*

Proof. We know that a metric space (X,d) is said to be complete if every Cauchy sequence in X converges.

Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. Then by definition, for given $\epsilon > 0$, there exists an integer N such that

$$\begin{aligned} n \geq N,\ m \geq N &\Rightarrow \|f_n - f_m\| < \epsilon. \\ n \geq N,\ m \geq N &\Rightarrow \sup_{x\in X}|f_n - f_m| < \epsilon. \\ n \geq N,\ m \geq N &\Rightarrow |f_n - f_m| < \epsilon. \end{aligned}$$

Thus by Cauchy's criteria, $\{f_n\}$ converges uniformly to f (say).

It remains to prove that $f \in \mathscr{C}(X)$.

Since $\{f_n\}$ is of continuous function, f is also continuous and also it is bounded.

For if, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$ and f_n is bounded.

$$\begin{aligned} |f(x)| &= |f(x) - f_N(x) + f_N(x)| \\ &< |f(x) - f_N(x)| + |f_N(x)| \\ &< 1 + |f_N(x)| \end{aligned}$$

and hence f is bounded.

Thus $f \in \mathscr{C}(X)$ and hence $f_n \to f$ in $\mathscr{C}(X)$ and hence $\mathscr{C}(X)$ is complete. ■

6.4. Uniform convergence and Integration:

Theorem 6.9. *Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, for $n = 1,2,3,\ldots$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and*

$$\int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n d\alpha \tag{6.17}$$

Proof. suppose that α is monotonically increasing on $[a,b]$; $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ for $n = 1,2,3,\ldots.$

It suffices to prove the theorem for the case of real f_n.

$$\begin{aligned} \text{Put } \epsilon_n &= \sup_{a\le x\le b}|f_n(x) - f(x)| \\ &= \sup_{a\le x\le b}|f(x) - f_n(x)| \\ \Rightarrow |f(x) - f_n(x)| &< \epsilon_n \\ \Rightarrow -\epsilon_n &< f(x) - f_n(x) < \epsilon_n \\ \Rightarrow f_n(x) - \epsilon_n &< f(x) < f_n(x) + \epsilon_n \end{aligned}$$

Since $f \in \mathscr{R}(\alpha)$ on $[a,b]$, so that the upper and lower integral of f satisfy

$$\int_a^b (f_n - \epsilon_n)\,d\alpha \le \underline{\int_a^b} f(x)\,\mathrm{d}\alpha \le \overline{\int_a^b} f(x)\,\mathrm{d}\alpha \le \int_a^b (f_n + \epsilon_n)\,d\alpha \tag{6.18}$$

$$\Rightarrow 0 \le \overline{\int_a^b} f(x)\,\mathrm{d}\alpha - \underline{\int_a^b} f(x)\,\mathrm{d}\alpha \le \int_a^b [f_n + \epsilon_n - f_n + \epsilon_n]\,d\alpha$$

$$\Rightarrow 0 \le \overline{\int_a^b} f(x)\,\mathrm{d}\alpha - \underline{\int_a^b} f(x)\,\mathrm{d}\alpha \le 2\epsilon_n\int_a^b d\alpha = 2\epsilon_n\,[\alpha(b) - \alpha(a)]$$

Letting $\epsilon_n \to \infty$, then the upper integral and lower integral are equal.

$$\begin{aligned} i.e., \quad \overline{\int_a^b} f(x)\,\mathrm{d}\alpha &= \underline{\int_a^b} f(x)\,\mathrm{d}\alpha \\ \Rightarrow \quad f \in \mathscr{R}(\alpha) \end{aligned}$$

Hence, (6.18) can be written as

$$\begin{aligned}
\int_a^b (f_n - \epsilon_n)\, d\alpha &\leq \int_a^b f(x)\, \mathrm{d}\alpha \leq \int_a^b (f_n + \epsilon_n)\, d\alpha \\
\Rightarrow \int_a^b f_n d\alpha - \epsilon_n [\alpha(b) - \alpha(a)] &\leq \int_a^b f(x) d\alpha \leq \int_a^b f_n d\alpha + \epsilon_n [\alpha(b) - \alpha(a)] \\
\Rightarrow -\epsilon_n [\alpha(b) - \alpha(a)] &\leq \int_a^b f(x) d\alpha - \int_a^b f_n d\alpha \leq \epsilon_n [\alpha(b) - \alpha(a)] \\
\Rightarrow \left| \int_a^b f(x) d\alpha - \int_a^b f_n d\alpha \right| &\leq \epsilon_n [\alpha(b) - \alpha(a)]
\end{aligned}$$

Letting, $n \to \infty$, we get

$$\int_a^b f(x) dx = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

This completes the proof of the theorem. ■

Corollary 6.1. *If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ and if*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \leq x \leq b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$$

In other words, the series may be integrated term by term.

Proof. Let $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, $n = 1, 2, 3, \ldots$ and $f(x) = \sum f_n(x)$, $a \leq x \leq b$ the series converges uniformly on $[a, b]$.

Let $s_k(f) = \sum_{n=1}^{k} f_n(x)$,

Then $\{s_k(f)\}$ be the sequence of partial sums converges uniformly to f on $[a, b]$.

Hence, by above theorem $f(x) \in \mathscr{R}(\alpha)$ on $[a, b]$ and

$$\begin{aligned}
\int_a^b f d\alpha &= \lim_{k\to\infty} \int_a^b s_k d\alpha \\
&= \lim_{k\to\infty} \int_a^b \sum_{n=1}^{k} f_n d\alpha \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \int_a^b f_n d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha
\end{aligned}$$

■

6.5. Uniform convergence and Differentiation:

We have already seen in Example (6.4) that uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$. Thus strong hypothesis is required for the claim that $f_n' \to f'$ if $f_n \to f$.

Theorem 6.10. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b) \tag{6.19}$$

Proof. Let $\epsilon > 0$ be given. Since $\{f_n(x_o)\}$ converges for some x_0 in $[a,b]$ and $\{f_n'\}$ converges uniformly on $[a,b]$.

Thus, both the series $\{f_n\}$ and $\{f_n'\}$ satisfies Cauchy's criteria for convergence.

Therefore, we can choose N such that $n \ge N$, $m \ge N$ such that

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2 \tag{6.20}$$

$$\text{and} \quad \left|f_n'(t) - f_m'(t)\right| < \frac{\epsilon}{2(b-a)} \qquad (a \le t \le b) \tag{6.21}$$

By mean value theorem, we have

$$[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)] = (x-t)\,[f_n'(\xi) - f_m'(\xi)] \tag{6.22}$$

for any x and t on $[a,b]$ and for some $\xi \in [x,t]$, if $n \ge N, m \ge N$.

Thus, the equation (6.22), can be written as

$$\begin{aligned} |f_n(x) - f_m(x) - f_n(t) + f_m(t)| &\le |x-t|[f_n'(\xi) - f_m'(\xi)] \\ &< |x-t|\frac{\epsilon}{2(b-a)} \qquad (\because \xi \in [a,b]) \\ &< (b-a)\frac{\epsilon}{2(b-a)} \qquad (\because x,t \in [a,b]) \\ &< \epsilon/2 \end{aligned} \tag{6.23}$$

The inequality

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \tag{6.24}$$

Using (6.20) and (6.23) in (6.24), we get

$$|f_n(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, $\{f_n\}$ converges uniformly on $[a,b]$.

Let $f(x) = \lim_{n\to\infty} f_n(x) \quad (a \le x \le b)$.

Fix a point x on $[a,b]$ and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}; \phi(t) = \frac{f(t)-f(x)}{t-x} \text{ for } a \le t \le b,\ t \ne x. \tag{6.25}$$

Then, $\lim_{t\to x} \phi_n(t) = f_n'(x) \ (n = 1,2,3,\ldots)$ (6.26)

Using (6.26) in (6.23), we get

$$|\phi_n(t) - \phi_m(t)| < \frac{\epsilon}{2(b-a)} \quad (n \ge N,\ \ m \ge N) \tag{6.27}$$

Therefore, $\{\phi_n\}$ converges uniformly for $t \ne x$. Since $\{f_n\}$ converges to f, we conclude from (6.25) that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t) \tag{6.28}$$

uniformly for $a \le t \le b,\ \ t \ne x$.

From (6.26) and (6.28),we get

$$\begin{aligned} \lim_{t\to x} \phi(t) &= \lim_{n\to\infty} f_n'(x) \\ i.e.,\ \lim_{t\to x} \frac{f(t)-f(x)}{t-x} &= \lim_{n\to\infty} f_n'(x) \\ i.e.,\ f'(x) &= \lim_{n\to\infty} f_n'(x) \end{aligned}$$

■

Theorem 6.11. *There exists a real continuous function on the real line which is nowhere differentiable. In other words, every where continuous but nowhere differentiable function exists on the real line.*

Proof. Define a function:

$$\phi(x) = |x| \quad (-1 \le x \le 1) \tag{6.29}$$

and we can extend the definition of $\phi(x)$ to entire real axis by periodicity such that $\phi(x+2) = \phi(x)$.

Then for all s and t, we have

$$|\phi(s) - \phi(t)| = |s| - |t| \le |s-t| \tag{6.30}$$

Clearly, ϕ is continuous on R^1.

$$\text{Define } f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \phi(4^n x) = \sum_{n=0}^{\infty} F_n(x) \tag{6.31}$$

$$\text{where } F_n(x) = \left(\frac{1}{4}\right)^n \phi(4^n x) \tag{6.32}$$

Since $0 \le \phi \le 1$, so that

$$|F_n(x)| = \left|\left(\frac{1}{4}\right)^n \phi(4^n x)\right| \le \left|\frac{1}{4}\right|^n = M_n \text{ (say)}$$

since $\sum M_n$ is a geometric series with common ratio less than 1 and hence $\sum M_n$ is convergent.

Thus, by Weierstrass's M test, we have $\sum F_n(x)$ converges uniformly and hence F is continuous at x.

Let m be a fixed positive integer and let x be a fixed real number. Put

$$\delta_m = \pm\frac{1}{2}4^{-m} \tag{6.33}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$.

This can be done because $4m\delta^m = \left|\pm\frac{1}{2}\right| = \frac{1}{2}$.

$$\text{Define } \gamma_n = \frac{\phi\,(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m} \tag{6.34}$$

When $n > m$, then $4^n\delta_m$ is an even integer, so that $\gamma_n = 0$.

When $0 \le n \le m$, then (6.30) gives

$$\begin{aligned} |\gamma_n| &= \frac{|\phi\,(4^n(x+\delta_m)) - \phi(4^n x)|}{|\delta_m|} \\ &\le \frac{|4^n(x+\delta_m) - 4^n x|}{|\delta_m|} = \frac{|4^n\delta_m|}{|\delta_m|} = 4^n. \end{aligned}$$

Also $|\gamma_m| = 4^m$.

Therefore, we conclude that

$$\begin{aligned} \left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| &= \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}\right| \\ &= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n + \sum_{n=m+1}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n\right| \\ &= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right| \qquad (\because \gamma_n = 0 \ (n > m)) \\ &= \left|\left(\frac{3}{4}\right)^m 4^m - \left(-\sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n\right)\right| \\ &\ge |3^m| - \left|\sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n\right| \\ &\ge 3^m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n 4^n \\ &= 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}\,(3^m + 1) \end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$, it follows that f is not differentiable at x. ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- interchange of limits and differentiation.
- interchange of limits and integration.

Check Your Progress:

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E.
3. Show by an example that for term by term differentiation, the condition of uniform convergence is sufficient but not necessary.
4. Show that the series $\sum \frac{1}{n^2 + n^4 x^2}$ is uniformly convergent for all real values of x and it can be differential term by term.

Glossaries:

Uniform convergence: It is a property involving the process of convergence of an order of continuous function.

Suggested Readings:

1. Rudin, W., “Principles of Mathematical Analysis”, Mc Graw-Hill, Third Edition, 1984.
2. Avner Friedman, “Foundations of Modern Analysis”, Hold Rinehart Winston, 1970.

Block-II

UNIT-7

Equicontinuous families of functions

Objectives

After completion of this unit, students will be able to

- ★ understand the concept of equicontinuous families of functions.
- ★ distinguish between uniformly pointwise bounded sequence of functions and pointwise bounded sequence of functions.
- ★ derive the Stone-Weierstrass theorem.

Overview

In this unit, we will illustrate the ideas of equicontinuous families of functions and explained in detail about uniformly pointwise bounded sequence of functions.

7.1. Equicontinuous families of functions:

We have seen that every bounded sequence of complex numbers has a convergent subsequence, and the question arises as to whether something similar is true for the sequence of functions.

Definition 7.1. Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is *pointwise* bounded on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \quad (x \in E,\ n = 1, 2, 3, \dots) \tag{7.1}$$

Definition 7.2. Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is *uniformly* bounded on E if there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E,\ \ n = 1, 2, 3, \dots) \tag{7.2}$$

Remark 7.1.

☞ If $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E then it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$.

☞ If $\{f_n\}$ is uniformly bounded sequence of continuous function on a compact set E, then it is not necessary that there exists a subsequence which converges pointwise on E.

Example 7.1. Let $f_n(x) = \sin nx \quad (0 \leq x \leq 2\pi, \quad n = 1, 2, 3, \dots)$. Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$.

$$\text{In that case } \lim_{n\to\infty} (\sin n_k x - \sin n_{k+1} x) = 0, \quad (0 \leq x \leq 2\pi) \tag{7.3}$$

$$\text{and hence } \lim_{n\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0, \quad (0 \leq x \leq 2\pi) \tag{7.4}$$

By Lebesgue's dominated convergence theorem, then (7.4) implies

$$\lim_{n\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2\, dx = 0 \tag{7.5}$$

But, a direct integration of (7.5), we have

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx = 2\pi \tag{7.6}$$

which is a contradiction to (7.5).

Hence, uniformly bounded sequence of continuous function on a compact set E, may not have a convergent subsequence.

Example 7.2. Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$ $\quad (0 \leq x \leq 1, \ \ n = 1, 2, 3, \ldots)$.

Then $|f_n(x)| \leq 1$, so that $\{f_n\}$ is uniformly bounded on compact set $[0, 1]$.

$$\text{Also,} \quad \lim_{n\to\infty} f_n(x) = 0 \qquad (0 \leq x \leq 1) \tag{7.7}$$

$$\text{but} \quad f_n\left(\frac{1}{n}\right) = 1 \qquad (n = 1, 2, 3, \ldots) \tag{7.8}$$

so that $\{f_n\}$ has no subsequence can converge uniformly on $[0, 1]$.

Hence a convergent and uniformly bounded sequence of functions on a compact set need not contain a uniformly convergent subsequence.

Definition 7.3. A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be *equicontinuous* on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \tag{7.9}$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$ and $f \in \mathscr{F}$. Here d denotes the metric of X.

Remark 7.2. From the definition, it is very clear that every member of equicontinuous is uniformly continuous. but the converse is not true. *i.e.*, Example (7.2) shows that $\{f_n\}$ is uniformly continuous but it is not equicontinuous.

Theorem 7.1. *If $\{f_n\}$ is a pointwise bounded sequence of complex function on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges for every $x \in E$.*

Proof. Let $\{f_n\}$ be a pointwise bounded sequence of complex valued function on a countable set E.

Let $\{x_i\}$, $i = 1, 2, 3 \ldots$ be the points of E, arranged in a sequence.

Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence $\{f_{1,k}\}$ such that $\{f_{1,k}\}$ converges as $k \to \infty$.

Let us consider sequence $S_1, S_2, S_3, \ldots$ which we represented by the array

$$\begin{array}{llllll}
S_1: & f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\
S_2: & f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\
S_3: & f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

and which have the following properties:

(a) S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, \ldots$

(b) $\{f_{n,k}(x_n)\}$ converges as $k \to \infty$.

(c) When going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go do down the diagonal of the array.

Consider the sequence $S: \; f_{1,1} \;\; f_{2,2} \;\; f_{3,3} \;\; f_{4,4} \ldots.$

By (c), the sequence S (except possibly, its first $n-1$ terms) is a subsequence of S_n for $n = 1, 2, 3, \ldots$

Hence (b) implies that $\{f_{n,n}(x_i)\}$ converges as $n \to \infty$ for every $x_i \in E$. ∎

Theorem 7.2. *If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.*

Proof. Let $\epsilon > 0$ be given.

Then by definition of uniformly convergent of $\{f_n\}$, there exists a positive integer N such that

$$n \geq N \;\; \Rightarrow \;\; |f_n - f_N| < \epsilon/3 \tag{7.10}$$

We know that every continuous function is uniformly continuous on the compact set, therefore $\exists \; \delta > 0$ such that

$$1 \leq i \leq N, \quad d(x, y) < \delta \quad \Rightarrow |f_i(x) - f_i(y)| < \epsilon/3 \tag{7.11}$$

If $n > N$ and $d(x, y) < \delta$, it follows that

$$\begin{aligned} |f_n(x) - f_n(y)| &= |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)| \\ &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Thus, the $\{f_n\}$ is equicontinuous on K. ∎

Theorem 7.3. *Let K be a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*

(a) $\{f_n\}$ is uniformly bounded on K.

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof.

(a) Let $\epsilon > 0$ be given.

Since each $\{f_n\}$ is equicontinuous on K, then by definition there exists a $\delta > 0$ such that

$$d(x,y) < \delta, \quad x \in K, y \in K \quad \Rightarrow |f_n(x) - f_n(y)| < \epsilon \quad \forall n \tag{7.12}$$

Since K is compact, therefore there are many points $p_1, p_2, \ldots, p_r$ in K such that to every $x \in K$, there corresponds at least one p_i with $d(x, p_i) < \delta$.

Also, $\{f_n\}$ is pointwise bounded, $\exists\ M_i < \infty$ such that

$$|f_n(p_i)| < M_i \quad \forall n = 1, 2, 3, \ldots \tag{7.13}$$

If $M = \max\{M_1, M_2, \ldots, M_r\}$, then we have

$$|f_n(p_i)| < M + \epsilon \quad \forall x \in K, \quad n = 1, 2, 3, \ldots \tag{7.14}$$

Thus $\{f_n\}$ is uniformly bounded on K.

(b) We know that if (X, d) is a compact metric space, then there always exist a countable dense subset.

Let E be a countable dense subset of K. Then $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}\}$ converges for every $x \in E$.

Put $f_{n_i} = g_i$. Now, we shall prove that $\{g_i\}$ is converges uniformly on K.

Let $\epsilon > 0$ and choose $\delta > 0$ as in (7.12). Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$.

Since E is dense subset in K and K is compact, there exists finitely many points $x_1, x_2, \ldots, x_m$ in E such that

$$K \subset V(x_1, \delta) \cup V(x_2, \delta) \cup \ldots \cup V(x_m, \delta) \tag{7.15}$$

Since $\{g_i(x)\}$ converges for every $x \in E$, then there is an integer N such that

$$i \geq N, \quad j \geq N, \quad 1 \leq s \leq m \quad \Rightarrow \quad |g_i(x_s) - g_j(x_s)| < \epsilon/3. \tag{7.16}$$

If $x \in K$, then (7.15) shows that $x \in V(x_s, \delta)$ for some s, so that

$$|g_i(x) - g_i(x_s)| < \epsilon/3 \quad \text{for every } i \tag{7.17}$$

If $i \geq N$, $j \geq N$, then from (7.16), we have

$$\begin{aligned} |g_i(x) - g_j(x)| &= |g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x)| \\ &\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Hence, for given $\epsilon > 0$, there exists a positive integer N such that

$$\begin{aligned} i \geq N, \quad j \geq N, \quad x \in E \quad &\Rightarrow \quad |f_{n_i}(x) - f_{n_j}(x)| < \epsilon \\ &\Rightarrow \quad \{f_{n_i}\} \text{ converges uniformly on } K \end{aligned}$$

Thus, $\{f_n\}$ contains a uniformly convergent subsequence. ■

7.2. The Stone-Weierstrass theorem:

Theorem 7.4. *If f is a continuous complex function on $[a,b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x) \tag{7.18}$$

uniformly on $[a,b]$. If f is real, then P_n may be taken real.

Proof. Without loss of generality, we may assume that $[a,b] = [0,1]$. Also, we may assume that $f(0) = f(1) = 0$.

Consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1) \tag{7.19}$$

$$\text{Then,}\quad g(0) = g(1) = 0 \tag{7.20}$$

Also, g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

We define $f(x)$ to be zero for x outside $[0,1]$. Then f would be uniformly continuous on the whole real line.

We put

$$Q_n(x) = c_n\left(1 - x^2\right)^n \qquad (n = 1, 2, 3, \ldots) \tag{7.21}$$

Where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)dx = 1 \qquad (n = 1, 2, 3, \ldots) \tag{7.22}$$

In order to determine, the magnitude of c_n, the following inequality is needed.

$$\left(1 - x^2\right)^n \ge 1 - nx^2$$

$$\begin{aligned}
\text{Thus,}\quad 1 &= \int_{-1}^{1} c_n\left(1 - x^2\right)^n dx = 2c_n \int_0^1 \left(1 - x^2\right)^n dx \\
&\ge 2c_n \int_0^{1/\sqrt{n}} (1 - x^2)^n dx \\
&\ge 2c_n \int_0^{1/\sqrt{n}} (1 - nx^2) dx \\
&= 2c_n \left[x - \frac{nx^3}{3}\right]_0^{1/\sqrt{n}} = \frac{4c_n}{3\sqrt{n}} \\
&> \frac{c_n}{\sqrt{n}} \\
\Rightarrow \quad c_n &< \sqrt{n}
\end{aligned} \tag{7.23}$$

Therefore, for any $\delta > 0$, equation (7.23) becomes

$$Q_n(x) \leq \sqrt{n}\left(1 - \delta^2\right)^n, \quad \text{when } \delta \leq |x| \leq 1 \tag{7.24}$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

$$\begin{aligned} \text{Let } P_n(x) &= \int_{-1}^{1} f(x+t)Q_n(t)dt \\ &= \int_{-1}^{-x} f(x+t)Q_n(t)dt + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + \int_{1-x}^{1} f(x+t)Q_n(t)dt \end{aligned}$$

If $-1 \leq t \leq -x$, then $-1 + x \leq x + t \leq 0$, so that $x + t$ lies outside $[0, 1]$ and hence $f(x + t) = 0$.

Thus, the first integral on the R.H.S. becomes zero. Similarly, the third integral also becomes zero.

$$\therefore \quad P_n(x) = \int_{-1}^{1-x} f(x+t)Q_n(t)dt \tag{7.25}$$

$$= \int_{0}^{1} f(t)Q_n(t-x)dt \tag{7.26}$$

which is a polynomial in x.

Thus, $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

It remains to show that $\{P_n(x)\}$ converges uniformly to f on $[0, 1]$.

Since the continuous function defined on a compact set $[0, 1]$ is bounded and uniformly continuous, therefore f is uniformly continuous on $[0, 1]$.

$$\Rightarrow \exists\ M \text{ such that } M = \sup_{x\in[0,1]} |f(x)| \tag{7.27}$$

and for any given $\epsilon > 0$, we can choose $\delta > 0$ such that for any two points $x, y \in [0, 1]$.

$$|f(x) - f(y)| < \epsilon/2 \quad \text{whenever } |x - y| < \delta \tag{7.28}$$

For $0 \leq x \leq 1$, we have

$$\begin{aligned} |P_n(x) - f(x)| &= \left|\int_{1}^{1} f(x+t)Q_n(t)dt - f(x)\right| \\ &= \left|\int_{1}^{1} \left[f(x+t) - f(x)\right] Q_n(t)dt\right| \qquad \text{(using (7.22))} \\ &\leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt \\ &= \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt \\ &\quad + \int_{\delta}^{1} |f(x+t) - f(x)|Q_n(t)dt \end{aligned}$$

$$\begin{aligned}
&\leq 2M\int_{-1}^{-\delta} Q_n(t)dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)dt + 2M\int_{\delta}^{1} Q_n(t)dt \\
&\leq 4M\sqrt{n}\left(1-\delta^2\right)^n + \frac{\epsilon}{2} \qquad \text{(using (7.22) and (7.24))} \\
&< \epsilon \qquad \text{for sufficiently large } n.
\end{aligned}$$

Therefore, for any given $\epsilon > 0$, $\exists\, N$ such that $|P_n(x) - f(x)| < \epsilon \quad \forall n \geq N$.

$\Rightarrow \lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[0, 1]$. ■

Corollary 7.1. *for every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that*

$$\lim_{n\to\infty} P_n(x) = |x| \tag{7.29}$$

uniformly on $[-a, a]$.

Proof. By theorem (7.4), there is a sequence of real polynomials $\{P_n^*\}$ which converges to $|x|$ uniformly on $[-a, a]$.

In particular, $P_n^*(0) \to 0$ as $n \to \infty$.

Consider $P_n(x) = P_n^*(x) - P_n^*(0)$.

Clearly, $P_n(x)$ will converge uniformly to $|x|$ such that $P_n(0) = 0$. Hence the proof. ■

Definition 7.4. A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra*, if

(i) $f + g \in \mathscr{A}$

(ii) $fg \in \mathscr{A}$

(iii) $cf \in \mathscr{A}$ for all $f, g \in \mathscr{A}$ and for all complex constants c.

Definition 7.5. If $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly bounded*

Definition 7.6. Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$.

For Example, the set of all polynomials is an algebra.

Remark 7.3. The Weierstrass theorem may be stated that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials of $[a, b]$.

Theorem 7.5. *Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.*

Proof. Let $f, g \in \mathscr{B}$, then there exists uniformly convergent sequence $\{f_n\}$ and $\{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathscr{A}$, $g_n \in \mathscr{A}$.

Since $f_n + g_n \to f + g$, $f_n g_n \to fg$, $cf_n \to cf$, where c is any constant, the convergent being uniform in each case.

Hence $f+g \in \mathscr{B}$, $fg \in \mathscr{B}$ and $cf \in \mathscr{B}$ and hence $\mathscr{B}$ is an algebra.

Since $\overline{\mathscr{A}} = \mathscr{B}$ and thus, $\mathscr{B}$ is a uniform closed algebra. ■

Definition 7.7. Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

Example 1: The algebra of all polynomials in one variable have separated points $E \subset R$.

Consider the function $P(x) = \dfrac{x-a}{b-a}$ is a polynomial in one variable. Choose two distinct points $a, b \in E$ such that $P(a) = 0;\quad P(b) = 1$

i.e., $P(a) \neq P(b)$. Hence $\mathscr{A}$ is a separate points on E.

Example 2: Consider the algebra of even polynomials.

$Let P(x) = x^2 + 1$. Choose two distinct points $x_1 = -\frac{1}{2}$ and $x_2 = \frac{1}{2}$ on $E = [-1, 1] \subset R^1$.

Clearly, $P\left(\frac{-1}{2}\right) = P\left(\frac{1}{2}\right)$. Thus, $\mathscr{A}$ have no separate points on $[-1, 1]$.

Definition 7.8. If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes* at no point of E.

Example: The algebra of polynomials in one variable vanishes at no point of $E \subset R^1$.

Consider $P(x) = x + 2$ is a polynomial of one variable, such that $P(0) \neq 0$. Thus $\mathscr{A}$ vanishes at no points of $[0, 1]$.

Theorem 7.6. *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separate points on E, and $\mathscr{A}$ vanishes at no points of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constant (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \quad f(x_2) = c_2. \tag{7.30}$$

Proof. Suppose $\mathscr{A}$ separate points on E, then there exists a function $g \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$.

Also, $\mathscr{A}$ vanishes at no points of E, then there exists a function $h(x)$, $k(x) \in \mathscr{A}$ such that $h(x_1) \neq 0$; $k(x_2) \neq 0$.

Put $u = gk - g(x_1)k;\quad v = gh - g(x_2)h$.

Then $u \in \mathscr{A}, v \in \mathscr{A}$ such that

$$\begin{aligned} u(x_1) &= g(x_1)k(x_1) - g(x_1)k(x_1) = 0 \\ u(x_2) &= g(x_2)k(x_2) - g(x_1)k(x_2) = [g(x_2) - g(x_1)]\, k(x_2) \neq 0 \\ v(x_1) &= g(x_1)h(x_1) - g(x_2)h(x_1) \neq 0 \\ v(x_2) &= g(x_2)h(x_2) - g(x_2)h(x_2) = 0 \end{aligned}$$

Let $f = \dfrac{c_1 v}{v(x_1)} + \dfrac{c_2 u}{u(x_2)}$. Then,

$$\begin{aligned} f(x_1) &= \frac{c_1 v(x_1)}{v(x_1)} + \frac{c_2 u(x_1)}{u(x_2)} = c_1 \\ f(x_2) &= \frac{c_1 v(x_2)}{v(x_1)} + \frac{c_2 u(x_2)}{u(x_2)} = c_2 \end{aligned}$$

This completes the proof. ■

Theorem 7.7 (Stone Weierstrass theorem)**.** *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separate points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

Proof. Suppose $\mathscr{A}$ be an algebra of real continuous function on compact set K. Suppose that $\mathscr{A}$ separate points of K and $\mathscr{A}$ vanishes at no point of K.

Our aim is to prove that the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

Now, we shall divide the proof into four steps.

Step 1: If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

Proof:

$$\text{Let } a = \sup|f(x)|, \quad (x \in K) \tag{7.31}$$

and let $\epsilon > 0$ be given. Then by Weierstrass theorem, there exist a real numbers $c_1, c_2, \ldots, c_n$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \epsilon \quad (-a \leq y \leq a) \tag{7.32}$$

Since $\mathscr{B}$ is an algebra, then the function $g = \sum_{i=1}^{n} c_i f^i \in \mathscr{B}$.

By (7.31) and(7.32), we have

$$|g(x) - |f(x)|| < \epsilon \quad (x \in K) \tag{7.33}$$

Since $\mathscr{B}$ is uniformly closed and hence $|f| \in \mathscr{B}$.

Step 2: If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.

Proof: By $\max(f, g)$ we mean the function h is defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

Similarly $\min(f,g)$ we mean the function k is defined by

$$k(x) = \begin{cases} g(x) & \text{if } f(x) \geq g(x) \\ f(x) & \text{if } f(x) < g(x) \end{cases}$$

Also, $\max(f,g)$ and $\min(f,g)$ are also defined by

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$
$$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$

Since $f, g \in \mathscr{B} \Rightarrow \frac{f+g}{2} \in \mathscr{B}$ and by step 1, we have $\frac{|f-g|}{2} \in \mathscr{B}$.

Thus, $\max(f,g) \in \mathscr{B}$. Similarly, we have $\min(f,g) \in \mathscr{B}$.

By iteration, the result can be extended to any finite set of functions.

Step 3: Given a real function f, a continuous on K, a point $x \in K$ and $\epsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \epsilon \quad (t \in K) \tag{7.34}$$

Proof: Since $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ separate points on K and $\mathscr{A}$ vanishes at no point of K. Thus, $\mathscr{B}$ separate points on K and $\mathscr{B}$ vanishes at no point of K.

Hence by theorem (7.6), for every $g \in K$ there exists a function $h_y \in \mathscr{B}$ such that

$$h_y(x) = f(x); \quad h_y(y) = f(y) \tag{7.35}$$

By the continuity of h_y there exists an open set J_y containing y, such that

$$h_y(t) > f(t) - \epsilon \qquad (t \in J_y) \tag{7.36}$$

Since K is compact, then there is a finite set of points $y_1, y_2, \ldots, y_n$ such that

$$K \subset \bigcup_{i=1}^{n} J_{y_i} \tag{7.37}$$

Put $g_x = \max\{h_{y_1}, h_{y_2}, \ldots, h_{y_n}$.

By step 2, we have $g_x \in \mathscr{B}$ and the relations (7.35), (7.36) and (7.37) show that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$.

Step 4: Given a real function f, continuous on K and $\epsilon > 0$, there exists a function $h \in \mathscr{B}$ such that

$$|h(x) - h(y)| < \epsilon \quad (x \in K) \tag{7.38}$$

Proof: Let us consider the function g_x, for each $x \in K$ constructed in step 3. By the continuity of g_x, there exists an open sets V_x containing x such that

$$g_x(t) < f(t) + \epsilon \quad (t \in V_x) \tag{7.39}$$

Since K is compact, there exists a finite set of points $x_1, x_2, \ldots, x_m$ such that

$$K \subset V_{x_1} \cup V_{x_2} \cup V_{x_3} \cup \ldots \cup V_{x_m} \tag{7.40}$$

put $h = \min\{g_{x_1}, g_{x_2}, \ldots, g_{x_m}\}$.

By step 2, we have $h \in \mathscr{B}$ and by (7.34), it follows that

$$h(t) > f(t) - \epsilon \quad (t \in K) \tag{7.41}$$

From (7.39) and (7.40), it follows that

$$h(t) < f(t) + \epsilon \quad (r \in K) \tag{7.42}$$

Thus, from (7.41) and (7.42), we have

$$|h(t) - f(t)| < \epsilon \quad (t \in K) \tag{7.43}$$

Since $\mathscr{B}$ is uniformly closed, by step (4), $\mathscr{B}$ is the set of all real continuous functions on K. Hence the proof. ■

Remark 7.4. Theorem (7.7) does not hold for complex algebra. However, the conclusion of the theorem hold good, even for complex algebra, if an addition condition is imposed on $\mathscr{A}$ *i.e.*, $\mathscr{A}$ be self-adjoint.

Definition 7.9. $\mathscr{A}$ is said to be *self-adjoint*, if for every $f \in \mathscr{A}$, its complex conjugate $\overline{f}$ must also belong to $\mathscr{A}$. $\overline{f}$ is defined by $\overline{f}(x) = \overline{f(x)}$.

Theorem 7.8. *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$ which belong to $\mathscr{A}$.*

Proof. Suppose $\mathscr{A}$ separate points on K and $\mathscr{A}$ vanishes at no points on K. Let $\mathscr{A}_R$ be the set of all real functions on K.

If $f \in \mathscr{A}$ and $f = u + iv$, with u, v real then $2u = f + \overline{f}$.

Since $\mathscr{A}$ is self-adjoint, it follows that $u \in \mathscr{A}_R$.

If $x_1 \neq x_2$ and $\mathscr{A}$ separate points on K, it follows that there exists $f \in \mathscr{A}$ such that $f(x_1) = 1$, $f(x_2) = 0$ and hence $0 = u(x_2) \neq u(x_1) = 1$.

Thus $\mathscr{A}_R$ separate points on K.

Since $\mathscr{A}$ vanishes at no points of K, if $x \in K$ then $g(x) \neq 0$ for some $g \in \mathscr{A}$ and there is a complex number λ such that $\lambda g(x) > 0$. If $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$ and hence $\mathscr{A}_R$ vanishes at no points of K.

Thus, $\mathscr{A}_R$ satisfies the hypothesis of the theorem (7.7). It follows that every real continuous function on K lies in the uniform closure of $\mathscr{A}_R$ and hence lies in $\mathscr{B}$.

If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathscr{B}$, $v \in \mathscr{B}$, hence $f \in \mathscr{B}$. Hence the proof. ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- pointwise bounded, uniform bounded, equicontinuuous and Stone Weierstrass theorem.

Check Your Progress:

1. Define Pointwise bounded sequence.
2. Define equicontinuous.
3. If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.
4. State and Prove Stone-Weierstrass theorem.
5. Define separation points.

Choose the correct or more suitable answer:

1. The algebra of polynomials in one variable vanishes at $\ldots\ldots$ of $E \subset R^1$

   (a) no point (b) one point
   (c) two points (d) three points

Answer:

(1) a

Glossaries:

Equicontinuous: Family of functions is equicontinuous if all the functions are continuous and they have equal variations over a given neighborhood.

Suggested Readings:

1. Rudin, W., "Principles of Mathematical Analysis", Mc Graw-Hill, Third Edition, 1984.

2. Avner Friedman, "Foundations of Modern Analysis", Hold Rinehart Winston, 1970.

Block-III

Unit-8: Measurable Sets.

Unit-9: Regularity.

Unit-10: Abstract Measure Spaces.

Block-III

UNIT-8

MEASURABLE SETS

Objectives

After completion of this unit, students will be able to

- ★ understand the concept of Lebesgue outer measure and its properties.
- ★ define measurable sets.

Overview

In this unit, we will discuss the basic concepts of measure theory.

8.1. Lebesgue Outer Measure:

All the sets consider in this chapter are subsets of the real line unless stated otherwise.

We will be concerned partial with intervals I of the form $I = [a, b)$, where a and b are finite and unless specified, by an interval we shall only mean an interval of the above type.

If $a = b$, then I is the empty set $\emptyset$, we will denote the length of the interval $i.e., (b - a)$ by $l(I)$.

Definition 8.1. Let A be a subset of R and let $[I_n]$ be a finite or countable collection of intervals such that $A \subseteq \bigcup I_n$. The outer measure of A, denoted by $m^*(A)$, defined as

$$m^*(A) = \inf \sum l(I_n)$$

where the infimum is taken over all $[I_n]$.

Note 8.1. In Definition (8.1),

1. A finite collection of intervals I_n means $I_n = \{I_1, I_2, \ldots, I_n\}$ (and)

   A countable collection of intervals I_n means $I_n = \{I_1, I_2, \ldots\}$

   Where each I_n is of the form $I_n = [a_n, b_n)$.

2. Without loss in generality, we may assume that the collection is countably infinite, the finite case is included we may take $I_n = \emptyset$ except for a finite number of integers n.

Theorem 8.1.

(i) $m^*(A) \geq 0$

(ii) $m^*(\emptyset) = 0$

(iii) $m^*(A) \leq m^*(B)$ *if* $A \subseteq B$

(iv) $m^*([x]) = 0$ *for any* $x \in R$.

Proof. By definition, $m^*(A) = \inf \sum l(I_n)$, where the infimum is taken over all countable collection of intervals $[I_n]$ such that $A \subseteq \cup I_n$.

$$\begin{aligned}
(i) \quad \text{Since, } l(I_n) &\geq 0 \quad \forall\ I_n \\
\Rightarrow \quad \sum l(I_n) &\geq 0 \quad \text{for every every countable collection } [I_n] \text{ of intervals} \\
\Rightarrow \quad \inf \sum l(I_n) &\geq 0 \\
i.e., \quad m^*(A) &\geq 0
\end{aligned}$$

(ii) Consider the collection $[I_n]$, where $I_n = \emptyset$ for all n. From, this we can easily see that $\emptyset \subseteq \bigcup I_n$.

$$\begin{aligned}
\text{Since, } l(I_n) &= 0 \quad \forall\ I_n \\
\Rightarrow \quad \sum l(I_n) &= 0 \quad \text{for every every countable collection } [I_n] \text{ of intervals} \\
\Rightarrow \quad \inf \sum l(I_n) &= 0 \\
i.e., \quad m^*(A) &= 0
\end{aligned}$$

(iii) Let $[I_n]$ be the countable collection of intervals such that $B \subseteq \cup I_n$.

Since $A \subseteq B$, then it follows that $A \subseteq \bigcup I_n$ and hence $m^*(A) \leq \sum l(I_n)$.

Thus, $m^*(B) = \inf \sum l(I_n) \geq m^*(A)$.

(iv) Let x be any real number.

Consider the intervals, $I_n = \left[x, x + \frac{1}{n}\right)$, $n = 1, 2, 3, \ldots$

We have $x \in I_n$ for each n and $l(I_n) = \frac{1}{n}$.

$\therefore$ By definition, $m^*([x]) = 0$. ■

Example 8.1. Show that for any set A, $m^*(A) = m^*(A+x)$, where $A+x = \{y+x : y \in A\}$ is invariant. In other words, the outer measure is translation invariant.

Solution: For any set A and $A + x = \{y + x : y \in A\}$. Now, we have to show that $m^*(A + x) = m^*(A)$.

By definition, $m^* = \inf \sum l(I_n)$, where $[I_n]$ is a countable collection of intervals such that $A \subseteq \bigcup I_n$.

For every $\epsilon > 0$, there exists $[I_n]$ is a countable collection of intervals such that

$$m^*(A) + \epsilon \geq \sum l(I_n) \tag{8.1}$$

$$\begin{aligned}
\text{Since } A \subseteq \cup I_n \quad &\Rightarrow \quad A + x \subseteq \cup(I_n + x) \\
\therefore \quad m^*(A + x) &\leq \sum l(I_n + x) \\
&= \sum l(I_n) \\
&\leq m^*(A) + \epsilon \qquad \text{(using (8.1))}
\end{aligned}$$

$$\text{So, for every } \epsilon > 0, \Rightarrow \quad m^*(A + x) \leq m^*(A) \tag{8.2}$$

To prove the opposite inequality, Take $A = (A + x) - x = B - x$, where $B = A + x$.

Hence by (8.2), we have

$$\begin{aligned} m^*(B - x) &\leq m^*(B) \\ \Rightarrow \quad m^*(A) &\leq m^*(A + x) \end{aligned} \tag{8.3}$$

Combining (8.2) and (8.3), we get

$$m^*(A + x) = m^*(A)$$

Theorem 8.2. *The outer measure of an interval equal its length.*

Proof. Let I be any interval in R, now we must show that $m^*(I) = l(I)$.

We consider all the cases separately.

Case 1: Suppose, that I is a closed interval *i.e.*, $[a, b]$, then we must show that $m^*([a, b]) = b - a$.

For every $\epsilon > 0$, we have $I = [a, b] \subseteq [a, b + \epsilon)$.

By (8.1), we have

$$\begin{aligned} m^*(I) &\leq m^*[a, b + \epsilon) \leq b - a + \epsilon \\ i.e., \quad m^*(I) &\leq b - a + \epsilon, \qquad \forall \epsilon > 0 \\ \Rightarrow \quad m^*(I) &\leq b - a = l(I) \end{aligned} \tag{8.4}$$

so, it remains to prove that $m^*(I) \geq l(I)$.

For given $\epsilon > 0$, by definition of outer measure, there is a sequence of intervals $[a_n, b_n)$ such that $I = [a, b] \subseteq [a_n, b_n)$ and

$$m^*(I) > \sum_{n=1}^{\infty} l(I_n) - \epsilon = \sum_{n=1}^{\infty} (b_n - a_n) - \epsilon$$

For each n, let $I'_n = \left(a_n - \frac{\epsilon}{2^n}, b_n\right)$, then $I \subseteq \bigcup_{n=1}^{\infty} I'_n$.

i.e., $[I'_n]$ is an open cover for I.

Since I is compact and hence by Heine-Borel theorem, $\exists$ a finite sub-collection of the intervals I'_n say $J_1, J_2, \ldots, J_N$ where $J_k = (c_k, d_k)$ covers I. *i.e.*, $I \subseteq \bigcup_{k=1}^{N} J_k$.

Without loss of generality, we may assume that no J_k is contained in any other. Suppose that $c_1 < c_2 < \ldots < c_N$. Then

$$\begin{aligned} d_N - c_1 &= \sum_{k=1}^{N}(d_k - c_k) - \sum_{k=1}^{N-1}(d_k - c_{k+1}) \\ &< \sum_{k=1}^{N}(d_k - c_k) \end{aligned}$$

$$i.e., \quad d_N - c_1 < \sum_{k=1}^{N}(d_k - c_k) \tag{8.5}$$

Hence

$$\begin{aligned} m^*(I) &> \sum_{n=1}^{\infty}(b_n - a_n) - \epsilon \\ &= \sum_{n=1}^{\infty}(b_n - a_n) + \sum_{n=1}^{\infty}\frac{\epsilon}{2^n} - 2\epsilon \\ &= \sum_{n=1}^{\infty}\left(b_n - a_n + \frac{\epsilon}{2^n}\right) - 2\epsilon \\ &= \sum_{n=1}^{\infty} l\left(a_n - \frac{\epsilon}{2^n}, b_n\right) - 2\epsilon \\ &= \sum_{n=1}^{\infty} l\,(I'_n) - 2\epsilon \\ &\geq \sum_{k=1}^{N} J_k - 2\epsilon \\ &> (b-a) - 2\epsilon \end{aligned}$$

Since ϵ is arbitrary, then we have

$$m^*(I) \geq (b-a) \tag{8.6}$$

From (8.4) and (8.6), we have

$$m^*(I) = l(I) \tag{8.7}$$

Case 2: Suppose that $I = (a, b]$.

If $a = b$, then $I = \emptyset$ and hence by theorem (8.1), we have

$$\therefore \quad m^*(I) = m^*(\emptyset) = 0 = l(I)$$

So, we may assume that $a < b$. Let $0 < \epsilon < b - a$.

Put $I' = [a + \epsilon, b]$. Then,

$$\begin{aligned} I' \subseteq I \quad &\Rightarrow \quad m^*(I') \leq m^*(I) \quad \text{(by theorem (8.1))} \\ \Rightarrow \quad l(I') &\leq m^*(I) \quad \text{(by using case(1))} \\ i.e., \quad b - a - \epsilon &\leq m^*(I) \\ i.e., \quad m^*(I) &\geq l(I) - \epsilon \end{aligned} \tag{8.8}$$

Let $I'' = [a, b + \epsilon)$, then $I \subseteq I''$.

$$\Rightarrow \quad m^*(I) \leq l(I'') = b + \epsilon - a$$
$$i.e., \quad m^*(I) \leq l(I) + \epsilon \qquad (8.9)$$

Since (8.8) and (8.9) are true for same ϵ.

$$\therefore \quad m^*(I) = l(I)$$

Similarly, the cases $I = [a, b)$ and $I = (a, b)$ are considered.

Case 3: Suppose that I is an infinite interval. Four types of intervals occur, say $(-\infty, a]$, $(-\infty, a)$, $[a, \infty)$, (a, ∞), where a is finite.

First, let us prove the theorem for the case $I = (-\infty, a]$, then the other cases are considered in a similar way.

Let $M > 0$ be arbitrary, then we can find a k such that the finite interval

$I_M = [k, k + M]$ is contained in I.

$$i.e., \quad [k, k + M] \subseteq I$$
$$\Rightarrow \quad m^*(I_M) < m^*(I)$$
$$i.e., \quad m^*(I) > M \qquad \text{for every } M > 0$$
$$\Rightarrow \quad m^*(I) = \infty = l(I)$$

This completes the proof of the theorem. ■

Theorem 8.3. *For any sequence of sets* $\{E_i\}$, $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$

Proof. If one the sets E_i has an infinite measure. $i.e., m^*(E_i) = \infty$ for some i, then $\sum_{i=1}^{\infty} m^*(E_i) = \infty$ and hence

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \infty = \sum_{i=1}^{\infty} m^*(E_I)$$

Suppose each $m^*(E_i)$ is finite. $i.e., m^*(E_i) < \infty$ for each i.

Let $\epsilon > 0$ be given. For each i, $\exists$ a sequence of intervals $\{I_{i,j}, j = 1, 2, \ldots\}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j} \quad \text{and} \quad m^*(E_i) \geq \sum_{j=1}^{\infty} l(I_{i,j}) - \frac{\epsilon}{2^i}$$

Then,

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$$

that is, the collection $[I_{i,j}]$ form a countable class of covering for $\bigcup_{i=1}^{\infty} E_i$. So

$$\begin{aligned}
m^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i,j=1}^{\infty} l(I_{i,j}) \\
&\leq \sum_{i=1}^{\infty}\left[m^*(E_i) + \frac{\epsilon}{2^i}\right] \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty}\frac{\epsilon}{2^i} \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \epsilon\sum_{i=1}^{\infty}\frac{1}{2^i} \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \epsilon \\
i.e., \quad m^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i=1}^{\infty} m^*(E_i) + \epsilon
\end{aligned}$$

Since ϵ is arbitrary and hence we have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i}^{\infty} m^*(E_i)$$

Hence the proof. ■

Example 8.2. Show that, for any set A and any $\epsilon > 0$, there is an open set E containing A and such that $m^*(E) \leq m^*(A) + \epsilon$.

Solution: Given $\epsilon > 0$, we can find a collection of intervals $[I_n]$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} l(I_n) - \frac{\epsilon}{2} \leq m^*(A)$$

If $I_n = [a_n, b_n)$, let $I'_n = \left(a_n - \frac{\epsilon}{2^{n+1}}, b_n\right)$, so that $A \subseteq \bigcup_{n=1}^{\infty} I'_n$.

Take $E = \bigcup_{n=1}^{\infty} I'_n$, then E is an open set and

$$\begin{aligned}
m^*(E) &\leq \sum_{n=1}^{\infty} l(I'_n) = \sum_{n=1}^{\infty}\left(b_n - a_n + \frac{\epsilon}{2^{n+1}}\right) \\
&= \sum_{n=1}^{\infty}(b_n - a_n)\sum_{n=1}^{\infty} + \frac{\epsilon}{2^{n+1}} \\
&= \sum_{n=1}^{\infty} l(I_n) + \frac{\epsilon}{2} \\
&\leq \left[m^*(A) + \frac{\epsilon}{2}\right] + \frac{\epsilon}{2} = m^*(A) + \epsilon
\end{aligned}$$

Hence, for a given set A and $\epsilon > 0$, there is an open set E containing A and $m^*(E) \leq m^*(A) + \epsilon$.

Example 8.3. suppose that in the definition of outer measure,
$m^*(E) = \inf\{\sum l(I_n) : E \subseteq I_n\}$ for sets $E \subseteq R$, we stipulate $(i)\ I_n = (a_n, b_n)$,

(ii) $I_n = [a_n, b_n)$, (iii) $I_n = (a_n, b_n]$ (iv) $I_n = [a_n, b_n]$ (or) (v) mixtures are allowed for different values of n. Show that the same m^* is obtained.

Solution: If we consider the intervals in case (ii) we obtain m^* of definition (8.1). We denote the corresponding m^* by m_o^* in case (i), m_{oc}^* in case (iii), m_c^* in case (iv) and m_m^* in case (v).

Now, we must prove that

$$m^*(E) = m_o^*(E) = m_{oc}^*(E) = m_c^*(E) = m_m^*(E)$$

We establish that each equals $m_m^*(E)$. For this, we consider m_o^*, the proof is similar for other cases.

Note that the set $\{[I_n] : I_n \text{ is any type of interval}\}$ contains the set $\{[I_n] : I_n \text{ is open}\}$.

Hence,

$$m_m^*(E) \leq m_o^*(E) \tag{8.10}$$

To prove the converse, let $\epsilon > 0$ be given and for each I_n and let I'_n be an open interval containing I_n with

$$l(I'_n) = (1+\epsilon)l(_n)$$

Suppose that the collection $[I_n]$ is such that $E \subseteq \bigcup I_n$ and $m_m^*(E) + \epsilon > \sum_{n=1}^{\infty} l(I_n)$. Then

$$\begin{aligned} m_m^*(E) + \epsilon &> \sum_{n=1}^{\infty} \frac{l(I'_n)}{1+\epsilon} \\ &= \frac{1}{1+\epsilon} \sum_{n=1}^{\infty} l(I'_n) \end{aligned}$$

But $E \subset \bigcup_{n=1}^{\infty} I'_n$, a union of open intervals. So

$$\begin{aligned} m_o^*(E) &\leq \sum_{n=1}^{\infty} l(I'_n) \\ &< (1+\epsilon)\left[m_m^*(E) + \epsilon\right] \\ &= (1+\epsilon)m_m^*(E) + (1+\epsilon)\epsilon \end{aligned}$$

Since, ϵ is arbitrary, we get the opposite inequality

$$m_o^*(E) \leq m_m^*(E) \tag{8.11}$$

Thus, we get $m_m^*(E) = m_o^*(E)$.

8.2. Measurable Sets

Definition 8.2. A set E is said to be *Lebesgue measure* or simply *measurable* if for each set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad (8.12)$$

where E^c is the complement of E.

Note 8.2. For any set A, we have

$$\begin{aligned}(A \cap E) \cup (A \cap E^c) &= A \cap (E \cup E^c) \\ &= A \cap R = A\end{aligned}$$

Since m^* is subadditive

$$\begin{aligned}\Rightarrow \quad m^*(A) &= m^*\left[(A \cap E) \cup (A \cap E^c)\right] \\ &\leq m^*(A \cap E) + m^*(A \cap E^c)\end{aligned}$$

Hence, to prove a set E is measurable, it is enough to show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \text{for every } A \quad (8.13)$$

Example 8.4. If $m^*(E) = 0$, then show that E is measurable.

Solution: For any set A, we have $A \cap E \subseteq E$.

$$\begin{aligned}A \cap E &\subseteq E \\ \Rightarrow \quad m^*(A \cap E) &\leq m^*(E) \quad \text{(using theorem (8.1))} \\ \Rightarrow \quad m^*(A \cap E) &= 0 \quad (\text{if } m^*(E) = 0) \\ \text{Also } A \cap E^c &\subseteq A \\ \Rightarrow \quad m^*(A \cap E^c) &\leq m^*(A) \\ \Rightarrow \quad m^*(A) &\geq m^*(A \cap E^c) + 0 \\ \Rightarrow \quad m^*(A) &\geq m^*(A \cap E^c) + m^*(A \cap E)\end{aligned}$$

Hence, E is measurable.

Definition 8.3. A collection S of subset of an arbitrary set X is said to be a σ-algebra or a σ-field if S has the following properties:

(i) $X \in S$.

(ii) If $A \in S$, then $A^c \in S$ where A^c is the complement of A relative to X.

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$, and if $A_n \in S$ for each n, then $A \in S$.

If (iii) is required for finite unions, then S is called an *algebra* of sets.

Notation: We will write $\mathfrak{M}$ for the collection of all measurable subsets of R.

Note 8.3.

1. Let τ be a σ-algebra on a set. The τ is closed for countable intersection.

   For if, let $\{A_i\}_{i=1}^{\infty}$ be any countable collection of sets in τ.

   For every i, $A_i \in \tau \Rightarrow A_i^c \in \tau$ ($\because$ closed with respect to complement)

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i^c \in \tau \quad (\because \text{ closed with respect to countable union})$$
$$\Rightarrow \left(\bigcap_{i=1}^{\infty} A_i\right)^c \in \tau$$
$$\Rightarrow \left(\bigcap_{i=1}^{\infty} A_i\right) \in \tau$$

2. We also note that empty set $\emptyset$ belong to a σ-algebra.

Theorem 8.4. *The class $\mathfrak{M}$ is a σ-algebra.*

Proof.

(i) First we show that $R \in \mathfrak{M}$.

For any set A,

$$\begin{aligned} A \cap R^c &= A \cap \emptyset = \emptyset \\ \Rightarrow m^*(A \cap R^c) &= m^*(\emptyset) = 0 \quad \text{(using theorem (8.1))} \\ \text{Also, } A \cap R \subseteq A &\Rightarrow m^*(A \cap R) \le m^*(A) \end{aligned}$$

Hence adding, we get

$$\begin{aligned} m^*(A) &\ge m^*(A \cap R) + m^*(A \cap R^c) \quad \text{for every set } A \\ \Rightarrow R & \quad \text{is measurable} \quad i.e., \quad R \in \mathfrak{M} \end{aligned}$$

(ii) Let $R \in \mathfrak{M}$. By definition E is measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \text{for every set } A \tag{8.14}$$

The symmetry in the relation (8.14) between E and E^c implies E^c is measurable. *i.e.*, $E^c \in \mathfrak{M}$.

(iii) Let $\{E_i\}$ be a countable collection of measurable sets and let $E = \bigcup_{j=1}^{\infty} E_j$.

Now, our wish is to prove that E is measurable.

Let A be any arbitrary set.

Since E_1 is measurable, then by definition we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \tag{8.15}$$

Sine E_2 is measurable with E replaced by E_2 and A by $A \cap E_1^c$ in (8.15), we get

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \tag{8.16}$$

Substitute (8.16) in (8.15), we get

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c) \tag{8.17}$$

Similarly, we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_3 \cap E_1^c \cap E_2^c) \\ &\quad + m^*(A \cap E_1^c \cap E_2^c \cap E_3^c) \end{aligned} \tag{8.18}$$

Continuing the process in this way, for $n \geq 2$

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \bigcap_{j<i} E_j^c\right) + m^*\left(A \cap \left(\bigcap_{j=1}^{n} E_j^c\right)\right) \\ &= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) \end{aligned}$$

Since $E^c = \left(\bigcup_{j=1}^{\infty} E_j\right)^c \subset \left(\bigcup_{j=1}^{n} E_j\right)^c$ for every n. Hence by theorem (8.1), we have

$$m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*(A \cap E^c)$$

The inequality being true for $n \geq 2$, it follows that

$$m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*(A \cap E^c)$$

Using the fact that

$$\begin{aligned} E &= \bigcup_{i=1}^{\infty} E_i = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap (E_1 \cup E_2)^c) \cup \ldots \\ &= \bigcup_{i=1}^{\infty} \left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) \end{aligned}$$

and that $A \cap \left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} (A \cap B_n)$, we get

$$\begin{aligned}
m^*(A) &\geq m^*(A\cap E_1) + \sum_{i=2}^{\infty} m^*\left(A\cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*(A\cap E^c)\\
&\geq m^*\left(\bigcup_{i=1}^{\infty}\left(A\cap E_i\cap\left(\bigcup_{j<i}E_j\right)^c\right)\right) + m^*(A\cap E^c)\\
&= m^*\left(A\cap\left(\bigcup_{i=1}^{\infty}E_i\cap\left(\bigcup_{j<i}E_j\right)\right)\right) + m^*(A\cap E^c)\\
m^*(A) &\geq m^*(A\cap E) + m^*(A\cap E^c) \qquad (8.19)
\end{aligned}$$

The opposite inequality always being true. we have equality in (8.19) and hence $E = \bigcup_{i=1}^{\infty} E_i$ is measurable.

Thus $\mathfrak{M}$ is σ-algebra. Hence the proof. ■

Note 8.4. If $A, B \in \mathfrak{M}$, then $A - B \in \mathfrak{M}$.

Proof. We know that $A - B = A \cap B^c$.

$$\begin{aligned}
\text{Now } B \in \mathfrak{M} &\Rightarrow B^c \in \mathfrak{M}\\
\therefore\ A, B^c \in \mathfrak{M} &\Rightarrow A\cap B^c \in \mathfrak{M}\\
\text{i.e., } A - B \in \mathfrak{M}
\end{aligned}$$

■

Example 8.5. Show that if $F \in \mathfrak{M}$ and $m^*(F\triangle G) = 0$, then G is measurable.

Solution: Suppose $F \in \mathfrak{M}$ and let $m^*(F\triangle G) = 0$.

Hence, by example (8.4), we have

$$m^*(F\triangle G) = 0 \Rightarrow F\triangle G \text{ is measurable}$$

Since, $F - G \subseteq F\triangle G$, we get $m^*(F - G) \leq m^*(F\triangle G) = 0$.

Hence, $m^*(F - G) = 0$ implies $F - G$ is measurable.

Similarly, we can prove that $G - F$ is also measurable.

We know that $F \cap G = F - (F - G)$, where both F and $F - G$ are measurable.

Thus, $F \cap G$ is measurable.

Now, $G = (F \cap G) \cup (G - F)$ where $F \cap G$ and $G - F$ are measurable and hence G is measurable.

Theorem 8.5. *If* $\{E_i\}$ *is any sequence of disjoint measurable sets, then*

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i) \qquad (8.20)$$

that is, m^* *is countably additive on disjoint sets of* $\mathfrak{M}$.

Proof. Since outer measure is subadditive

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

and hence it is enough to prove the theorem, it is sufficient to show that

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(E_i)$$

Let $A = \bigcup_{i=1}^{\infty} E_i$, since $E_i \cap E_j = \emptyset$ for $i \neq j$ and $A \cap E_i = E_i$ and $A \cap E_i = E_i$ for all i.

Also, $A \cap \left(\bigcup_{i=1}^{k} E_i\right)^c = \bigcup_{i=k+1}^{\infty} E_i, \quad k = 1, 2, \ldots$

Using the fact that each E_i is measurable and thus we have

$$\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c)\\
&= m^*(E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)\\
&= m^*(E_1) + m^*(E_2) + m^*(A \cap E_1^c \cap E_2^c)\\
&\ldots\\
&= \sum_{i=1}^{n} m^*(E_i) + m^*\left(A \cap \left(\bigcap_{i=1}^{n} E_i^c\right)\right)\\
&= \sum_{i=1}^{n} m^*(E_i) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^c\right)\\
&\geq \sum_{i=1}^{n} m^*(E_i) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^n\right)\\
&= \sum_{i=1}^{n} m^*(E_i) + m^*\left(A \cap A^n\right)\\
&= \sum_{i=1}^{n} m^*(E_i) + m^*\left(\emptyset\right)\\
&= \sum_{i=1}^{n} m^*(E_i)
\end{aligned}$$

Since this holds for all n, letting $n \to \infty$, we have

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(E_i)$$

$$i.e., \quad m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(E_i)$$

Hence the proof. ∎

Note 8.5. Put $E_{n+1} = E_{n+2} = \ldots = \emptyset$ in (8.20), we get the same results for finite unions as a special case. So, if $E_1, E_2, \ldots, E_n$ are disjoint measurable sets, then

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$$

If E is a measurable set we write $m(E)$ in place of $m^*(E)$ and $m(E)$ is called the *Lebesgue measure* of E. Thus, if $E \in \mathfrak{M}$ then $m(E) = m^*(E)$. Since the Lebesgue measure m is defined for each $E \in \mathfrak{M}$, m is a set function defined on the σ-algebra $\mathfrak{M}$. Theorem (8.5) states that m is countably additive set function.

Theorem 8.6. *Every interval is measurable.*

Proof. First, let us we consider the interval of the form $I = [a, \infty)$. Now, our aim is to prove that the interval I is measurable.

Hence, we have to prove that for any set A,

$$m^*(A) \geq m^*(A \cap [a, \infty)) + m^*(A \cap [a, \infty)^c)$$

$$i.e., \; m^*(A) \geq m^*(A \cap [a, \infty)) + m^*(A \cap (-\infty, a])$$

Let $\epsilon > 0$ be given. Then there exists intervals I_n such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \tag{8.21}$$

$$\text{and } m^*(A) + \epsilon > \sum_{n=1}^{\infty} l(I_n) \tag{8.22}$$

From (8.21), we have

$$A \cap [a, \infty) \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap [a, \infty)$$

$$\Rightarrow A \cap [a, \infty) \subseteq \bigcup_{n=1}^{\infty} (I_n \cap [a, \infty))$$

$$\text{Similarly, we have } A \cap (-\infty, a] \subseteq \bigcup_{n=1}^{\infty} (I_n \cap (-\infty, a])$$

$$\text{Let } I_n \cap (-\infty, a) = I_n'$$

$$\text{and } I_n \cap [a, \infty) = I_n'' \text{ so that}$$

$$\left.\begin{aligned} A \cap (-\infty, a) &\subseteq \textstyle\bigcup_{n=1}^{\infty} I_n' \\ \text{and } A \cap [a, \infty) &\subseteq \textstyle\bigcup_{n=1}^{\infty} I_n'' \end{aligned}\right\} \tag{8.23}$$

Note that I_n' and I_n'' are disjoint and

$$\begin{aligned} I_n' \cup I_n'' &= (I_n \cap (-\infty, a)) \cup (I_n \cap [a, \infty)) \\ &= I_n \cap [(-\infty, a) \cup [a, \infty)] \\ &= I_n \cap (-\infty, \infty) = I_n \\ \therefore \quad l(I_n') + l(I_n'') &= l(I_n) \end{aligned}$$

Thus, from (8.23), we have
$$m^*(A \cap (-\infty, a)) \leq \sum_{n=1}^{\infty} l(I_n')$$
$$\text{and} \quad m^*(A \cap [a, \infty)) \leq \sum_{n=1}^{\infty} l(I_n'')$$

Adding the above two inequalities, we get
$$\begin{aligned} m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)) &\leq \sum_{n=1}^{\infty} l(I_n') + \sum_{n=1}^{\infty} l(I_n'') \\ &= \sum_{n=1}^{\infty} l(I_n \\ &< m^*(A) + \epsilon \quad (\text{u}sing \ (8.22)) \end{aligned}$$

This is true for every $\epsilon > 0$ and hence we have
$$m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)) \leq m^*(A)$$

This shows that the interval $[a, \infty)$ is measurable.

Now theorem (8.4), gives the result for other types of intervals. ■

Theorem 8.7. *Let $\mathscr{A}$ be a class of subsets of a space X. Then there exists a smallest σ-algebra $\mathscr{S}$ containing $\mathscr{A}$. In this case, we say that $\mathscr{S}$ is the σ-algebra generated by $\mathscr{A}$.*

Proof. Let $\{\mathscr{S}_\alpha\}$ be any collection of σ-algebras of subsets of X. Then by definition, we get $\bigcap_\alpha \mathscr{S}_\alpha$ is a σ-algebra.

For if,

(i) $X \in \mathscr{S}_\alpha$ for every α
$\Rightarrow X \in \bigcap_\alpha \mathscr{S}_\alpha$.

(ii) $A_1, A_2, \ldots \in \bigcap_\alpha \mathscr{S}_\alpha$
$\Rightarrow A_1, A_2, \ldots \in \mathscr{S}_\alpha \ \forall \alpha$
$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{S}_\alpha \ \forall \alpha$
$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \bigcap_\alpha \mathscr{S}_\alpha$.

(iii) $A \in \bigcap_\alpha \mathscr{S}_\alpha$
$\Rightarrow A \in \mathscr{S}_\alpha \ \forall \alpha$
$\Rightarrow A^c \in \mathscr{S}_\alpha \ \forall \alpha$
$\Rightarrow A^c \in \bigcap_\alpha \mathscr{S}_\alpha$.

Consider the family of all σ-algebras containing $\mathscr{A}$. This family is non-empty as the class of all subsets of X belong to this family. Let $\mathscr{S}$ be the intersection of all these σ-algebras. Then clearly $\mathscr{S}$ is a σ-algebra containing $\mathscr{A}$ and in fact it is the smallest such σ-algebra. Hence the proof. ■

Definition 8.4. The Borel σ-algebra $\mathscr{B}$ is defined to be the smallest σ-algebra of interval of the form $[a, b)$. In other words, $\mathscr{B}$ is the σ-algebra generated by the class of intervals of the form $[a, b)$. The elements of $\mathscr{B}$ are called the *Borel sets*.

Definition 8.5. A countable union of closed sets is called an F_α set.

Definition 8.6. A countable intersection of open sets is called G_δ set.

Theorem 8.8.

(i) $\mathscr{B} \subseteq \mathfrak{M}$, *that is every Borel set is measurable.*

(ii) $\mathscr{B}$ *is the* σ*-algebra generated by each of the following classes: the open intervals, the open sets, the* G_δ*-sets, the* F_α *sets.*

Proof. (i) By theorem (8.6), every interval is measurable and also by theorem (8.4), the class of measurable sets $\mathfrak{M}$ is a σ-algebra. Hence $\mathscr{B}$, the smallest σ-algebra of intervals of the form $[a, b) \subset \mathfrak{M}$. Thus, every Borel set is measurable.

(ii) Let $\mathscr{B}_1$ be the σ-algebra generated by the open intervals. Every open interval (a, b) is a Borel set, since (a, b) is union of sequence of intervals $\left[a + \frac{1}{n},\ b\right)$, $n = 1, 2, 3 \ldots$ So $\mathscr{B}_1 \subset \mathscr{B}$. But every interval $[a, b)$ is the intersection of sequence of open intervals $\left(a - \frac{1}{n},\ b\right)$, $n = 1, 2, \ldots.$ So $\mathscr{B} \subset \mathscr{B}_1$. Thus $\mathscr{B} = \mathscr{B}_1$.

A set is open if and only if it is union of a sequence of open intervals. Hence the σ-algebra generated by the open sets is equal to σ-algebra generated by the open intervals and hence it is equal to $\mathscr{B}$.

By definition, a G_δ set is formed from open sets using countable intersection and a F_α set is formed from complements of open sets using countable union. Hence $\mathscr{B}$ is equals to both the σ-algebra generated by G_δ sets and by F_σ sets. ■

Example 8.6. For any set A there exists a measurable set E containing A such that $m^*(A) = m(E)$.

Solution: Given a set A and every $\epsilon = \frac{1}{n}, n = 1, 2, \ldots$ there exists open sets G_n such that $A \subset G_n$ and $m^*(G_n) \leq m^*(A) + \frac{1}{n}$.

Let $E = \bigcap_{n=1}^{\infty} G_n$. Clearly E is a G_δ set and it is measurable. Hence

$$m(E) \leq m^*(G_n) < m^*(A) + \frac{1}{n} \quad \text{for every } n$$

$$\text{Thus, } m(E) \leq m^*(A)$$

Since $A \subset G_n$ for each n, $A \subset \bigcap G_n = E$. So,

$$m^*(A) \leq m(E)$$

Thus, $m^*(A) = m(E)$.

Definition 8.7. For any sequence of sets $\{E_i\}$, we define

$$\limsup E_i = \bigcap_{n=1}^{\infty}\left(\bigcup_{i\geq n} E_i\right) = \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right) \tag{8.24}$$

$$\liminf E_i = \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right) = \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \tag{8.25}$$

If $\limsup E_i = \liminf E_i = E$ (say), then E is called the *limit* of E, and we write $\lim E_i = E$.

Note 8.6.

1. From the definition, we can easily see that $\liminf E_i \subseteq \limsup E_i$.

Proof.

$$\begin{aligned}
\text{Let} \quad x &\in \liminf E_i, \quad \text{then by definition} \\
\Rightarrow \quad x &\in \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \\
\Rightarrow \quad x &\in \bigcap_{i=n}^{\infty} E_i \quad \text{for some positive integer } n \\
\Rightarrow \quad x &\in E_n,\ E_{n+1}, \ldots \\
\Rightarrow \quad x &\in \bigcup_{i=1}^{\infty} E_i,\ \bigcup_{i=2}^{\infty} E_i,\ \ldots, \bigcup_{i=n}^{\infty} E_i, \ldots \\
\Rightarrow \quad x &\in \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right) \\
\Rightarrow \quad x &\in \limsup E_i
\end{aligned}$$

Thus,we get $\liminf E_i \subseteq \limsup E_i$ ■

2. $\limsup E_i$ is the set of points belonging to infinitely many of the sets E_i.

Proof.

$$\begin{aligned}
\text{Let} \quad x &\in \limsup E_i, \quad \text{then by definition} \\
\Leftrightarrow \quad x &\in \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right) \\
\Leftrightarrow \quad x &\in \bigcup_{i=n}^{\infty} E_i \quad \text{for } n = 1, 2, 3, \ldots \\
\Leftrightarrow \quad x &\in E_i \quad \text{for some } i \geq n, \quad n = 1, 2, \ldots \\
\Leftrightarrow \quad x &\in \text{infinitely many of the sets } E_i
\end{aligned}$$

■

3. $\liminf E_i$ is the set of points belonging to all but finitely many of the sets E_i.

Proof.

$$\begin{aligned}
\text{Let} \quad x &\in \liminf E_i, \quad \text{then by definition} \\
\Leftrightarrow \quad x &\in \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \\
\Leftrightarrow \quad x &\in \bigcap_{i=N}^{\infty} E_i \quad \text{for some positive integer } N \\
\Leftrightarrow \quad x &\in E_N, E_{N+1} \ldots \quad \text{for some positive integer } N \\
\Leftrightarrow \quad x &\in \text{belongs to all but finitely many of the sets } E_i
\end{aligned}$$

■

Theorem 8.9. *Let E_i be a sequence of measurable sets. Then*

(i) if $E_1 \subseteq E_2 \subseteq \ldots,$ we have $m(\lim E_i) = \lim m(E_i)$.

(ii) if $E_1 \supseteq E_2 \supseteq \ldots$ and $m(E_i) < \infty$, then we have $m(\lim E_i) = \lim m(E_i)$.

Proof.

(i) Suppose $E_1 \subseteq E_2 \subseteq E_3 \ldots$

Put $F_1 = E_1,\ \ F_i = E_i - E_{i-1}$ for $i = 1, 2, 3, \ldots$

Then the sets $F_1, F_2, \ldots$ are measurable and are disjoints and $\bigcup_{i=1}^{\infty} F_i$ is measurable and

$$m\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{\infty} m(F_i)$$

Since $E_1 \subseteq E_2 \subseteq E_3 \ldots$

$$\bigcup_{i=1}^{\infty} F_i = E_1 \cup \bigcup_{i=2}^{\infty} (E_i - E_{i-1}) = \bigcup_{i=1}^{\infty} E_i$$

and $\lim E_i = \bigcup_{n=1}^{\infty} E_i$. Hence,

$$\begin{aligned}
m(\lim E_i) &= m\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} F_i\right) \\
&= \sum_{i=1}^{\infty} m(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} m(F_i) \\
&= \lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} F_i\right) \\
&= \lim_{n\to\infty} m\left(E_1 \cup (E_2 - E_1) \cup (E_3 - E_2) \cup \ldots \cup (E_n - E_{n-1})\right) \\
&= \lim_{n\to\infty} m(E_n)
\end{aligned}$$

(ii) Suppose $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ Then

$$E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \ldots$$

Hence by (i), we have

$$m(\lim E_1 - E_i) = \lim m(E_1 - E_i) \tag{8.26}$$

Since $E_i \subseteq E_1$, E_i are measurable and $E_1 - E_i = E_1 \cap E_i^c$, we have

$$\begin{aligned}
m(E_1) &= m(E_1 \cap E_i) + m(E_1 \cap E_i^c) \\
&= m(E_i) + m(E_1 - E_i) \\
\text{Hence, } m(E_1 - E_i) &= m(E_1) - m(E_i)
\end{aligned}$$

Thus, (8.26) implies

$$m(\lim(E_1 - E_i)) = m(E_1) - \lim m(E_i)$$

But,

$$\lim(E_1 - E_i) = \bigcup_{i=1}^{\infty}(E_1 - E_i)$$
$$= E_1 - \bigcap_{i=1}^{\infty} E_i$$
$$= E_1 - \lim E_i$$

Thus,

$$m(\lim(E_1 - E_i) = m(E_1 - \lim E_i)$$
$$= m(E_1) - m(\lim E_i) \quad (8.27)$$

From (8.26) and (8.27) and $m(E_1) < \infty$, we have

$$m(\lim E_i) = \lim m(E_i)$$

■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- lebesgue outer measure and σ -algebra.
- measurable sets.

Check Your Progress:

1. Show that every countable set has measure zero.
2. Show that every non-empty open set has positive measure.
3. Show that there exists uncountable sets of zero measure.

Choose the correct or more suitable answer:

1. If $m^*(A) = 0$, then ...... for any set B.

   (a) $m^*(A \cap B) = m^*(B)$

   (b) $m^*(A \cup B) = m^*(B)$

   (c) $m^*(B) = 0$

   (d) none of these

Answer:

(1) *b*

Glossaries:

Measure: A measure is a function that assigns a number to certain subsets of a given set. Their number is said to be measuring of the set.

Suggested Readings:

1. G. de Barra, “Measure Theory and Integration”, New Age International Pvt. Ltd, Second Edition, 2013.

2. Rana I. K., “An Introduction to Measure and Integration”, Narosa Publishing House Pvt. Ltd., Second Edition, 2007.

3. Royden H. L., “Real Analysis”, Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-III

UNIT-9

REGULARITY

Objectives

After completion of this unit, students will be able to

★ define measurable function.

★ classify measurable sets and Borel sets.

Overview

In this unit, we will illustrate the classification between measurable sets and Borel sets.

9.1. Regularity:

Theorem 9.1. *The following statements regarding the set E are equivalent:*

(i) E *is measurable.*

(ii) $\forall\, \epsilon > 0, \exists\ G,$ *an open set,* $E \subseteq G$ *such that* $m^*(G - E) \leq \epsilon$.

(iii) $\exists$ *a* G_δ*-set,* $E \subseteq G$ *such that* $m^*(G - E) = 0$

*(ii)** $\forall\ \epsilon > 0, \exists\ F,$ *a closed set,* $F \subseteq E$ *such that* $m^*(E - F) \leq \epsilon$.

*(iii)** $\exists\, F,$ *an* F_σ*-set,* $F \subseteq E$ *such that* $m^*(E - F) = 0$.

Proof.

$(i) \Rightarrow (ii)$: Let E be a Lebesgue measurable and $\epsilon > 0$ be given.

Suppose that $m^*(E) = m(E) < \infty$. Then by definition, we can find intervals $I_1, I_2, \ldots$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and $m(E) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} m(I_n)$.

For every n, choose open intervals $J_n \supseteq I_n$ such that

$$m(J_n) \leq \frac{\epsilon}{2^{n+1}} + m(I_n) \tag{9.1}$$

Let $G = \bigcup_{n=1}^{\infty} J_n$. Then G is an open set with $E \subseteq G$ and $m(G) < \infty$. Also,

$$\begin{aligned} m^*(G - E) &= m(G - E) = m(G) - m(E) \\ &\leq \sum_{n=1}^{\infty} m(J_n) - m(E) \\ &\leq \sum_{n=1}^{\infty} \left[\frac{\epsilon}{2^{n+1}} + m(I_n)\right] - m(E) \qquad \text{(using (9.1))} \\ &= \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} + \sum_{n=1}^{\infty} m(I_n) - m(E) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence (ii) proved in the case $m(E) < \infty$.

Assume that $m(E) = m^*(E) = \infty$. Then, by definition we can find finite intervals I_n such that $I_n \cap I_m = \emptyset$ for $n \neq m$ and $E = \bigcup_{n=1}^{\infty} I_n$. *i.e.,* $l(I_n) < \infty$.

If $E_n = E \cap I_n$, then we have $m(E_n) < \infty$. Hence by the previous considered for $m^*(E) = m(E) < \infty$, $\exists$ an open set G_n such that $E_n \subseteq G_n$ and

$$m^*(G_n - E_n) < \frac{\epsilon}{2^n}, \quad n = 1, 2, 3, \ldots$$

Put $G = \bigcup_{n=1}^{\infty} G_n$. Clearly, G is an open set, $E \subseteq G$ and

$$\begin{aligned} G - E &= \bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty}(G_n - E_n) \\ \Rightarrow \quad m^*(G - E) &\leq \sum_{n=1}^{\infty} m^*(G_n - E_n) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \end{aligned}$$

Thus (ii) proved for $m^*(E) = \infty$.

$(ii) \Rightarrow (iii)$ For each n, choose an open set G_n, $E \subseteq G_n$ such that $m^*(G_n - E) < \frac{1}{n}$.

Take $G = \bigcap_{n=1}^{\infty} G_n$. Clearly G is a G_δ-set, $E \subseteq G$ and $m^*(G - E) \leq m^*(G_n - E) < \frac{1}{n}$ for every n.

Hence $m^*(G - E) = 0$.

$(iii) \Rightarrow (i)$ Given that $\exists$ a G_δ -set with $E \subseteq G$ with $m^*(G - E) = 0$.

Now, we have $E = G - (G - E)$ and $m^*(G - E) = 0$. Hence, $G - E$ is measurable and also we know that every G_δ is measurable. Thus G is measurable.

Hence $E = G - (G - E)$ is measurable.

$(i) \Rightarrow (ii)^*$ Suppose E is measurable, then E^c is measurable.

Hence, by (ii), $\exists$ an open set G such that $E^c \subseteq G$ and $m^*(G - E^c) \leq \epsilon$.

$$G - E^c = G \cap E = E \cap G = E - G^c$$

Now, G is open $\Rightarrow$ G^c is closed

$\Rightarrow$ F is closed (Take $G^c = F$)

Thus,

$$m^*(E - F) = m^*(E - G^c) = m^*(G - E^c) \leq \epsilon$$

$(ii)^* \Rightarrow (iii)^*$ For each n, choose closed sets F_n such that $F_n \subseteq E$ and $m^*(E - F_n) < \frac{1}{n}$.

Put $F = \bigcup_{n=1}^{\infty} F_n$. Then F is a F_σ -set, $F \subseteq E$ and $m^*(E - F_n) < \frac{1}{n}$ for every n.

Hence $m^*(E - F) = 0$.

$(iii)^* \Rightarrow (i)$ Given there exists an F_σ -set, $F \subseteq E$ such that $m^*(E - F) = 0$.

Since $m^*(E - F) = 0$ which implies, $E - F$ is measurable.

Also, we know that every F_σ -set is measurable and hence F is measurable.

Hence, $E = F \cup (E - F)$, which is measurable. ■

Theorem 9.2. *If* $m^*(E) < \infty$, *then* E *is measurable if and only if,* $\forall\ \epsilon > 0$, $\exists$ *disjoint finite intervals* $I_1, I_2, \ldots, I_n$ *such that* $m^*\left(E \triangle \bigcup_{i=1}^{n} I_i\right) < \epsilon$. *We may stipulate that the intervals* I_i *be open, closed or half-open.*

Proof. Assume that E is measurable, by theorem (9.1) for every $\epsilon > 0$ there exists an open set G, $G \subset E$ such that $m(G - E) < \epsilon$. Also, $m^*(E) = m(E) < \infty$ which implies $m(G) < \infty$.

Every open set is union of disjoint open intervals, so we can write open set G as $G = \bigcup_{i=1}^{\infty} I_i$, where I_i' s are disjoint open intervals and hence we have

$$\begin{aligned} m(G) &= m\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} l(I_i) \\ i.e., \sum_{i=1}^{\infty} l(I_i) &= m(G) < \infty \end{aligned}$$

Thus, there exist an integer n such that $\sum_{i=n+1}^{\infty} l(I_i) < \epsilon$.

Put $U = \bigcup_{i=1}^{n} I_i$. Then

$$E \triangle U = (E - U) \cup (U - E) \subseteq (G - U) \cup (G - E)$$

Hence,

$$\begin{aligned} m^*(E \triangle U) &\leq m^*(G - U) + m^*(G - E) \\ &= m\left(\bigcup_{i=n+1}^{\infty} I_i\right) + \epsilon \\ &= \sum_{i=n+1}^{\infty} l(I_i) + \epsilon \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

If we wish the intervals to be say, half-open, we first obtain open intervals $I_1, I_2, \ldots, I_n$ as above and then for each i choose a half-open interval $J_i \subset I_i$ such that $m(l_i - J_i) < \frac{\epsilon}{n}$. Then the intervals J_i are disjoint and we have

$$\begin{aligned}
m\left(E\triangle\bigcup_{i=1}^{n}J_i\right) &= m\left(\left(E\triangle\bigcup_{i=1}^{n}I_i\right)\cup\left(\bigcup_{i=1}^{n}I_i\triangle\bigcup_{i=1}^{n}J_i\right)\right)\\
&\leq m\left(E\triangle\bigcup_{i=1}^{n}I_i\right)+m\left(\bigcup_{i=1}^{n}I_i\triangle\bigcup_{i=1}^{n}J_i\right)\\
&< \epsilon+m\left(\bigcup_{i=1}^{n}(I_i\triangle J_i\right)\\
&= \epsilon+m\left(\bigcup_{i=1}^{n}(I_i-J_i\right)\\
&= \epsilon+\sum_{i=1}^{n}m(I_i-J_i)\\
&< \epsilon+\sum_{i=1}^{n}\frac{\epsilon}{n}\\
&= \epsilon+\epsilon=2\epsilon
\end{aligned}$$

Thus, the intervals I_i be open, closed or half-open.

Converse part: Assume that for every $\epsilon > 0$ there exists disjoint intervals $I_1, I_2, \ldots, I_n$ such that

$$m^*\left(E\triangle\bigcup_{i=1}^{n}I_i\right)<\epsilon$$

Put $J=\bigcup_{i=1}^{n}I_i$, then $m^*(E\triangle J)<\epsilon$.

By example (8.2), there exist an open set G, $E\subseteq G$ such that

$$m(G) \leq m^*(E)+\epsilon \tag{9.2}$$

In order to prove E is measurable, it is sufficient to prove that $m^*(G-E)$ can be arbitrarily small. Let $U=G\cap J$, we have

$$G\triangle E = (G\triangle U)\cup(U\triangle E)$$

By subadditivity, we have

$$m^*(G\triangle E) \leq m^*(G\triangle U)+m^*(U\triangle E) \tag{9.3}$$

Since, $U\subseteq J$, we have $U-E\subseteq J-E$ and also $E\subseteq G$, we have $E-U=E-(G\cap J)=E-J.$

Hence,

$$\begin{aligned}
U\triangle E &= (U-E)\cup(E-U)\\
&\subseteq (J-E)\cup(E-J)\\
&= J\triangle E
\end{aligned}$$

Given that $m^*(U\triangle E)<\epsilon$, But, $E\subseteq U\cup(U\triangle E)$, so $m^*(E)<m^*(U)+\epsilon$.

Hence,

$$\begin{aligned} m(G\triangle U) &= m(G-U) \\ &= m(G)-m(U) \\ &\leq m^*(E)+\epsilon-m(U) \\ &< m(U)+\epsilon+\epsilon-m(U)=2\epsilon \end{aligned}$$

Thus, from (9.3), we have

$$\begin{aligned} m^*(G-E) &= m^*(G\triangle E) \\ &\leq m^*(G\triangle U)+m^*(U\triangle E) \\ &< 2\epsilon+\epsilon=3\epsilon \end{aligned}$$

Thus, E is measurable. Hence the proof. ∎

9.2. Measurable Functions:

Definition 9.1. Let f be an extended real-valued function defined on a measurable set E. Then f is a *Lebesgue measurable function* or simply, a *measurable function* if, for each $\alpha \in R$, the set $[x : f(x) > \alpha]$ is measurable.

Note 9.1. The domain of definition of f will usually be either R or $R - F$, where $m(F) = 0$.

Note 9.2. The definition of measurable function states that f is a measurable function if for every real number α, the inverse image (α, ∞) under f *i.e.*, $f^{-1}(\alpha, \infty)$ is a measurable set.

Theorem 9.3. *The following statements are equivalent:*

(i) f *is a measurable function.*

(ii) $[x : f(x) \geq \alpha]$ *is measurable for all* α.

(iii) $[x : f(x) < \alpha]$ *is measurable for all* α

(iv) $[x : f(x) \leq \alpha]$ *is measurable for all* α.

Proof.

$(i) \Rightarrow (ii)$: Assume that f is measurable.

Then for each of the sets $\left[x; f(x) > \alpha - \frac{1}{n}\right]$ $(n = 1, 2, \ldots)$.

Thus, $\bigcap_{n=1}^{\infty}\left[x : f(x) > \alpha - \frac{1}{n}\right]$ is also measurable.

But,

$$\begin{aligned}[x : f(x) \geq \alpha] &= f^{-1}[\alpha, \infty) \\ &= f^{-1}\left(\bigcap_{n=1}^{\infty}\left(\alpha - \frac{1}{n}, \infty\right)\right) \\ &= \bigcap_{n=1}^{\infty} f^{-1}\left(\alpha - \frac{1}{n}, \infty\right) \\ &= \bigcap_{n=1}^{\infty}\left[x : f(x) > \alpha - \frac{1}{n}\right], \quad \text{which is measurable.}\end{aligned}$$

Thus, $[x : f(x) \geq \alpha]$ is measurable.

$(ii) \Rightarrow (iii)$ Suppose $[x : f(x) \geq \alpha]$ is measurable.

We know that if E is measurable then E^c is measurable and hence it follows that $[x : f(x) \geq \alpha]^c$ is measurable. *i.e.*, $[x : f(x) < \alpha]$ is measurable. This proves (iii).

$(iii) \Rightarrow (iv)$: Suppose $[x : f(x) < \alpha]$ is measurable.

Then for each of the sets $\left[x : f(x) < \alpha + \frac{1}{n}\right]$ $(n = 1, 2, \ldots)$.

Thus, $\bigcap_{n=1}^{\infty}\left[x : f(x) < \alpha + \frac{1}{n}\right]$ is also measurable.

But,

$$\begin{aligned}[x : f(x) \leq \alpha] &= f^{-1}(-\infty, \alpha] \\ &= f^{-1}\left(\bigcap_{n=1}^{\infty}\left(-\infty, \alpha + \frac{1}{n}\right)\right) \\ &= \bigcap_{n=1}^{\infty} f^{-1}\left(-\infty, \alpha + \frac{1}{n}\right) \\ &= \bigcap_{n=1}^{\infty}\left[x : f(x) < \alpha + \frac{1}{n}\right], \quad \text{which is measurable.}\end{aligned}$$

Thus, $[x : f(x) \leq \alpha]$ is measurable.

$(iv) \Rightarrow (i)$: Suppose $[x : f(x) \leq \alpha]$ is measurable.

Then, it complement is $[x : f(x) > \alpha]$. Hence, by the definition of measurable function, f is measurable. Hence the proof. ■

Example 9.1. Show that if f is measurable, then $[x : f(x) = a]$ is measurable for any extended real number α.

Solution: Assume that α is finite, then we have

$$[x : f(x) = \alpha] = [x : f(x) \geq \alpha] \cap [x : f(x) \leq \alpha]$$

By theorem (9.3), $[x : f(x) \geq \alpha]$ and $[x : f(x) \leq \alpha]$ are measurable.

We know that, intersection of two measurable is measurable and hence it follows that $[x : f(x) \geq \alpha] \cap [x : f(x) \leq \alpha]$ is measurable.

Thus, $[x : f(x) = \alpha]$ is measurable.

If $\alpha = \infty$, then

$$[x : f(x) = \infty] = \bigcap_{n=1}^{\infty}\left[x : f(x) > \frac{1}{n}\right]$$

Since, a countable intersection of measurable set is measurable and hence $[x : f(x) = \infty]$ is measurable.

Similarly, we can prove for the case $\alpha = -\infty$.

Example 9.2. The constant functions are measurable.

Solution: Given that $f(x) = c$ for all x. Then the set

$$[x : f(x) > \alpha] = \begin{cases} R & \text{if } \alpha < c \\ \emptyset & \text{if } \alpha \geq c \end{cases}$$

Thus, the set $[x : f(x) > \alpha]$ is measurable for all α and hence the constant function is measurable.

Example 9.3. The characteristic function χ_A of the set A, is measurable if and only if A is measurable.

Solution: The set

$$[x : f(x) > \alpha] = \begin{cases} \emptyset & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ R & \text{if } \alpha < 0 \end{cases}$$

So, $[x;\chi_A(x) > \alpha$ is measurable for all α if and only if A is measurable. Thus χ_A is measurable if and only if A is measurable.

Example 9.4. Continuous functions are measurable.

Solution: Assume that f is continuous. For every $\alpha \in R$, the interval (α, ∞) is an open set.

We know that an inverse image of an open set is open under a continuous mapping. That is $f^{-1}(\alpha, \infty)$ is open.

Further, every open set is measurable and hence $f^{-1}(\alpha, \infty) = [x : f(x) > \alpha]$ is measurable.

Thus, f is a measurable function.

Theorem 9.4. *Let c be any real number and let f and g be real-valued measurable functions defined on the same measurable set E. Then $f + c$, cf, $f + g$, $f - g$ and fg are also measurable.*

Proof. For any $\alpha \in R$. Consider the set

$$[x : f(x) + c > \alpha] \quad = \quad [x : f(x) > \alpha - c]$$

Since, f is the measurable function on the right hand side and thus, it follows that the set on the left hand side is also measurable. Thus, $c + f$ is measurable.

For any $\alpha \in R$, then we have

$$[x : cf(x) > \alpha] = \begin{cases} \left[x : f(x) > \dfrac{\alpha}{c}\right] & \text{for } c > 0 \\ \left[x : f(x) < \dfrac{\alpha}{c}\right] & \text{for } c < 0 \\ E \text{ or } \emptyset & \text{for } c = 0 \end{cases}$$

In any cases, the set $[x : cf(x) > \alpha]$ is measurable. Thus, the function cf is measurable.

Now, our wish is to prove that $f + g$ is measurable. In order to prove this, we shall apply some simple idea. *i.e.*, Two real numbers a and b satisfy $a > b$ if and only if there is some rational number r with $a > r > b$. Suppose $r_1, r_2, \ldots$ is a countable set of rational numbers.

Then $x \in [x : f(x) + g(x) > \alpha]$ only if there exists a rational number r_i such that

$$f(x) \quad > \quad r_i > \alpha - g(x)$$

So, for every $\alpha \in R$, we have

$$\begin{aligned} [x : f(x) + g(x) > \alpha] &\subseteq \bigcup_{n=1}^{\infty} ([x : f(x) > r_n] \cap [x : r_n > \alpha - g(x)]) \\ &= \bigcup_{n=1}^{\infty} ([x : f(x) > r_n] \cap [x : g(x) > \alpha - r_n]) \end{aligned}$$

Since f and g are measurable and hence the sets $[x : f(x) > r_n]$ and $[x : g(x) > \alpha - r_n]$ are measurable for each n. So, $[x : f(x) + g(x) > \alpha]$ is measurable. Hence $f + g$ is a measurable function.

If g is a measurable function and if $c \in R$, then cg is also a measurable function. In particular, taking $c = -1$, then $-g$ is a measurable function. So, $f - g = f + (-g)$ is also a measurable function.

Finally, we have to prove that the function fg is measurable. For this, first we shall prove that f^2 is measurable, if f is measurable.

If $\alpha < 0$, then the set $\left[x : f^2(x) > \alpha\right] = R$ is measurable.

If $\alpha \geq 0$, then

$$\left[x : f^2(x) > \alpha\right] \quad = \quad \left[x : f(x) > \sqrt{\alpha}\right] \cup \left[x : f(x) < \sqrt{\alpha}\right]$$

Since f is measurable and hence each of the sets on the right hand side is measurable. Thus $\left[x : f^2(x) > \alpha\right]$ is measurable for every $\alpha \in R$, which shows that f^2 is a measurable function whenever f is a measurable function.

If f and g are measurable functions, then $f+g$ is a measurable function and also $(f+g)^2$ is a measurable function. Similarly, $(f-g)^2$ is also a measurable function.

Now,

$$fg = \frac{1}{4}\left[(f+g)^2-(f-g)^2\right]$$

Hence, fg is a measurable function, since the right hand side is a difference of two measurable functions. ■

Theorem 9.5. *Let $\{f_n\}$ be a sequence of measurable functions defined on the same measurable set. Then*

(i) $\sup\limits_{1\le i\le n} f_i$ *is measurable for each* n.

(ii) $\inf\limits_{1\le i\le n} f_i$ *is measurable for each* n.

(iii) $\sup f_n$ *is measurable.*

(iv) $\inf f_n$ *is measurable.*

(v) $\limsup f_n$ *is measurable.*

(vi) $\liminf f_n$ *is measurable.*

Proof.

(i) For any n, we have

$$\left[x : \sup_{1\le i\le n} f(x) > \alpha\right] = \bigcup_{n=1}^{\infty}[x : f(x) > \alpha]$$
$$= \text{a countable union of measurable sets}$$
$$= \text{a measurable set}$$

Hence, $\sup\limits_{1\le i\le n} f_i$ is measurable for each n.

(ii) We know that

$$\inf_{1\le i\le n} f_i = -\sup_{1\le i\le n}(-f_i)$$

Thus, from (i), we can easily see that $\inf\limits_{1\le i\le n} f_i$ is measurable for each n.

(iii) For any $\alpha \in R$ and any x, $\sup f_n(x) > \alpha$ means that $f_n(x) > \alpha$ for $n = 1,2,3,\ldots$ and hence

$$[x : \sup f_n(x) > \alpha] = \bigcup_{n=1}^{\infty}[x : f_n(x) > \alpha]$$
$$= \text{a countable union of measurable sets}$$
$$= \text{a measurable set}$$

Thus, $\sup f_n$ is measurable.

(iv) We know that

$$\inf f_n = -\sup(-f_n)$$

Hence, $\inf f_n$ is measurable, by using (iii).

(v) We know that

$$\limsup f_n = \inf_n \left(\sup_{i\geq n} f_i\right)$$

Put $F_i = \sup_{i\geq n} f_i$ which is measurable, by using (iii).

Now, by applying (iv), we have $\limsup f_n = \inf F_n$ is measurable.

(vi) Since $\liminf f_n = -\limsup(-f_n)$.

Hence by applying (v), we have $\liminf f_n$ is measurable. Hence the proof. ■

Definition 9.2. Let f be an extended real valued function defined on a Borel set. We say that f is *Borel measurable* or a *Borel function* if for each $\alpha \in R$ the set $[x : f(x) > \alpha]$ is a Borel set.

Definition 9.3. A property is said to hold almost every where $(a.e)$ if it holds everywhere except for a set of measure zero.

Theorem 9.6. *Let f be a measurable function and let $f = g$ a.e. Then g is measurable.*

Proof. Suppose f is measurable and $f = g$ $a.e.$ Now, our wish is to prove that g is measurable.

Since f is measurable and by definition $[x : f(x) > \alpha]$ is measurable.

Since $f = g$ $a.e$ which implies $m[x : f(x) \neq g(x)] = 0$.

Now,

$$\begin{aligned} [x : f(x) > \alpha]\triangle[x : g(x) > \alpha] &= ([x : f(x) > \alpha] - [x : g(x) > \alpha]) \\ &\quad \cup ([x : g(x) > \alpha] - [x : f(x) > \alpha]) \\ &\subseteq [x : f(x) \neq g(x)] \\ \Rightarrow \quad m([x : f(x) > \alpha]\triangle[x : g(x) > \alpha]) &= 0 \end{aligned}$$

By example (8.4), we have $[x : f(x) > \alpha]\triangle[x : g(x) > \alpha]$ is measurable.

Since f is measurable and by example (8.5), we have g is measurable. ■

Example 9.5. Let $\{f_i\}$ be a sequence of measurable functions converging $a.e.$ to f, then f is measurable.

Solution: Suppose $\{f_i\}$ be a sequence of measurable functions converging $a.e$ to f.

By theorem (9.5), we have $\limsup f_i$ is measurable.

Since, $f = \limsup f_i\ a.e.$, then by theorem (9.6), f is measurable.

Example 9.6. If f is a measurable function, then so are $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$.

Solution:For any n,

If $f_1, f_2, f_3, \ldots, f_n$ are sequence of measurable functions, then by theorem (9.5) both $\lim \sup\limits_{1\le i\le n} f_i$ and $\lim \inf\limits_{1\le i\le n} f_i$ are measurable.

Put $f_1 = f$ and $f_2 \equiv 0$. Then f_1 and f_2 are measurable.

Thus,

$$f^+ = \max(f, 0) = \max(f_1, f_2) \quad \text{which is measurable}$$

$$f^- = \min(f, 0) = \min(f_1, f_2) \quad \text{which is measurable}$$

Example 9.7. The set of points on which a sequence of measurable functions $\{f_n\}$ converges, is measurable.

Solution: Suppose the set of points on which a sequence of measurable functions $\{f_n\}$ converges.

Thus, the sequence $\{f_n(x)\}$ is converges for a fixed x if and only if

$$\limsup f_n(x) = \liminf f_n(x)$$

Now, our wish is to prove that $[x : \limsup f_n(x) - \liminf f_n(x)] = 0$ is measurable.

Since each f_n is measurable, then by theorem (9.5), both $\limsup f_n$ and $\liminf f_n$ are measurable.

Thus, $\limsup f_n - \liminf f_n$ is measurable.

We know that constant functions are measurable, thus the set $[x : \limsup f_n - \liminf f_n = 0]$ is measurable.

Definition 9.4. Let f be a measurable function; then $\inf[\alpha : f \le \alpha\ a.e]$ is called the *essential supremum* of f, denoted by $ess \sup f$.

Example 9.8. Show that $f \le ess \sup f,\ a.e.$

Solution:

If $ess \sup f = \infty$, then clearly $f \le ess \sup f$ and the result is obvious.

Suppose $ees \sup f = -\infty$. Then by definition (9.4), $f \le n\ a.e$ for all $n \in Z$. Thus $f = -\infty\ a.e.$ and hence the relation $f \le ess \sup f$ holds.

Suppose $ess \sup f$ is finite.

Put

$$E_n = \left[x : f(x) > ess \sup f + \frac{1}{n}\right]$$
$$\text{and } E = [x : f(x) > ess \sup f]$$

Then, $E = \bigcup_{n=1}^{\infty} E_n$. But, by definition of $ess \sup f$, for every $n = 1, 2, \ldots$ there exists α_n such that $ess \sup f + \frac{1}{n} > \alpha_n$ and $f \leq \alpha_n$ $a.e.$

Hence $ess \sup f + \frac{1}{n} > f$ $a.e.$ and hence $m(E_n) = 0$ for all n.

Thus, $m(E) = 0$. So, $f \leq ess \sup f$.

Example 9.9. Show that for any measurable functions f and g

$$ess \sup(f + g) \leq ess \sup f + ess \sup g$$

and give an example of strict inequality.

Solution:

From example (9.8), we have

$$f \leq ess \sup f \ a.e$$
$$\text{and } g \leq ess \sup g \ a.e$$
$$\Rightarrow f + g \leq ess \sup f + \sup g \ a.e.$$

Hence, by definition (9.4), we have

$$ess \sup f + g \leq ess \sup f + ess \sup g$$

Example for strict inequality: Consider $f = \chi_{[-1,0]} - \chi_{[0,1]}$ and $g = -f$, Then $f + g = 0$. So, $ess \sup(f + g) = 0$.

But,

$$ess \sup f = 1 = ess \sup g.$$

Hence,

$$ess \sup(f + g) = 0 < 2 = ess \sup f + ess \sup f$$

Definition 9.5. Let f be a measurable function; then $sup[\alpha : f(x) \geq \alpha \ a.e.]$ is called the *essential infimum* of f.

Example 9.10.

$$ess \sup f = -ess \inf(-f)$$

Solution: By definition (9.4), we have

$$\begin{aligned} ess \sup f &= \inf[\alpha : f \leq \alpha \ \ a.e] \\ &= \inf[\alpha : -f \geq -\alpha \ \ a.e.] \\ &= -\sup[-\alpha : -f \geq -\alpha \ \ a.e.] \\ &= -ess \inf(-f) \end{aligned}$$

Definition 9.6. If f is a measurable function and $ess \sup|f| < \infty$, then f is said to be *essentially bounded*.

Example 9.11. Let f be a measurable function and B a Borel set. Then $f^{-1}(B)$ is a measurable set.

Solution: We have

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) \quad \text{and} \quad f^{-1}(A^c) = \left(f^{-1}(A)\right)^c.$$

So the class of sets whose inverse image under f are measurable forms a σ-algebra. Since f is measurable, the inverse image of an interval under f is measurable. So this class contains intervals. Thus, it must contain all Borel sets.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- measurable functions and its properties.
- $\lim sup$ and $\lim inf$, essential supremum and essential infimum.

Check Your Progress:

1. Show that for any measurable function f, $ess \sup f \leq \sup f$.
2. Show that $f \leq ess \sup f$, $a.e.$

Choose the correct or more suitable answer:

1. The characteristic function χ_A of the set A is

   (a) not measurable, if A is measurable.

   (b) measurable if A is not measurable.

   (c) measurable if and only if A is measurable.

   (d) none of these.

Answer:

(1) c

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.

2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.

3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-III

UNIT-10

ABSTRACT MEASURE SPACES

Structure

Objectives

After completion of this unit, students will be able to

- ★ define σ-algebra and σ-ring.
- ★ understand the concept of hereditary.
- ★ understand the concept of measure space and its properties.

Overview

In this unit, we will illustrate the basic concepts of σ-algebra and σ-ring. Further we discuss in detail about the measure spaces and its properties.

10.1. Measures and Outer Measures:

Definition 10.1. A class of sets $\mathscr{R}$ of some fixed space is called a *ring* if whenever $E \in \mathscr{R}$ and $F \in \mathscr{R}$ then $E \cup F$ and $E - F$ belong to $\mathscr{R}$.

Example 10.1. The class of finite union of intervals of the form $[a, b)$ forms a ring.

Definition 10.2. A ring is called a σ*-ring* if it is closed under the formation of countable unions.

Example 10.2. Show that every algebra is a ring and every σ-algebra is a σ-ring but that the converse is not true.

Solution: Let A be algebra on a set. Let $E \in A$ and $F \in A$, then

$$\Rightarrow \quad E \cup F \in A \quad (\because \text{finite union is closed in algebra})$$

$$\text{Also, } E \in A \Rightarrow E^c \in A \quad (\because \text{complement is closed})$$

$$\Rightarrow \quad E^c \cup F \in A \quad (\because \text{finite union is closed in algebra})$$

$$\Rightarrow \quad (E^c \cup F)^c \in A$$

$$i.e., \quad E - F \in A$$

Thus, A is a *ring* on a set.

Let A be a σ-algebra on a set. Then, it is closed with respect to countable union. Thus, A is a ring and it is closed with respect to countable union. Hence, A is a σ-ring.

For proving the converse part, consider the σ-ring of all subset of $[0, 1]$ which are at most countable.

If $\bigcup A_\alpha \in \mathscr{S}$, where the A_α are the sets of the σ-ring $\mathscr{S}$, then $\mathscr{S}$ may be regarded as a σ-algebra on the space $\bigcup A_\alpha$.

Theorem 10.1. *There exists a smallest ring and a smallest σ-ring contained a class of subsets of a space; we refer to these as the generated ring and the generated σ-ring respectively.*

Proof. Let $\{\mathscr{S}_\alpha\}$ be any collection of σ-rings of subsets of X. Then by definition, we get $\bigcap\limits_{\alpha} \mathscr{S}_\alpha$ is a σ-ring.

For if,

(i) $X \in \mathscr{S}_\alpha$ for every α

$\Rightarrow \; X \in \bigcap\limits_{\alpha} \mathscr{S}_\alpha.$

(ii) $A_1, A_2, \ldots \in \bigcap_\alpha \mathscr{S}_\alpha$

$\Rightarrow \; A_1, A_2, \ldots \in \mathscr{S}_\alpha \quad \forall \alpha$

$\Rightarrow \; \bigcup\limits_{n=1}^{\infty} A_n \in \mathscr{S}_\alpha \quad \forall \alpha$

$\Rightarrow \; \bigcup\limits_{n=1}^{\infty} A_n \in \bigcap\limits_{\alpha} \mathscr{S}_\alpha.$

(iii) $A \in \bigcap_\alpha \mathscr{S}_\alpha$

$\Rightarrow \; A \in \mathscr{S}_\alpha \quad \forall \alpha$

$\Rightarrow \; A^c \in \mathscr{S}_\alpha \quad \forall \alpha$

$\Rightarrow \; A^c \in \bigcap_\alpha \mathscr{S}_\alpha.$

Consider the family of all σ-rings containing $\mathscr{A}$. This family is non-empty as the class of all subsets of X belong to this family. Let $\mathscr{S}$ be the intersection of all these σ-rings. Then clearly $\mathscr{S}$ is a σ-ring containing $\mathscr{A}$ and in fact it is the smallest such σ-ring. Hence the proof. ■

Notation: We will write $\mathscr{S}(\mathscr{R})$ for the σ-ring $\mathscr{S}$ generated by the ring $\mathscr{R}$; we write $\mathscr{H}(\mathscr{R})$ for the class consisting of $\mathscr{S}(\mathscr{R})$ together with all subsets of the sets of $\mathscr{S}(\mathscr{R})$. A class of sets with this property, namely that every subsets of one of its members belongs to the class, is said to be *hereditary*.

Clearly $\mathscr{H}(\mathscr{R})$ is a σ-ring and is the smallest hereditary σ-ring containing $\mathscr{R}$. Indeed $\mathscr{H}(\mathscr{R}) = \mathscr{H}(\mathscr{S}(\mathscr{R})) = \mathscr{H}(\mathscr{H}(\mathscr{R}))$.

Definition 10.3. A set function μ defined on a ring $\mathscr{R}$ is a *measure* if

(i) μ is non-negative

(ii) $\mu(\emptyset) = 0$

(iii) for any sequence $\{A_n\}$ of disjoint sets of $\mathscr{R}$ such that $\bigcup\limits_{n=1}^{\infty} A_n \in \mathscr{R}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 10.4. A measure μ on $\mathscr{R}$ is *complete* if whenever $E \in \mathscr{R}$, $F \subseteq E$ and $\mu(E) = 0$, then $F \in \mathscr{R}$.

Definition 10.5. A measure μ on $\mathscr{R}$ is *σ-finite*, if for every $E \in \mathscr{R}$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_r\}$ such that $E_n \in \mathscr{R}$ and $\mu(E_n) < \infty$ for each n.

Example 10.3. Show that Lebesgue measure m defined on $\mathscr{M}$, the class of measurable sets of R, is σ-finite and complete.

Solution: By Theorem 5.4, we have $\mathscr{M}$ is a σ-algebra. But, every σ-algebra is a σ-ring and hence it is ring on which m is defined.

Take $E_n = E \cap (-n, n)$, then we have $\mu(E_n) < \infty$. Hence $\mathscr{M}$ is σ-finite.

Let $F \subseteq E$, then

$$\begin{aligned} \mu^*(F) &\leq \mu^*(E) = m(E) = 0 \\ \therefore \quad \mu*(F) &= 0 \end{aligned}$$

Since $\mu^*(F) = 0$, we have that F is a Lebesgue measurable set. Hence every subset of Lebesgue measurable set E with $\mu(E) = 0$ is also a Lebesgue measurable. Thus, $\mathscr{M}$ is complete.

Definition 10.6. If $\mathscr{R}$ is a ring, a set function μ^* defined on the class $\mathscr{H}(\mathscr{R})$ is an *outer measure* if

(i) μ^* is non-negative.

(ii) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$,

(iii) $\mu^*(\emptyset) = 0$

(iv) for any sequence $\{A_n\}$ of sets of $\mathscr{H}(\mathscr{R})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

that is, μ^* is countably subadditive.

Example 10.4. Show that if $A, B \in \mathscr{R}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

Solution: Since, $B = A \cup (B - A)$, then clearly, $\mu(B) \geq \mu(A)$.

10.2. Extension of a Measure:

Theorem 10.2. *Let $\{A_i\}$ be a sequence in a ring $\mathscr{R}$, then there is a sequence $\{B_i\}$ of disjoint sets of $\mathscr{R}$ such that $B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N, so that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.*

Proof. Let us prove the result by induction.

Define

$$\begin{aligned} B_1 &= A_1 \\ B_n &= A_n - \bigcup_{i=1}^{n-1} B_i \quad \text{for } n > 1. \end{aligned}$$

Since $B_i \in \mathscr{R}$ and $B_i \subseteq A_i$.

Further, $\{B_i\}$ is a sequence of disjoint set and thus B_n and $\bigcup_{i=1}^{n-1} B_i$ are disjoint sets.

Hence, we have $B_n \cap B_m = \emptyset$ for $n > m$.

Finally, if we have $B_1 = A_1$ and $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$, then

$$\begin{aligned} B_{k+1} \cup \left(\bigcup_{i=1}^{k} B_i\right) &= \left(A_{k+1} - \bigcup_{i=1}^{k} B_i\right) \cup \bigcup_{i=1}^{k} B_i \\ &= A_{k+1} \cup \bigcup_{i=1}^{k} B_i = A_{k+1} \cup \bigcup_{i=1}^{k} A_i \\ i.e., \bigcup_{i=1}^{k+1} B_i &= \bigcup_{i=1}^{k+1} A_i \end{aligned}$$

Thus, we have $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. ■

Example 10.5. Show that $\mathscr{H}(\mathscr{R}) = \left[E : E \bigcup_{n=1}^{\infty} E_n, E_n \in \mathscr{R}\right]$

Solution: It is easily verified that the right hand side defines a class of sets which is hereditary, contains $\mathscr{R}$ and is a σ-ring. So, it contains $\mathscr{H}(\mathscr{R})$.

But, if $E_n \in \mathscr{R}$ for each n, we have $\bigcup_{n=1}^{\infty} E_n \in \mathscr{S}(\mathscr{R})$ and so each subset belong to $\mathscr{H}(\mathscr{R})$. So, we get equality.

Theorem 10.3. *If μ is a measure on a ring $\mathscr{R}$ and if the set μ^* is defined by $\mathscr{H}(\mathscr{R})$ by*

$$\mu^*(E) = \int \left[\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathscr{R}, n = 1, 2, \ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right] \qquad (10.1)$$

then (i) for $E \in \mathscr{R}$, $\mu^*(E) = \mu(E)$, (ii) μ^* is an outer measure on $\mathscr{H}(\mathscr{R})$.

Proof. (i) If $E \in \mathscr{R}$, then from equation (10.1) we have $\mu^*(E) \leq \mu(E)$.

Suppose that $E \in \mathscr{R}$ and $E \subseteq \bigcup_{n=1}^{\infty} E_n$ where $E_n \in \mathscr{R}$.

By Theorem (10.2), we may replace the sequence $\{E_i \cap E\}$ by a sequence $\{F_i\}$ of disjoint sets in $\mathscr{R}$ such that $F_i \subseteq E_i \cap E$ and $\bigcup_{i=1}^{\infty} F_i = E$. Then by example (10.4), we have $\mu(F_i) < \mu(E_i)$ for each i.

So,

$$\begin{aligned} \mu(E) &= \mu\left(\bigcup_{n=1}^{\infty} F_i\right) \\ &= \sum_{i=1}^{\infty} \mu(F_i) \\ &\leq \sum_{i=1}^{\infty} \mu(E_i) = \mu^*(E) \\ i.e., \quad \mu(E) &\leq \mu^*(E) \end{aligned}$$

Thus, $\mu(E) = \mu^*(E)$. This proves (i).

(ii) Clearly, μ^* is non-negative. Also, $\mu^*(\emptyset) = \mu(\emptyset) = 0$. Next, we have to prove that μ^* is countably subadditive.

Suppose that $\{E_i\}$ is a sequence of sets in $\mathscr{K}(\mathscr{R})$. By definition of μ^*, for each $\epsilon > 0$ we can find for each i a sequence $\{E_{i,j}\}$ of sets of $\mathscr{R}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} E_{i,j}$ and $\sum_{j=1}^{\infty} \mu(E_{i,j}) \leq \mu^* E_i) + \frac{\epsilon}{2^i}$.

The sets $\{E_{i,j}\}$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$. So

$$\begin{aligned} \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(E_{i,j}) \\ &\leq \sum_{i=1}^{n} \mu^*(E_i) + \epsilon \end{aligned}$$

Since ϵ is arbitrary, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} \mu^*(E_i)$$

Thus, μ^* is an outer measure on $\mathscr{K}(\mathscr{R})$. Hence the proof. ∎

Definition 10.7. Let μ^* be an outer measure on $\mathscr{K}(\mathscr{R})$. Then $E \in \mathscr{K}(\mathscr{R})$ is μ^* measurable if for each $A \in \mathscr{K}(\mathscr{R})$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \tag{10.2}$$

Theorem 10.4. *Let μ^* be an outer measure on $\mathscr{K}(\mathscr{R})$ and let $\mathscr{S}^*$ denote the class of μ^* measurable sets. Then $\mathscr{S}^*$ is a σ-ring and μ^* is restricted to $\mathscr{S}^*$ is a complete measure.*

Proof. Suppose μ^* be an outermeasure of $\mathscr{K}(\mathscr{R})$ and $\mathscr{S}^*$ denote the class of μ^*-measurable sets. First, we shall prove that μ^* is a measure on the σ-ring $\mathscr{S}^*$.

As in theorem (8.4), $\mathscr{S}^*$ is closed under countable union. It remains to prove that if $E, F \in \mathscr{S}^*$ then $E - F \in \mathscr{S}^*$.

Let $A \in \mathscr{H}(\mathscr{R})$, then A can be written as union of the four disjoint sets.

$$\begin{aligned} i.e.,\ A &= A_1 \cup A_2 \cup A_3 \cup A_4 \\ \text{Where } A_1 &= A - (E \cup F) \\ A_2 &= A \cap E \cap F \\ A_3 &= A \cap (F - E) \\ A_4 &= A \cap (E - F) \end{aligned}$$

Since F is measurable, then equation (10.2) gives

$$\mu^*(A) = \mu^*(A_1 \cup A_4) + \mu^*(A_2 \cup A_3) \tag{10.3}$$

E is measurable and for any set $A_1 \cup A_4$, then (10.2) gives

$$\mu^*(A_1 \cup A_4) = \mu^*(A_1) + \mu^*(A_4) \tag{10.4}$$

F is measurable and for any set $A_1 \cup A_2 \cup A_3$, then (10.2) gives

$$\mu^*(A_1 \cup A_2 \cup A_3) = \mu^*(A_1) + \mu^*(A_2 \cup A_3) \tag{10.5}$$

Substitute equations (10.4) and (10.5) in (10.3), we have

$$\begin{aligned} \mu^*(A) &= \mu^*(A_4) + \mu^*(A_1 \cup A_2 \cup A_3) \\ i.e.,\ \mu^*(A) &= \mu^*(A \cap (E - F)) + \mu^*(A \cap (E - F)^c \end{aligned}$$

i.e., $E - F$ is measurable.

Suppose that $\{E_i\}$ is a sequence of disjoint sets in $\mathscr{S}^*$, Then exactly as in theorem (8.5), we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i)$$

Also, for every set $E \in \mathscr{H}(\mathscr{R})$ such that $\mu^*(E) = 0$ is μ^* measurable, for if $A \in \mathscr{H}(\mathscr{R})$,

$$\begin{aligned} \mu^*(A) &\leq \mu^*(A \cap E) + \mu^*(A \cap E^c) \\ &\leq \mu^*(E) + \mu^*(A) = \mu^*(A) \end{aligned}$$

So equality holds good and E is μ^*-measurable.

If $E \in \mathscr{S}^*$ and $\mu^*(E) = 0$ and $F \subseteq E$ then it follows that $F \in \mathscr{S}^*$. Thus, μ^* is a complete measure on $\mathscr{S}^*$. Hence the proof. ∎

Theorem 10.5. *Let μ^* be the outer measure on $\mathscr{H}(\mathscr{R})$ defined by μ on $\mathscr{R}$, then $\mathscr{S}^*$ contains $\mathscr{S}(\mathscr{R})$, the σ-ring generated by $\mathscr{R}$.*

Proof. Since S^* is a σ-ring. It is enough to prove that $\mathscr{R} \subseteq \mathscr{S}^*$.

If $E \in \mathscr{R}$, $A \in \mathscr{H}(\mathscr{R})$ and $\epsilon > 0$ then by the definition of μ^* there exists a sequence $\{E_n\}$ of sets of $\mathscr{R}$ such that $A \subseteq \bigcup_{n=1}^{\infty} E_n$ and

$$\begin{aligned} \mu^*(A) + \epsilon &\geq \sum_{n=1}^{\infty} \mu(E_n) \\ &= \sum_{n=1}^{\infty} \mu(E_n \cap E) + \sum_{n=1}^{\infty} \mu(E_n \cap E^c) \\ \Rightarrow \quad \mu^*(A) + \epsilon &\geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \text{(as } \mu \text{ is a measure)} \end{aligned}$$

Since ϵ is arbitrary and hence we have

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Since the opposite inequality is quite obvious, thus we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Thus, $E \in \mathscr{S}^*$ and hence $\mathscr{R} \subseteq \mathscr{S}^*$. This completes the proof. ■

Example 10.6. Show that if μ is a σ-finite measure on $\mathscr{R}$, then the extension $\overline{\mu}$ of μ to $\mathscr{S}^*$ is also σ-finite.

Solution: Let $E \in \mathscr{S}^*$.

By definition of $\overline{\mu}$ there is a sequence $\{E_n\}$ of sets of $\mathscr{R}$ such that

$$\overline{\mu}(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

By hypothesis, each E_n is the union of sequence $\{E_{n,i}\ i = 1, 2, \ldots\}$ of sets of $\mathscr{R}$ such that $\mu(E_{n,i}) < \infty$ for each n and i. So

$$\overline{\mu}(E) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu(E_{n,i})$$

Thus, E is the union of countable collection of sets of finite $\overline{\mu}$-measure and hence $\mathscr{S}^*$ is σ-finite.

10.3. Measure Spaces:

Definition 10.8. A pair $[[X, \mathscr{S}]]$ where $\mathscr{S}$ is a σ-algebra of subsets of a space X, is called a *measurable space*. The sets of $\mathscr{S}$ are called measurable sets.

Definition 10.9. A triple $[[X, \mathscr{S}, m]]$ is called a *measure space* if $[[X, \mathscr{S}]]$ is a measurable space and μ is a measure on $\mathscr{S}$.

Example 10.7. $[[R, \mathscr{M}, m]]$ and $[[R, \mathscr{B}, m]$ are measurable spaces, where $\mathscr{B}$ denotes the Borel sets and where in the second example m restricted to $\mathscr{B}$.

Theorem 10.6. *Let $\{E_i\}$ be a sequence of measurable sets. We have*

(i) If $E_1 \subseteq E_2 \subseteq \dots$ then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.

(ii) If $E_1 \supseteq E_2 \supseteq \dots$ and $\mu(E_1)\infty$, then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.

Proof. See Theorem 8.9 ■

Definition 10.10. Let f be an extended real-valued function defined on X. Then f is said to be *measurable* if $\forall \alpha$, $[x : f(x) \geq \alpha] \in \mathscr{S}$.

Example 10.8. Let $[[X, \mathscr{S}]]$ be a measurable space and let $X = \bigcup_{n=1}^{\infty} X_n$ where, for each n, $X_n \in \mathscr{S}$ and $X_n \cap X_m = \emptyset$ for $n \neq m$. Write $\mathscr{S}_n = B \cap X_n; B \in \mathscr{S}]$. Show that f is measurable with respect to $[[X, \mathscr{S}]]$ only if, for each n its restriction f_n to X_n is measurable with respect to $[[X_n, \mathscr{S}_n]]$ and conversely, if for each n the functions f_n are measurable with respect to $[[X_n, \mathscr{S}_n]]$ and f is defined by $f(x) = f_n(x)$ when $x \in X_n$, then f is measurable with respect to $[[X, \mathscr{S}]]$.

Solution: For each α, $[x : f_n(x) > \alpha] = [x : f(x) > \alpha] \cap X_n$. So f_n is measurable with respect to the measurable set $[[X_n, \mathscr{S}_n]]$. The converse follows from $[x : f(x) > \alpha] = \bigcup_{n=1}^{\infty} [x : f_n(x) > \alpha]$.

Theorem 10.7. *The measurability of f is equivalent to*

(i) $\forall \alpha$, $[f(x) \geq \alpha] \in \mathscr{S}$,

(ii) $\forall \alpha$, $[f(x) < \alpha] \in \mathscr{S}$,

(iii) $\forall \alpha$, $[f(x) \leq \alpha] \in \mathscr{S}$.

Proof. See Theorem 9.3 ■

Example 10.9.

(i) if f is measurable, then $[x : f(x) = \alpha]$ is measurable for each extended real number α,

(ii) the constant functions are measurable,

(iii) the characteristic function χ_A is measurable, if and only if $A \in \mathscr{S}$;

(iv) a continuous function of a measurable function is measurable.

Theorem 10.8. *If c is a real number and f, g measurable functions, then $f + c$, cf, $f + g, g - f$ and fg are also measurable.*

Proof. See Theorem 9.4 ■

Theorem 10.9. *If $f_i (i = 1, 2, \ldots)$ is measurable, then* $\sup_{1 \le i \le n} f_i$, $\inf_{1 \le i \le n} f_i$, $\sup f_n$, $\inf f_n$, $\limsup f_n$ *and* $\liminf f_n$ *are also measurable.*

Proof. See Theorem 9.5 ■

Definition 10.11. If a property holds except on a measurable set E such that $\mu(E) = 0$, we say that it holds *almost everywhere* with respect to μ, written $a.e.(\mu)$

Example 10.10. The limit of a pointwise convergent sequence of measurable function is measurable.

Example 10.11. Let $f = g\ a.e.(\mu)$, where μ is a complete measure. Show that if f is measurable, so is g.

Solution: Write

$$\begin{aligned} E &= [x : g(x) > \alpha] \\ E_1 &= [x : f(x) > \alpha] \\ E_2 &= [x : f(x) \neq g(x)] \end{aligned}$$

Clearly E_1 and E_2 are measurable.

As μ is measure, so $E_1 \cap E_2$ is also measurable.

So $E = (E_1 - E_2) \cup (E \cap E_2)$ is measurable.

Hence g is measurable.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- σ-algebra, σ-ring, measure space and its properties.

Check Your Progress:

1. Describe the ring generated by the finite open intervals.

2. Show that if μ is a non-negative set function on a ring and is countably additive and is finite on some set, then μ is a measure.

3. Show that if μ is not complete, then f measurable and $f = g\ a.e.$ do not imply g measurable.

Choose the correct or more suitable answer:

1. if $A, B \in \mathscr{R}$ and $A \subseteq B$, then

   (a) $\mu(A) < \mu(B)$.

   (b) $\mu(A) > \mu(B)$.

   (c) $\mu(A) \leq \mu(B)$.

   (d) $\mu(A) \geq \mu(B)$.

Answer:

(1) c

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.

2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.

3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-IV

Block-IV

UNIT-11

LEBESGUE INTEGRAL

Objectives

After completion of this unit, students will be able to

- ★ define Lebesgue integral for non-negative function.
- ★ derive Fatou's lemma, Lebesgue's Monotone Convergence theorem.

Overview

In analysis, it is often convenient to replace an expression of the form $\int \sum f_n dx$ by $\sum \int f_n dx$, or $\int \lim f_n dx$ by $\lim \int f_n dx$ or $\int \lim_{\alpha \to \alpha_0} f_\alpha dx$ by $\lim_{\alpha \to \alpha_0} \int f_\alpha dx$. In this chapter, we define a definition of an integral which applies to a large Lebesgue measurable function and which allows the interchange of integral and sum or limit in very general circumstances. First we define Lebesgue integral, so called simple function and then we extend to non-negative measurable functions.

11.1. Integration of Non-negative functions:

We first define integral for the class of non-negative measurable functions and study the properties of the integral.

Definition 11.1. A non-negative finite-valued function $\varphi(x)$, taking only a finite number of different values is called a *simple function*.

If $a_1, a_2, \ldots, a_n$ are the distinct values of φ and $A_i = [x : \varphi(x) = a_i]$ then

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

Definition 11.2. Let φ be a measurable simple function. Then the integral of φ is defined as

$$\int \varphi dx = \sum_{i=1}^{n} a_i m(A_i)$$

where $a_1, a_2, \ldots, a_n$ are the distinct values that φ assumes and $A_i = [x : \varphi(x) = a_i]$.

Example 11.1. Let the sets A_i be defined by $A_i = [x : \varphi(x) = a_i]$. Then $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} A_i = R$.

Proof. Let if possible $A_i \cap A_j \neq \emptyset$ if $i \neq j$. Then there exist x such that $x \in A_i \cap A_j$.

$$x \in A_i \Rightarrow \varphi(x) = a_i$$
$$x \in A_j \Rightarrow \varphi(x) = a_j$$
$$\Rightarrow a_i = a_j, \text{ where } i \neq j$$

a contradiction.

Thus, $A_i \cap A_j = \emptyset$ if $i \neq j$. Further $\bigcup_{i=1}^{n} A_i = R$.

For any $x \in R, \varphi(x) = a_i$, for some i.

$$\Rightarrow \quad x \in A_i \text{ for some } i$$
$$\Rightarrow \quad x \in \bigcup_{i=1}^{n} A_i$$
$$\Rightarrow \quad R \subset \bigcup_{i=1}^{n} A_i$$
$$\text{Hence, } R = \bigcup_{i=1}^{n} A_i$$
■

Definition 11.3. Let f be a non-negative measurable function. Then *integral* of f denoted by $\int f dx$, is defined as

$$\int f dx = \sup \int \varphi(x) dx$$

where the supremum is taken over all measurable simple functions φ such that $\varphi \leq f$.

Definition 11.4. For any measurable set E and any non-negative measurable function f, we define the *integral* of f over E by

$$\int_E f dx = \int f \chi_E dx$$

Note 11.1. If the set E in definition (11.4) is an interval, say $[a, b]$ then in place of $\int_E f dx$ we write $\int_a^b f dx$ if $a > b$ we use the usual convection

$$\int_a^b f dx = -\int_b^a f dx$$

The integral defined in definition (11.4) is called the *Lebesgue integral* of the function f.

Example 11.2. If φ is a measurable simple function, Definition (11.2) and Definition (11.3) both give a value for its integral. Show that these values are the same.

Solution:

Let $\int^* dx$ be the integral value of φ as given by Definition (11.2) and let $\int \varphi dx$ be the integral value of φ as given by Definition (11.3).

If $a_1, a_2, \ldots, a_n$ are the distinct values of φ, then by definition

$$\int \varphi dx = \sum_{n=1}^{\infty} a_i m(A_i)$$

where $A_i = [x : \varphi(x) = a_i]$.

Also, by definition

$$\int \varphi dx = \sup \int \psi dx$$

where ψ is a measurable simple function such that $\psi \leq \varphi$.

So,

$$\int^{*} \varphi dx \quad \leq \quad \int \varphi dx \tag{11.1}$$

Since φ is a measurable simple function.

Also, if $\psi \neq \varphi$ is a measurable simple function with distinct values b_j $(j = 1, 2, \ldots, m)$ and $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$, then

$$\int \psi dx \quad = \quad \sum_{j=1}^{m} b_j m(B_j)$$

Since $A_i \cap A_k = \emptyset$ for $i \neq k$, $B_j \cap B_k = \emptyset$ for $j \neq k$ and $\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j = R$.

Also, we have $A_i \cap B_j$, $(i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots m)$ are disjoint sets and $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap B_j) = R$.

Hence
$$\int \psi dx \quad = \quad \sum_{j=1}^{m} b_j m(B_j) = m(B_j \cap A_i)$$

where $b_j < a_i$ if $m(B_j \cap A_i) > 0$. Thus,

$$\begin{aligned}
\int \psi dx \quad &\leq \quad \sum_{j=1}^{m} \sum_{i=1}^{n} a_i m(B_j \cap A_i) \\
&= \quad \sum_{i=1}^{n} a_i m(A_i) = \int \varphi dx \\
\Rightarrow \sup_{\psi \leq \varphi} \int \psi dx \quad &\leq \quad \int \varphi dx \\
i.e., \quad \int^{*} \varphi dx \quad &\leq \quad \int \varphi dx
\end{aligned} \tag{11.2}$$

From (11.1) and (11.2), we get

$$\int^{*} \varphi dx \quad = \int \varphi dx$$

Theorem 11.1. *If φ is a measurable simple function, then*

(i) $\int_E = \sum_{i=1}^{n} a_i m(A_i \cap E)$ *for any measurable set E.*

(ii) $\int_{A \cup B} \varphi dx = \int_A \varphi dx + \int_B \varphi dx$ *for any disjoint measurable sets A and B.*

(iii) $\int a\varphi dx = a \int \varphi dx$ *if $a > 0$.*

Proof.

$$\begin{aligned}
\chi_E(X) &= \begin{cases} 1 & \text{if } x \leq E \\ 0 & \text{if } 0 < x \leq E^c \end{cases} \\
\varphi(x) &= a_i \quad \text{and} \quad A_i = [x : \varphi(x) = a_i] \\
\therefore \quad \varphi\chi_E(X) &= \begin{cases} a_i \cdot 1 & \text{if } x \in A_i \cap E \\ a_i \cdot 0 & \text{if } x \in A_i \cap E^c \end{cases}
\end{aligned}$$

(i) $\varphi\chi_E$ is a simple measurable function for which takes the value a_i on $A_i \cap E$ and 0 on $A_i \cap E^c$.

$$\begin{aligned}
\int_E \varphi dx &= \int \varphi\chi_E dx \quad \text{(by definition (11.4))} \\
&= \sum_{i=1}^{n} a_i m(A_i \cap E) \quad \text{(by definition (11.2))}
\end{aligned}$$

$$\begin{aligned}
(ii) \int_A \varphi dx + \int_B \varphi dx &= \sum_{i=1}^{n} a_i m(A_i \cap A) + \sum_{i=1}^{n} a_i m(A_i \cap B) \quad \text{(by definition (11.2))} \\
&= \sum_{i=1}^{n} a_i \left[m(A_i \cap A) + m(A_i \cap B)\right] \\
&= \sum_{i=1}^{n} a_i \left[m\left((A_i \cap A) \cup m(A_i \cap B)\right)\right] \quad (\because A \text{ and } B \text{ are disjoint}) \\
&= \sum_{i=1}^{n} a_i \left[m(A_i \cap (A \cup B))\right] \\
&= \int_{A \cup B} \varphi dx \quad \text{(by (i))}
\end{aligned}$$

(iii) φ takes the values a_i on A_i for $1 \leq i \leq n$ which implies $a\varphi$ takes the values aa_i on A_i for $1 \leq i \leq n$. Further, $a\varphi$ is a simple measurable function, if $a > 0$. Then by definition (11.2), we have

$$\begin{aligned}
\int a\varphi dx &= \sum_{i=1}^{n} aa_i m(A_i) \\
&= a \sum_{i=1}^{n} a_i m(A_i) \\
&= a \int \varphi \quad \text{(by definition (11.2))}
\end{aligned}$$

Hence the proof. ■

Example 11.3. Show that if f is a non-negative measurable function, then $f = 0\ a.e.$ if and only if $\int f dx = 0$.

Solution: Suppose f is a non-negative measurable function and $f = 0\ a.e.$ Now, our wish is to prove that $\int f dx = 0$.

Since $f = 0$ *a.e.*, then $m[x : f(x) \neq 0] = 0$. Suppose φ is a measurable simple function such that $\varphi \leq f$. Then $m[x : \varphi(x) \neq 0] = 0$.

Hence,

$$m(A_i) = m[x : \varphi(x) = a_i] = 0 \ \forall i$$

Thus, for every measurable simple function $\varphi \leq f$, $\int \varphi dx = 0$.

So, by definition (11.3), we have

$$\int f dx = \sup_{\varphi \leq f} \int \varphi dx = 0$$

Conversely, let $\int f dx = 0$, we have to prove that $f = 0$ *a.e.* Let

$$E = [x : f(x) > 0] \text{ and } E_n = \left[x : f(x) \geq \frac{1}{n}\right], \quad n = 1, 2, 3, \ldots$$

Then $E = \bigcup_{n=1}^{\infty} E_n$, if $x \in E_n, f(x) \geq \frac{1}{n}$ and $\chi_{E_n}(x) = 1$.

Thus,

$$f(x) \geq \left(\frac{1}{n}\right)\chi_{E_n}(x) \quad \forall\ x \tag{11.3}$$

$$\text{By definition} \int f dx = \sup_{\varphi \leq f} \int \varphi dx$$

$$\Rightarrow \int \varphi dx \leq \int f dx$$

for every measurable simple function φ. Since $\left(\frac{1}{n}\right)\chi_{E_n}$ is a simple measurable function and thus (11.3) implies that

$$0 = \int f dx \geq \int \left(\frac{1}{n}\right)\chi_{E_n}(x) = \frac{1}{n}\int \chi_{E_n} dx = \frac{1}{n} m(E_n)$$

So $m(E_n) = 0$. But

$$0 \leq m[x : f(x) > 0] = m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n) = 0$$

$$\Rightarrow m[x : f(x) > 0] = 0$$

Hence $f = 0$ *a.e.*

Theorem 11.2. *Let f and g be non-negative measurable functions.*

(i) If $f \leq g$, then $\int f dx \leq \int g dx$.

(ii) If A is measurable set and $f \leq g$, then $\int_A f dx \leq \int_A g dx$.

(iii) If $a \geq 0$, then $\int a f dx = a \int f dx$.

(iv) If A and B are measurable sets and $A \supseteq B$, then $\int_A f dx \geq \int_B f dx$.

Proof. (i) Let $f \leq g$ $\quad (i.e.,)\ f(x) \leq g(x)\ \forall x.$

Let φ be a measurable simple functions such that $\varphi \leq f.$

Similarly, ψ is a measurable simple functions such that $\psi \leq g.$

Since, $f \leq g,$ then $\{\varphi : \varphi \leq f\} \subseteq \{\psi : \psi \leq g\}.$

Hence,

$$\int f dx = \sup_{\varphi \leq f} \int \varphi dx \leq \sup_{\psi \leq g} \int \psi dx = \int g dx.$$

(ii) If $f \leq g$ on $A,$ then $f\chi_A \leq g\chi_B.$ then by definition (11.3) and $(i),$ we have

$$\int_A f dx = \int f\chi_A dx \leq \int g\chi_A dx = \int_A g dx$$

(iii) If $a = 0$ then the result is quite obvious. So, assume that $a > 0.$ Then $\varphi \leq af$ if and only if $\varphi = a\psi,$ where ψ is a simple and $\psi \leq f.$ But if ψ is a measurable simple functions, then $a\psi$ is also a measurable function and $\int a\psi dx = a \int \psi dx.$ Thus,

$$\begin{aligned} \int af dx &= \sup_{\varphi \leq af} \int \varphi dx = \sup_{a\psi \leq af} \int a\psi dx \\ &= \sup_{\psi \leq f} a \int \psi dx = a \sup_{\psi \leq f} \int \psi dx \\ &= a \int f dx \end{aligned}$$

$(iv),$ If $B \subseteq A$ implies $\chi_B \leq \chi_A.$ So, $f\chi_B \leq f\chi_A.$ Then, by (i) we have

$$\int_A f dx = \int f\chi_A\, dx \geq \int f\chi_B\, dx = \int_B f dx$$

Hence the proof. ■

The following result will be basic in proving convergence theorem.

Theorem 11.3 (Fatou's lemma)**.** *Let* $\{f_n, n = 1, 2, \ldots\}$ *be a sequence of non-negative measurable functions. Then*

$$\liminf \int f_n dx \geq \int (\liminf f_n)\, dx$$

Proof. Let $f = \liminf f_n.$ As each f_n is non-negative then $f = \liminf f_n$ is also non-negative.

Also, each f_n is measurable, then by theorem (9.5) $\liminf f_n$ is also measurable.

Thus, $\liminf f_n = f$ is non-negative measurable function.

Let φ be a measurable simple function such that $\varphi \leq f.$

Now, our aim is to prove that

$$\begin{aligned}
\int \liminf f_n dx &\leq \liminf \int f_n dx \\
i.e., \int f dx &\leq \liminf \int f_n dx \\
i.e., \sup_{\varphi \leq f} \int \varphi dx &\leq \liminf \int f_n dx \qquad \left(\because \int f dx = \sup_{\varphi \leq f} \int \varphi dx\right) \\
i.e., \int \varphi dx &\leq \liminf \int f_n dx \qquad (11.4)
\end{aligned}$$

Case (i)**:** Let $\int \varphi dx = \infty$.

Then from the definition of integral of a measurable simple function, there exists a measurable set A with $m(A)$ and φ is a constant on A.

Choose a such that $\varphi > a > 0$ on A.

$$\begin{aligned}
\text{Define } g_k(x) &= \inf\{f_k(x), f_{k+1}(x), \ldots\} \\
&= \inf_{j \geq k}\{f_k(x)\}
\end{aligned}$$

Since each f_n is a non-negative measurable function, then by theorem (9.5), each $g_k(x)$ is a non-negative measurable function.

$$\begin{aligned}
\text{Let } A_n &= [x : g_k(x) > a, \forall k \geq n] \\
&= \bigcap_{k=n}^{\infty} [x : g_k(x) > a]
\end{aligned}$$

Since each g_k is a measurable function, then it follows that each A_n is a measurable set.

Also, $A_n \subseteq A_{n+1} \quad \forall n = 1, 2, 3, \ldots$

$$\begin{aligned}
g_k(x) &= \inf\{f_k(x), f_{k+1}(x), \ldots\} \\
i.e., \ g_1(x) &= \inf\{f_1(x), f_2(x), \ldots\} \\
g_2(x) &= \inf\{f_2(x), f_3(x), \ldots\}
\end{aligned}$$

Thus, from the definition of $g_k(x)$, each x, $g_k(x)$ is monotonic increasing sequence and

$$\begin{aligned}
\lim_{k \to \infty} g_k(x) &= \liminf f_k(x) \\
&= f(x) \geq \varphi(x) \quad (\because \varphi \leq f) \qquad (11.5)
\end{aligned}$$

So, $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

Now, we have $\{A_n\}$ to be a sequence of measurable sets such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$

$$\begin{aligned}
\lim_{n \to \infty} m(A_n) &= m\left(\lim A_n\right) \\
\Rightarrow \lim_{n \to \infty} m(A_n) &= m\left(\bigcup_{n=1}^{\infty} A_n\right) \qquad (11.6)
\end{aligned}$$

Since, if $E_1 \subseteq E_2 \subseteq \ldots,$ then $\lim E_i = \bigcup_{i=1}^{\infty} E_i$.

Now,

$$A \subseteq \bigcup_{n=1}^{\infty} A_n \tag{11.7}$$

$$\Rightarrow m(A) \leq m\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$\Rightarrow \infty \leq m\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$$

$$\therefore \quad \lim_{n\to\infty} m(A_n) = \infty \tag{11.8}$$

From the definition of g_n, we have $g_n \leq f_n$ and

$$\begin{aligned}
\Rightarrow \quad \int f_n dx &\geq \int g_n dx > a\, m(A_n) \\
\Rightarrow \liminf \int f_n dx &\geq \liminf a\, m(A_n) \\
\Rightarrow \liminf \int f_n dx &\geq a \lim_{n\to\infty} m(A_n) \\
\Rightarrow \liminf \int f_n dx &\geq a.\infty \quad \text{(by using (11.8))} \\
\Rightarrow \liminf \int f_n dx &= \infty \\
\Rightarrow \liminf \int f_n dx &= \int \varphi dx
\end{aligned}$$

This proves case (i).

Case (ii): Let $\int \varphi dx < \infty$.

Write $B = \{x : \varphi(x) > 0\}$ is of finite measure .

Let M be the largest value of the function φ. *i.e.*, $\varphi(x) \leq M \quad \forall x$.

Let $0 < \epsilon < 1$ be arbitrary.

Put $B_n = \{x : g_k(x) > (1-\epsilon)\varphi(x) \quad \forall k \geq n\}$

where g_k is defined as in case (i).

Then B_n are all measurable sets and $B_n \subseteq B_{n+1} \ \forall n$.

Note that from the definition of g_k, $\{g_k(x)\}$ is an increasing sequence for every x and

$$\lim_{k\to\infty} g_k(x) = f(x) \geq \varphi(x)$$

Further $B \subseteq \bigcup_{n=1}^{\infty} B_n$

$$\begin{aligned}
\therefore \quad \lim_{k\to\infty} g_k(x) &= f(x)\\
\Rightarrow \quad \text{given } \epsilon\varphi(x) &> 0, \exists\, n \text{ such that } |g_k(x) - f(x)| < \epsilon\varphi(x) \ \ \forall k \geq n\\
\Rightarrow \quad f(x) - g_k(x) &< \epsilon\varphi(x), \quad \forall k \geq n\\
\Rightarrow \quad \varphi(x) - g_k(x) &\leq f(x) - g_k(x) < \epsilon\varphi(x), \quad \forall k \geq n\\
\Rightarrow \quad g_k(x) &> (1-\epsilon)\varphi(x), \quad \forall k \geq n\\
\Rightarrow \quad x \in B_n & \quad \text{for some } n\\
\Rightarrow \quad x \in \bigcup_{n=1}^{\infty} B_n &
\end{aligned}$$

$$B_n \subseteq B_{n+1} \ \Rightarrow \ B - B_n \supseteq B - B_{n+1} \qquad \forall n$$

Also $B - B_n$ is a measurable set $\forall n$.

$$B - B_n \subseteq B \Rightarrow \ m(B - B_n) \leq m(B) < \infty.$$

On the other hand,

$$\begin{aligned}
\lim(B - B_n) &= m\left[\bigcap_{n=1}^{\infty}(B - B_n)\right]\\
&= m\left(B - \bigcup_{n=1}^{\infty} B_n\right)\\
&= m(\emptyset) = 0
\end{aligned}$$

$$\begin{aligned}
\therefore \ \text{given } \epsilon > 0, \ \exists\, N \text{ such that } |m(B - B_n) - 0| &< \epsilon, \quad \forall n \geq N\\
\Rightarrow \quad m(B - B_n) &< \epsilon \ \ \forall n \geq N
\end{aligned}$$

Hence $n \geq N \Rightarrow$

$$\begin{aligned}
\int g_n dx &\geq \int_{B_n} g_n dx > \int_{B_n} (1-\epsilon)\varphi(x)dx\\
&= (1-\epsilon)\int_{B_n}\varphi(x)dx\\
&= (1-\epsilon)\left[\int_B \varphi(x)dx - \int_{B-B_n}\varphi(x)dx\right]\\
i.e., \quad \int g_n dx &\geq (1-\epsilon)\int_B \varphi dx - (1-\epsilon)\int_{B-B_n}\varphi dx\\
&\geq (1-\epsilon)\int_B \varphi dx - Mm(B - B_n)\\
&= (1-\epsilon)\left[\int_B \varphi dx - \int_{B^c}\varphi dx\right] - Mm(B - B_n)\\
&> (1-\epsilon)\int \varphi dx - M\epsilon\\
&= \int \varphi dx - \epsilon\left[\int \varphi dx + M\right]
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary

$$n \geq N \quad \Rightarrow \quad \int g_n dx \quad \geq \quad \int \varphi dx$$

$$\therefore \quad \liminf \int g_n dx \quad \geq \quad \int \varphi dx \tag{11.9}$$

$$\because g_n \leq f_n, \ \forall n \quad \Rightarrow \quad \int g_n dx \quad \leq \quad \int f_n dx$$

$$\Rightarrow \quad \liminf \int g_n dx \quad \leq \quad \liminf \int f_n dx \tag{11.10}$$

From (11.9) and (11.10), we have

$$\Rightarrow \quad \liminf \int f_n dx \quad \geq \quad \int \varphi dx$$

$$\Rightarrow \quad \sup_{\varphi} \int \varphi dx \quad \leq \quad \liminf \int f_n dx$$

$$\Rightarrow \quad \int f dx \quad \leq \quad \liminf \int f_n dx$$

$$\Rightarrow \quad \int \lim f_n dx \quad \leq \quad \liminf \int f_n dx \qquad \blacksquare$$

Theorem 11.4 (Lebesgue's Monotone Convergence Theorem)**.** *Let $\{f_n, n = 1, 2, \ldots\}$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotonically increasing for each x. Let $f = \lim f_n$. Then $\int f dx = \lim \int f_n dx$.*

Proof.

$$\lim f_n = f \quad \Rightarrow \quad \liminf f_n = f \tag{11.11}$$

Therefore, By Fatou's lemma

$$\liminf \int f_n dx \quad \geq \quad \int \lim f_n dx$$

$$\Rightarrow \quad \liminf \int f_n dx \quad \geq \quad \int f dx \quad \text{(using (11.11))} \tag{11.12}$$

f_n is increasing and convergent to f

$$\Rightarrow \quad f_n \quad \leq \quad f \quad \forall n$$

$$\Rightarrow \quad \int f_n dx \quad \leq \quad \int f dx$$

$$\Rightarrow \quad \limsup \int f_n dx \quad \leq \quad \int f dx \tag{11.13}$$

Combining (11.12) and (11.13), we get

$$\limsup \int f_n dx \quad \leq \quad \int f dx \leq \liminf \int f_n dx \leq lim\sup \int f_n dx$$

$$\Rightarrow \quad \int f dx \quad = \quad \limsup \int f_n dx = \liminf \int f_n dx$$

$$\Rightarrow \quad \int f dx \quad = \quad \lim_{n \to \infty} \int f_n dx \qquad \blacksquare$$

Theorem 11.5. *Let f be a non-negative measurable function. Then there exists a sequence $\{\varphi_n\}$ of measurable simple functions such that for each x, $\varphi_n(x) \uparrow f(x)$.*

Proof. Since f is a non-negative measurable function, range of f is a subset of $[0,\infty)$.

For every n, Consider the partition P_n of $[0,\infty)$ given by the points $0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{(2^n-1)}{2^n}, n, \infty$.

For $1 \le k \le n2^n$, let

$$E_{nk} = \left[x : \frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\right]$$
$$\text{and} \quad F_n = [x : f(x) > n]$$

Since f is measurable, thus the sets E_{nk}, F_n are measurable for all n and $1 \le k \le n2^n$.

For every $n = 1, 2, \ldots,$ Put

$$\varphi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\chi_{E_{nk}} + n\chi_{F_n}$$

Then the function φ_n are measurable simple functions. The partition P_{n+1} giving φ_{n+1} is a refinement of φ_n. So, for each x, we have $\varphi_n(x) \le \varphi_{n+1}(x)$.

If $f(x) = \infty$, then for all n, $x \in F_n$ and hence $\varphi_n(x) = n$. Thus, We have $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

Further, if $f(x) < \infty$, then for $n = 1, 2, \ldots,$ $x \in E_{nk}$ for some k, $1 \le k \le 2^n$, $i.e., \varphi_n(x) = \frac{k-1}{2^n}$. Since $\frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}$ and thus we have $\varphi_n(x) \le f(x)$ and

$$|f(x) - \varphi_n(x)| < \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n}$$

Hence, $\lim_{n\to\infty} \varphi_n(x) = f(x)$. This completes the proof. ■

Corollary 11.1. *Suppose $\{\varphi_n\}$ is a sequence of measurable simple functions such that for each x, $\varphi_n(x) \uparrow f(x)$, where f is a non-negative measurable function. Then*

$$\lim_{n\to\infty} \int \varphi_n dx = \int f dx$$

Proof. Suppose, for each x, $\varphi_n(x) \uparrow f(x)$, we have $\{\varphi_n(x)\}$ is monotone increasing sequence for each x.

Hence, by Lebesgue monotone convergence theorem, we have

$$\int f dx = \lim_{n\to\infty} \int \varphi_n dx$$ ■

Theorem 11.6. *Let f and g be non-negative measurable functions. Then*

$$\int f dx + \int g dx = \int (f+g) dx$$

Proof. Let φ and ψ be two measurable simple functions. Let the values of φ be $a_1, a_2, \ldots, a_n$ taken on the sets $A_1, A_2, \ldots, A_n$ and let the values of ψ be $b_1, b_2, \ldots, b_m$ on the sets $B_1, B_2, \ldots B_m$. Then the simple function $\varphi + \psi$ has the values $a_i + b_j$ on the measurable set $A_i \cap B_j$. Thus, we have

$$\begin{aligned}
\int_{A_i\cap B_j}(\varphi+\psi)dx &= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i+b_j)m(A_i\cap B_j)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}a_i m(A_i\cap B_j)+\sum_{i=1}^{n}\sum_{j=1}^{m}b_j m(A_i\cap B_j)\\
&= \int_{A_i\cap B_j}\varphi dx+\int_{A_i\cap B_j}\psi dx
\end{aligned}$$

But the union of nm disjoint sets $A_i \cap B_j$ is R. Hence,

$$\begin{aligned}
\int(\varphi+\psi)dx &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{A_i\cap B_j}(\varphi+\psi)dx\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{A_i\cap B_j}\varphi dx+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{A_i\cap B_j}\psi dx\\
&= \int\varphi dx+\int\psi dx
\end{aligned}$$

This proves the theorem for measurable simple functions φ and ψ.

Let f and g be any non-negative measurable functions. Let $\{\varphi_n\}, \{\psi_n\}$ be sequences of measurable simple functions such that $\varphi \uparrow f$ and $\psi \uparrow g$. Then,we have $\varphi+\psi \uparrow f+g$. But

$$\int(\varphi_n+\psi_n)dx = \int\varphi_n dx+\int\psi_n dx$$

Hence, by applying Lebesgue monotone convergence theorem, we have

$$\begin{aligned}
\int(f+g)dx &= \lim_{n\to\infty}\int(\varphi_n+\psi_n)dx\\
&= \lim_{n\to\infty}\left(\int\varphi_n dx+\int\psi_n dx\right)\\
&= \lim_{n\to\infty}\int\varphi_n dx+\lim_{n\to\infty}\int\psi_n dx\\
&= \int f dx+\int g dx
\end{aligned}$$

■

Theorem 11.7. *Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then*

$$\int\sum_{n=1}^{\infty}f_n dx = \sum_{n=1}^{\infty}\int f_n dx$$

Proof. Let us prove the theorem by the method of induction.

By theorem (11.6), we have

$$\int(f+g)dx = \int f dx+\int g dx$$

Let $S_n = f_1+f_2+\ldots+f_n = \sum_{k=1}^{n}f_k$ denote the n^{th} partial sum of $\sum f_n$. Thus, we have

$$\int S_n = \int \sum_{k=1}^{n} f_k dx = \sum_{k=1}^{n} \int f_k dx$$

Taking limits as $n \to \infty$, we get

$$\lim_{n\to\infty} \int S_n dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k dx$$

$$\Rightarrow \int \lim_{n\to\infty} S_n dx = \sum_{k=1}^{\infty} \int f_k dx$$

$$\Rightarrow \int \lim_{n\to\infty} \sum_{k=1}^{n} f_k dx = \sum_{k=1}^{\infty} \int f_k dx$$

$$\Rightarrow \int \sum_{n=1}^{\infty} f_n dx = \sum_{n=1}^{\infty} \int f_n dx$$

■

Example 11.4. Give an example of a sequence $\{f_n\}$ of non-negative measurable functions such that

$$\liminf \int f_n dx > \int (\liminf f) dx$$

In other words, give an example where strict inequality occurs in Fatou's lemma.

Solution:

Let $f_{2n-1} = \chi_{[0,1]}$ and $f_{2n} = \chi_{[1,2]}$, (n=1,2,...)

Then each f_n is non-negative measurable function.

For any x, $\liminf f_n(x) = 0$ and $\int f_n(x)dx = 1 \ \forall n$.

Hence,

$$\liminf \int f_n dx = 1 > 0 = \int (\liminf f_n) dx$$

Example 11.5. Show that $\int_1^{\infty} \frac{dx}{x} = \infty$.

Solution: Let $f(x) = \frac{1}{x}$.

Clearly, $f(x)$ is continuous for $x > 0$ and also it is measurable function.

Thus, $f(x)$ is non-negative measurable function ($\because f(x)$ is non-negative for $x > 0$)

Therefore, $\int_1^{\infty} \frac{dx}{x}$ is well defined.

For any n, we have $\int_1^{\infty} \frac{dx}{dx} > \int_1^{n} \frac{dx}{x}$.

If $k - 1 \le x < k$, then

$$\frac{1}{x} > \frac{1}{k} \quad \text{So, for } n = 2, 3, \ldots$$

So,

$$\int_1^n \frac{dx}{x} > \sum_{k=2}^{n} \int_1^n k^{-1} \chi_{[k-1,k)} dx$$
$$> \sum_{k=2}^{n} k^{-1}$$

$$\text{As } n \to \infty, \ \sum_{k=2}^{n} k^{-1} \to \infty$$
$$\Rightarrow \int_1^{\infty} \frac{dx}{x} = \infty$$

Example 11.6. Define $f(x)$ on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ n & \text{if } x \text{ is irrational} \end{cases}$$

where n is the number of zeros immediately after the decimal point in the representation of x on the decimal scale. Show that f is measurable and find $\int_0^1 f dx$.

Solution:

For $x \in (0,1]$, let

$$g(x) = \begin{cases} n & 10^{-(n+1)} \leq x \leq 10^{-n}, n = 1, 2, \ldots \\ n & x = 1 \end{cases}$$

and $g(0) = 0$; $g(1) = 0$. That is,

$$g(x) = 0 \text{ if } 0.1 \leq x \leq 1$$
$$g(x) = 1 \text{ if } 0.01 \leq x \leq 0.1$$
$$g(x) = 2 \text{ if } 0.001 \leq x \leq 0.01 \vdots$$

So, if $0 < x < 1$, $g(x)$ is the number of zero immediately after decimal point in the representation of x on the decimal scale. Hence

$$f(x) = g(x) \quad \text{if } x \text{ is irrational}$$
$$f(x) \leq g(x) \quad \text{if } x \text{ is rational}$$

So, $f = g \ a.e,$ g is measurable implies that f is measurable and

$$\int_0^1 f dx = \int_0^1 g dx$$

But,

$$\int_0^1 g dx = \int \left(\sum_{n=0}^{\infty} g \chi_{I_n} \right) dx$$

where $I_n = \left(\frac{1}{10^{n+1}} - \frac{1}{10^n} \right)$. So

$$\int_0^1 f dx = \int_0^1 g dx = \sum_{n=0}^{\infty} n\left(\frac{1}{10^n} - \frac{1}{10^{n+1}}\right)$$
$$= \sum_{n=1}^{\infty} \frac{9n}{10^{n+1}} = \frac{1}{9}.$$

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of simple function and measurable function.
- derive Fatou's lemma and Lebesgue monotone convergence theorem.

Check Your Progress:

1. Define simple function.
2. Define Lebesgue integral of the function f.
3. State and Prove Fatou's lemma.
4. State and Prove Lebesgue's monotone convergence theorem.

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.
2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.
3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-IV

UNIT-12

THE GENERAL INTEGRAL

Objectives

After completion of this unit, students will be able to

★ derive Lebesgue dominated convergence theorem.

★ evaluate definite integral.

Overview

In this unit, we will illustrate the definition of integral to real valued function and also explain in detail for proving Lebesgue dominated convergence theorem.

12.1. The General Integral:

The definition of the integral will now be extended to a wide class of real-valued functions, not necessarily non-negative.

Definition 12.1. If $f(x)$ is any real function,

$$\begin{aligned} f^+(x) &= \max(f(x), 0) \\ f^-(x) &= \max(-f(x), 0) \end{aligned}$$

are said to be the *positive* and *negative parts* of f, respectively.

Theorem 12.1.

(i) $f = f^+ - f^-$; $|f| = f^+ + f^-$; $f^+, f^- \geq 0$.

(ii) f *is measurable if and only if* f^+ *and* f^- *are both measurable.*

Proof. (i) To Prove: $f = f^+ - f^-$

Case (a)**:** Let $f(x) > 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = f(x) \\ f^-(x) &= \max(-f(x), 0) = 0 \\ \Rightarrow \quad f^+ - f^- &= f \end{aligned}$$

Case (b)**:** Let $f(x) < 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = 0 \\ f^-(x) &= \max(-f(x), 0) = -f(x) \\ \Rightarrow \quad f^+ - f^- &= f \end{aligned}$$

To Prove: $|f| = f^+ + f^-$.

Case (a)**:** Let $f(x) > 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = f(x) \\ f^-(x) &= \max(-f(x), 0) = 0 \\ \Rightarrow \quad f^+ + f^- &= f \qquad (12.1) \end{aligned}$$

Case (b)**:** Let $f(x) < 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = 0 \\ f^-(x) &= \max(-f(x), 0) = -f(x) \\ \Rightarrow \quad f^+ + f^- &= f \end{aligned} \qquad (12.2)$$

From (12.1) and (12.2), we get

$$\begin{aligned} f^+ + f^- &= \pm f \\ \textit{i.e.,} \quad f^+ + f^- &= |f| \end{aligned}$$

To Prove: $f^+ \geq 0$; $f^- \geq 0$.

Case (a): Let $f(x) > 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = f(x) \geq 0 \\ f^-(x) &= \max(-f(x), 0) = 0 \\ \Rightarrow \quad f^+ &\geq 0; \quad f^- = 0 \end{aligned}$$

Case (b): Let $f(x) < 0$

$$\begin{aligned} f^+(x) &= \max(f(x), 0) = 0 \\ f^-(x) &= \max(-f(x), 0) = -f(x) \geq 0 \\ \Rightarrow \quad f^+ &= 0; \quad f^- \geq 0 \end{aligned}$$

(ii) To prove f is measurable if and only if f^+ and f^{-1} is measurable.

Necessary part: Suppose f is measurable. Now, our aim is to prove that f^+ and f^- are measurable. By definition,

$$\begin{aligned} f^+ &= \max(f(x), 0) \quad \text{which is measurable} \\ f^- &= \max(-f(x), 0) \quad \text{which is measurable} \end{aligned}$$

Thus, f^+ and f^- are measurable.

Sufficient part: Suppose f^+ and f^- are measurable. Now we have to prove that f is measurable.

By case (i), $f = f^+ - f^-$ which is the difference of two measurable functions and hence f is measurable. This completes the proof. ■

Definition 12.2. If f is a measurable function and $\int f^+ dx < \infty$, $\int f^- dx < \infty$, we say that f is *integrable*, and its integral is given by

$$\int f dx = \int f^+ dx - \int f^- dx$$

Note 12.1. If measurable function f is integrable then f^+ and f^- are measurable non-negative functions and $\int f^+ dx < \infty$, $\int f^- dx < \infty$. So,

$$\int f^+ dx + \int f^- dx = \int (f^+ + f^-) dx = \int |f| dx \quad \text{exists}$$

Hence, $|f|$ is integrable and $\int |f| dx = \int f^+ dx + \int f^- dx$.

Definition 12.3. If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable, we say that f is *integrable over* E, and its integral is given by

$$\int_E f dx = \int f \chi_E dx$$

Notation: $L(E)$ denotes the class of integrable functions over the measurable set E.

Definition 12.4. If f is a measurable function such that at least one of $\int f^+ dx, \int f^- dx$ is finite, then

$$\int f dx = \int f^+ dx - \int f^- dx$$

Note 12.2. Integrals of real-valued functions are allowed to take infinite values by Definition (12.4). So Definition (12.4) is an extension of Definition (11.3). But, it is said to be *integrable* only if $\int f^+ dx$ and $\int f^{-1} dx$ are finite. In other words, f is integrable only if $|f|$ has a finite integral.

Theorem 12.2. *Let f and g be integrable functions and let a be a real number. Then*

(i) af is integrable and $\int af dx = a \int f dx$.

(ii) $f + g$ is integrable and $\int (f + g) dx = \int f dx + \int g dx$.

(iii) If $f = 0$ a.e., then $\int f dx = 0$.

(iv) If $f \leq g$ a.e., then $\int f dx \leq \int g dx$.

(v) If A and B are disjoint measurable sets, then

$$\int_A f dx + \int_B f dx = \int_{A \cup B} f dx$$

Proof. Given that f and g are integrable functions, which implies that $\int f^+ dx < \infty$, $\int f^- dx < \infty$, $\int g^+ dx < \infty$, $\int g^- dx < \infty$.

Also,

$$\int f dx = \int f^+ dx - \int f^- dx$$
$$\int g dx = \int g^+ dx - \int g^- dx$$

(i) To Prove: af is integrable. It is enough to prove that $\int (af)^+ < \infty$ and $\int (af)^- < \infty$.

Case (a): $a > 0$.

$$\begin{aligned} \text{Given that} \quad & f \quad \text{is measurable} \\ \Rightarrow \quad & af \quad \text{is measurable} \\ \text{Now, } (af)^+ &= \max\{(af), 0\} \\ &= \max\{af(x), 0\} \\ &= a \max\{f(x), 0\} = af^+ \end{aligned}$$

In a similar way, we can prove that $(af)^- = af^-$, $(ag)^+ = ag^+$, $(ag)^- = ag^-$.

Now, consider

$$\begin{aligned} \int (af^+)dx &= \int af^+dx \\ &= a\int f^+dx < \infty \qquad (\because f^+ \text{ is measurable}) \end{aligned}$$

Similarly, we can prove that $\int (af^-)dx < \infty$.

$$\begin{aligned} \int afdx &= \int (af)^+dx - \int (af)^-dx \\ &= \int af^+dx - \int af^-dx \\ &= a\left[\int f^+dx - \int f^-dx\right] \\ &= a\int fdx \end{aligned}$$

Case (b): If $a = -1$, then $af = -f$.

$$\begin{aligned} (af)^+ &= (-f)^+ = \max(-f, 0) = f^- \\ (af)^- &= (-f)^- = \max(-(-f), 0) = \max(f, 0) = f^+ \\ \therefore \quad (-f)^+ &= f^- \\ (-f)^- &= f^+ \\ \text{Thus,} \quad \int (-f)^+dx &= \int f^-dx < \infty \\ \text{i.e.,} \quad \int (-f)^+dx &< \infty \end{aligned}$$

i.e., $(-f)^+$ is integrable

Similarly,we can prove that $(-f)^-$ is integrable.

Hence, when $a = -1$, af is integrable. Further,

$$\begin{aligned} \int (-f)dx &= \int (-f)^+dx - \int (-f)^-dx \\ &= \int f^-dx - \int f^+dx \\ &= -\left[\int f^+dx - \int f^-dx\right] = -\int fdx \\ \therefore \quad \int afdx &= a\int fdx \quad \text{when} \quad a = -1. \end{aligned}$$

Case (c): If $a < 0$, then $a = -|a|$.

Since f is measurable, which implies that $-|a|f$ is measurable and hence af is measurable ($a < 0$).

Also. af is integrable and hence

$$\int afdx = \int -|a|fdx$$
$$= -|a|\int fdx = a\int fdx$$

This proves (i).

(ii) Suppose f and g are integrable. Then

$\int f^+dx < \infty$; $\int f^-dx < \infty$; $\int g^+dx < \infty$; $\int g^-dx < \infty$. Also $f+g$ is measurable.

Our claim is to prove that $\int(f+g)^+ < \infty$ and $\int(f+g)^- < \infty$. For this, consider

$$
\begin{aligned}
(f+g)^+(x) &= \max\{(f+g)(x), 0\} \\
&= \max\{f(x)+g(x), 0\} \\
&= \max\{f(x), 0\} + \max\{g(x), 0\} \\
&= f^+ + g^+ \\
i.e.,\quad (f+g)^+(x) &= f^+ + g^+ \\
\text{Also,}\quad \int (f+g)^+(x) &= \int (f^+ + g^+)dx \\
&= \int f^+dx + \int g^+dx < \infty
\end{aligned}
$$

In a similar way, we can prove that $\int(f+g)^-dx < \infty$.

Hence $f+g$ is integrable.

$$
\begin{aligned}
\int (f+g)dx &= \int [(f+g)^+ - (f+g)^-]\,dx \\
&= \int [f^+ + g^+ - f^- - g^-]\,dx \\
&= \int (f^+ - f^-)dx + \int (g^+ - g^-)dx \\
&= \int fdx + \int gdx
\end{aligned}
$$

This proves (ii).

(iii) Suppose $f = 0\ a.e.$, then

$$
\begin{aligned}
f^+ &= \max\{f(x), 0\} = 0 \qquad (\because f = 0\ \ a.e) \\
f^- &= \max\{-f(x), 0\} = 0
\end{aligned}
$$

So, $f = 0\ a.e.$ implies $f^+ = 0\ \ a.e$ and $f^- = 0\ \ a.e.$

Since, f^+ is a non-negative measurable function such that $f^+ = 0\ a.e.$ and hence we have $\int f^+dx = 0$.

Similarly we can prove that $\int f^-dx = 0$.

Hence $\int fdx = \int f^+dx - \int f^-dx = 0$. This proves (iii).

(iv) Suppose $f \leq g\ a.e.$.

By (i) and (ii), we have $g - f$ is measurable.

Since $f \leq g$ *a.e.* which implies that $g - f \geq 0$ *a.e.*

Now, $(g-f)^- = \max\{-(g-f)(x), 0\} = 0$. So,

$$\begin{aligned}
\int g dx &= \int (f + g - f) dx \\
&= \int f dx + \int (g-f) dx \\
&= \int f dx + \int (g-f)^+ dx - \int (g-f)^- dx \\
&= \int f dx + \int (g-f)^+ dx \\
&\geq \int f dx
\end{aligned}$$

(v) Since A and B are disjoint, we have

$$\chi_{A\cup B} = \chi_A + \chi_B$$

Hence

$$\begin{aligned}
\int_{A\cup B} f dx &= \int f\chi_{A\cup B} dx \\
&= \int f[\chi_A + \chi_B] dx \\
&= \int f\chi_A dx + \int f\chi_B dx \\
&= \int_A f dx + \int f_B f dx
\end{aligned}$$

■

Example 12.1. Show that if f is an integrable function, then $|f| \leq |g|$ *a.e.* and g is measurable, then f is integrable.

Solution: Since g is integrable, $|g|$ is integrable and hence

$$\int |g| dx = \int g^+ dx + \int g^- dx < \infty$$

Now,

$$\begin{aligned}
|f| &\leq |g| \ a.e. \\
\Rightarrow f^+ + f^- &\leq |g| \ a.e. \\
\Rightarrow f^+ &\leq |g| \text{ and } f^- \leq |g| \ a.e.
\end{aligned}$$

Since f^+ and f^- are non-negative functions. So by theorem (12.2), we have

$$\begin{aligned}
\int f^+ dx &\leq \int |g| dx < \infty \quad \text{and} \quad \int f^- dx \leq \int |g| dx < \infty \\
\Rightarrow \int f dx &= \int f^+ dx + \int f^- dx < \infty
\end{aligned}$$

Thus, f is integrable.

Example 12.2. Show that if f is an integrable function, then $\left|\int f dx\right| \leq \int |f| dx$. When does equality occur?

Solution: Suppose f is an integrable function.

$$\begin{aligned}
|f|-f &= f^+ + f^- - (f^+ - f^-) \\
&= f^+ + f^- - f^+ + f^- = 2f^- \geq 0 \\
\Rightarrow \int |f|dx &\geq \int f dx \qquad (12.3) \\
\text{Also, } |f|+f &= f^+ + f^- + (f^+ - f^-) \\
&= 2f^+ > 0 \\
\Rightarrow -f &< |f| \\
\Rightarrow f &\geq -|f| \\
\Rightarrow \int f dx &\geq -\int |f|dx \qquad (12.4)
\end{aligned}$$

Thus, from (12.3) and (12.4), we have

$$\begin{aligned}
-\int |f|dx &\leq \int f dx \leq \int |f|dx \\
\Rightarrow \left|\int f dx\right| &\leq \int |f|dx
\end{aligned}$$

Now, we shall prove the necessary condition for equality.

Case (i)**:** If $\int f dx \geq 0$, then

$$\begin{aligned}
\left|\int f dx\right| &= \int |f|dx \\
\Leftrightarrow \int f dx &= \int |f|dx \\
\Leftrightarrow \int (f - |f|)dx &= 0 \\
\Leftrightarrow f - |f| &= 0 \quad a.e. \\
\Leftrightarrow f &= |f| \quad a.e. \\
\Leftrightarrow f &\geq 0 \quad a.e.
\end{aligned}$$

Case (ii)**:** If $\int f dx \le 0$, then

$$\begin{aligned}
\left|\int f dx\right| &= \int |f| dx \\
\Leftrightarrow -\int f dx &= \int |f| dx \\
\Leftrightarrow \quad \int (f+|f|) dx &= 0 \\
\Leftrightarrow \quad f+|f| &= 0 \quad a.e. \\
\Leftrightarrow \quad f &= -|f| \quad a.e. \\
\Leftrightarrow \quad f &\le 0 \quad a.e.
\end{aligned}$$

Thus, $f \ge 0 \quad a.e.$ (or) $f \le 0 \quad a.e.$ is a necessary condition for $\left|\int f dx\right| = \int |f| dx$. This is also a sufficient condition.

Example 12.3. If f is measurable and g is measurable and α, β are real numbers such that $\alpha \le f \le \beta \ a.e.$, then there exists γ, $\alpha \le \gamma \le \beta$ such that $\int f|g| dx = \gamma \int |g| dx$.

Solution: Given

$$\begin{aligned}
\alpha &\le f \le \beta \quad a.e. \\
\Rightarrow f &\le |\beta| \quad \text{and} - f \le |\alpha| \ a.e. \\
\Rightarrow |f| &\le |\alpha| + |\beta| \quad a.e.
\end{aligned}$$

So,

$$|fg| = |f||g| \le (|\alpha|+|\beta|)\,|g| \quad a.e.$$

Then, by example (12.1), we have fg is integrable.

Also, $\alpha|g| \le f|g| \le \beta|g| \ a.e.$ So,

$$\alpha \int |g| dx \le \int f|g| dx \le \beta \int |g| dx \tag{12.5}$$

If $\int |g| dx = 0$, then $g = 0 \ a.e.$ and hence

$$\int f|g| dx = 0 = \gamma \int |g| dx \quad \text{for every } \alpha \le \gamma \le \beta.$$

If $\int |g| dx \ne 0$, take $\gamma = \dfrac{\int f|g| dx}{\int |g| dx}$. Then by equation (12.5), $\alpha \le \gamma \le \alpha$ and

$$\int f|g| dx = \gamma \int |g| dx$$

Thus, there exists a γ, $\alpha \le \gamma \le \beta$ such that $\int f|g| dx = \gamma \int |g| dx$.

Example 12.4. Show that if f is integrable, then f is finite valued $a.e.$

Solution:

Assume that f is not-finite valued $a.e.$ Then there exists a set E with $m(E) > 0$ so that $|f| = \infty$ on E. So $|f| > n$ for all n on the set E. This gives

$$\int |f| dx \quad > \quad \int_E n dx = n\, m(E) \ \ \forall n$$

This contradicts our hypothesis that f is integrable.

Thus, f is integrable, then f is finite-valued $a.e.$

Example 12.5. If f is measurable, $m(E) < \infty$ and $A \leq f \leq B$ on E, then $A\, m(E) \leq \int_E f dx \leq B\, m(E)$.

Solution: Suppose f is measurable and $m(E) < \infty$.

$$\begin{aligned} m(E) < \infty \ &\Rightarrow \ E \text{ is measurable.} \\ &\Rightarrow \ \chi_E \text{ is measurable.} \\ \Rightarrow \ A\chi_E \text{ and } B\chi_E &\text{ are measurable.} \end{aligned}$$

$$\begin{aligned} \text{Given } A &\leq f \leq g \\ \Rightarrow A\chi_E &\leq f\chi_E \leq B\chi_E \end{aligned} \tag{12.6}$$

Now, we shall prove that χ_E is integrable.

$$\begin{aligned} \chi_E^+ &= \max(\chi_E, 0) = \chi_E \\ \Rightarrow \int \chi_E^+ dx &= \int \chi_E dx = m(E) < \infty \end{aligned}$$

Similarly, we can prove that $\int \chi_E^- dx < \infty$.

Thus, $\int \chi_E dx = \int \chi_E^+ dx + \chi_E^- dx < \infty$ and hence χ_E is integrable.

Also, $A\chi_E$ and $B\chi_E$ are integrable.

Thus, from equation (12.6), we have

$$\begin{aligned} A\chi_E &\leq f\chi_E \leq B\chi_E \\ \Rightarrow \int A\chi_E dx &\leq \int f\chi_E dx \leq \int B\chi_E dx \\ \Rightarrow A\int \chi_E dx &\leq \int f\chi_E dx \leq B\int \chi_E dx \\ \Rightarrow A\, m(E) &\leq \int_E f dx \leq B\, m(E) \end{aligned}$$

Theorem 12.3 (Lebesgue's Dominated Convergence Theorem).

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable and let $\lim f_n = f$ a.e. Then f is integrable and

$$\lim \int f_n dx \quad = \quad \int f dx$$

Proof. Suppose $\{f_n\}$ be a sequence of measurable functions such that $|f_n|\leq g$ and g is integrable.

Since, $|f_n|\leq g$, f_n is integrable for all n.

Also,

$$\begin{aligned} \left|\lim_{n\to\infty} f_n(x)\right| &\leq g(x) \\ \Rightarrow |f(x)| &\leq g(x) \\ i.e., |f| &\leq |g| \\ \Rightarrow \int |f|dx &\leq \int |g|dx < \infty \quad (\because g \text{ is integrable}) \\ \Rightarrow \int |f|dx &< \infty \end{aligned}$$

Thus, $|f|$ is integrable and hence f is integrable.

Also, $f_n \leq g$ which implies that $f_n + g \leq 2g$.

Hence, $\{f_n + g\}$ is a sequence of non-negative measurable functions.

By Fatou's lemma, we have

$$\begin{aligned} \liminf \int (f_n + g)dx &\geq \int \liminf(f_n + g)dx \\ \Rightarrow \liminf\left[\int f_n dx + \int g dx\right] &\geq \int (\liminf f_n + \liminf g dx \\ \Rightarrow \liminf \int f_n dx &\geq \int \liminf f_n \\ \Rightarrow \liminf \int f_n dx &\geq \int f dx \end{aligned} \tag{12.7}$$

Since, each $f_n's$ are non-negative measurable functions and $f_n \leq g$, $g - f_n$ are non-negative measurable functions.

i.e., $\{g - g_n\}$ is a sequence of non-negative measurable functions.

Hence, by Fatou's lemma, we have

$$\begin{aligned} \liminf \int (g - f_n)dx &\geq \int \liminf(g - f_n)dx \\ \Rightarrow \liminf\left[\int g dx + \int (-f_n)dx\right] &\geq \int (\liminf g + \liminf(-f_n)dx \\ \Rightarrow \int g dx + \liminf \int (-f_n)dx &\geq \int g dx + \int \liminf(-f_n)dx \\ \Rightarrow -\limsup \int f_n dx &\geq \int -\limsup f_n dx \\ \Rightarrow -\limsup \int f_n dx &\leq \int \limsup f_n dx \\ \Rightarrow \limsup \int f_n dx &\leq \int f dx \end{aligned} \tag{12.8}$$

From equation (12.7) and (12.8), we get

$$\liminf \int f_n dx \geq \int f dx \geq \limsup \int f_n dx$$

$$\Rightarrow \lim \int \int f_n dx \geq \limsup \int f_n dx \quad (12.9)$$

$$\text{But always} \quad \limsup \int f_n dx \geq \liminf \int f_n dx \quad (12.10)$$

From (12.9) and (12.10), we have

$$\liminf \int f_n dx = \limsup \int f_n dx = \lim \int f_n dx$$

$$\therefore \quad \lim \int f_n dx = \int f dx$$

■

Example 12.6. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable and let $\lim f_n = f \;\; a.e.$ Then f is integrable and

$$\lim \int |f_n - f|\, dx = 0$$

Solution:

$$|f_n - f| \leq |f_n| + |f|$$

$$\leq g + g$$

$$\leq 2g$$

$$\Rightarrow \int |f_n - f| dx \leq \int 2g dx < \infty \quad (\because \; g \text{ is measurable}) \Rightarrow |f_n - f| \text{ is measurable}$$

$$\text{Also,} \quad \lim f_n = f \;\; a.e.$$

$$\Rightarrow |f_n - f| = 0 \;\; a.e.$$

$$\lim_{n \to \infty} \int |f_n - f| dx = 0$$

Theorem 12.4. *Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| dx < \infty$ then the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e., its sum $f(x)$ is integrable and*

$$\int f dx = \sum_{n=1}^{\infty} \int f_n dx$$

Proof.

$$\text{Let } \varphi(x) = \sum_{n=1}^{\infty}|f_n|.$$

$$\begin{aligned}
\text{Then } \int \varphi(x)dx &= \int \sum_{n=1}^{\infty}|f_n|dx \\
&= \sum_{n=1}^{\infty}|f_n|dx < \infty \\
&\Rightarrow \int \varphi(x)dx < \infty \\
&\Rightarrow \varphi \text{ is finite value } a.e. \\
&\Rightarrow \sum_{n=1}^{\infty}|f_n| \text{ converges } a.e. \\
&\Rightarrow \sum_{n=1}^{\infty} f_n \text{ converges absolutely } a.e.
\end{aligned}$$

$$\begin{aligned}
\text{Let } f &= \sum_{n=1}^{\infty} f_n \\
|f| &= \left|\sum_{n=1}^{\infty} f_n\right| \\
&\leq \sum_{n=1}^{\infty}|f_n| = \varphi(x) \\
i.e., \quad |f| &\leq \varphi(x) \\
\Rightarrow \int |f|dx &\leq \int \varphi(x)dx \\
\Rightarrow \int |f|dx &< \infty \\
\Rightarrow |f| \text{ is integrable } &\Rightarrow f \text{ is integrable.} \\
\text{Let } g_n &= \sum_{i=1}^{n} f_i \\
\Rightarrow |g_n| = \left|\sum_{i=1}^{n} f_i\right| \leq \sum_{i=1}^{n}|f_i| &\leq \varphi(x)
\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\int g_n dx &= \int \lim_{n\to\infty} g_n dx = \int f dx \\
\Rightarrow \lim_{n\to\infty}\int \sum_{i=1}^{n} f_i dx &= \int \lim_{n\to\infty}\left(\sum_{i=1}^{n} f_i\right) dx \\
\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\int f_i dx &= \int \sum_{i=1}^{\infty} f_i dx \\
\Rightarrow \sum_{i=1}^{\infty}\int f_i dx &= \int \sum_{i=1}^{\infty} f_i dx = \int f dx \qquad \blacksquare
\end{aligned}$$

Example 12.7. Lebesgue Dominated Convergence Theorem deals with a sequence of functions $\{f_n\}$. State and prove a continuous parameter version of the theorem.

Theorem: For each $\xi \in [a, b]$, $-\infty \leq a \leq b < \infty$, let f_ξ be a measurable function,

$|f_\xi(x)| \leq g(x)$, where g is an integrable function and let $\lim_{\xi\to\xi_0} f_\xi(x) = f(x)$ *a.e.*, where $\xi_0 \in [a,b]$. Then f is integrable $\lim_{\xi\to\xi_0} \int f_\xi(x)dx = \int f(x)dx$.

Solution: Let $\{\xi_n\}$ be any sequence in $[a,b]$ such that $\lim_{n\to\infty} \xi_n = \xi_0$.
$\Rightarrow$ $\{f_{\xi_n}\}$ satisfies the hypothesis of Lebesgue Dominated Theorem ((12.3)).
$\Rightarrow$ $f = \lim_{n\to\infty} f_{\xi_n}$ is integrable.
Let $\lim_{\xi\to\xi_0} \int f_\xi dx \neq \int f dx$
$\Rightarrow$ $\exists$ an $\delta > 0$ and a sequence $\{\beta_n\} \in [a_n, b_n]$, $\lim \beta_n = \xi_0$ such that for all n $\left|\int f\beta_n dx - \int f dx\right| > \delta$. But by applying Lebesgue dominated convergence theorem to $\{f_{\beta_n}\}$, we get a contradiction.
Hence our assumption is wrong.

$$\therefore \quad \lim_{\xi\to\xi_0} f_\xi dx = \int f dx$$

Example 12.8.

(i) If f is integrable then $\int f dx = \lim_{a\to\infty} \lim_{b\to-\infty} \int_b^a f dx = \lim_{b\to-\infty} \lim_{a\to\infty} \int_b^a f dx$.

(ii) If f is integrable on $[a,b]$ and $0 < \epsilon < b - a$ then $\int_b^a f dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^b f dx$.

Solution: $\int_b^a f dx = \int_{-\infty}^a \chi_{[b,+\infty)} f dx$.
f is an integrable function and thus by above example, we have

$$\lim_{b\to-\infty} \int_{-\infty}^a \chi_{[b,+\infty)} f dx = \int_{-\infty}^a f dx$$

A second application of the above example gives the first equation and the second follows in the same way.

Similarly, we can prove result (ii).

Theorem 12.5. *If f is continuous on the finite interval $[a,b]$, then f is integrable and $F(x) = \int_a^x f(t)dt$ $(a < x < b)$ is a differentiable function such that $F'(x) = f(x)$.*

Proof. We know that continuous functions are measurable, so f is measurable. Since f is continuous function defined on the finite interval $[a,b]$, we have $|f|$ is bounded. Hence f is integrable on $[a,b]$.

If $a < x < b$, then we have $x + h \in (a,b)$ for small h.

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

Let $m = \min[f(x) : x \in [a,b]]$ and $M = \max[f(x) : x \in [a,b]]$. Then $m \le f(x) \le M \ \forall x$.

This implies

$$mh \le \int_x^{x+h} f(t)dt \le Mh$$

Since f is continuous, then by intermediate mean value theorem we have

$$\int_x^{x+h} f(t)dt = hf(\xi)$$

where $\xi = x + \theta h, \ 0 \le \theta \le 1$.

So, supposing $h \neq 0$, dividing by h and letting $h \to 0$, we get

$$\begin{aligned}
\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt \\
&= \lim_{h\to 0} f(\xi) \\
&= \lim_{h\to 0} f(x+\theta h) = f(x) \\
i.e., \ F'(x) &= f(x)
\end{aligned}$$

■

Example 12.9. Show that if $\alpha > 1$

$$\int_0^1 \frac{x \sin x}{1 + (nx)^\alpha} dx = o(n^{-1}) \text{ as } n \to \infty.$$

Solution: We write $x_n = o(n^p)$ if $x_n n^{-p} \to 0$ as $n \to \infty$. So, we wish to show that

$$\lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1 + (nx)^\alpha} dx = 0 \tag{12.11}$$

Now, for any fixed x

$$\frac{nx \sin x}{1 + (nx)^\alpha} = \frac{\frac{1}{n^{\alpha-1}} x \sin x}{\frac{1}{n^\alpha} + x^\alpha} \to 0 \text{ as } n \to \infty \quad (\alpha > 1)$$

Since $\lim_{n\to\infty} \frac{nx \sin x}{1 + (nx)^{\alpha)}} = 0$. So, we get (12.11), if we are allowed to interchange limit and integral. Hence, it is sufficient to show that dominated convergence theorem as applicable to the sequences

$$f_n(x) = \frac{nx \sin x}{1 + (nx)^\alpha}, \quad n = 1, 2, \ldots$$

Consider the function $h(x) = 1 + (nx)^\alpha - nx^{3/2}$. We have $h(0) = 1$, $h(1) = 1 + n^\alpha - n > 1$ since $\alpha > 1$. Now

$$h'(x) = \alpha n^\alpha x^{\alpha-1} - (3/2)nx^{1/2}$$

$h'(x) \neq 0$ in $[0,1]$ for $1 < \alpha \leq 3/2$ and for all large n. If $\alpha > 3/2$, then there is a point in $[0,1]$ at which $h'(x) = 0$. At that point h approaches 1 for large n. So, it follows that for large n, $h(x) > 0$ on $[0,1]$. Hence

$$\begin{aligned} |f_n(x)| &= \left|\frac{nx\sin x}{1+(nx)^\alpha}\right| \\ &\leq \frac{nx}{1+(nx)^{1/2}} \leq \frac{nx}{nx^{3/2}} = \frac{1}{\sqrt{x}} \end{aligned}$$

Since $\frac{1}{\sqrt{x}}$ is integrable over $(0,1]$, dominated convergence theorem is applicable for $\{f_n\}$ and we have

$$\begin{aligned} \lim_{n\to\infty}\int_0^1 f_n(x)dx &= \int_0^1 \lim_{n\to\infty} f_n(x)dx \\ i.e., \quad \lim_{n\to\infty}\int_0^1 \frac{nx\sin x}{1+(nx)^\alpha}dx &= 0 \end{aligned}$$

Example 12.10. Show that $\lim \int_0^1 \frac{dx}{(1+x/n)^n x^{1/n}} = 1.$

Solution: We have

$$\lim \frac{1}{(1+x/n)^n - x^{1/m}} = \frac{1}{e^x} = e^{-x}$$

and $\int_0^\infty e^{-x}dx = 1$. So, we wish to apply dominated convergence theorem.

For $n > 1$, $x > 0$,

$$\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{n(n-1)x^2}{2n^2} + \ldots > \frac{x^2}{4}$$

So, for $x \geq 1$ and $n > 1$,

$$\frac{1}{(1+x/n)^n x^{1/n}} \leq \frac{1}{x^2/4} = \frac{4}{x^2} \quad (\because x^{1/n} \geq 1)$$

If $0 < x < 1$, then $x^{1/n} \geq x^{1/2}$ for $n > 1$. So

$$\frac{1}{(1+x/n)^n x^{1/n}} \leq \frac{1}{x^{1/2}} = x^{-1/2}$$

Hence, for $x > 0$, $n > 1$,

$$\frac{1}{(1+x/n)^n x^{1/n}} \leq g(x)$$

where

$$g(x) = \begin{cases} 4/x^2 & \text{if } x \geq 1 \\ x^{-1/2} & \text{if } 0 < x \leq 1 \end{cases}$$

But g is integrable over $(0,\infty)$. So, if $f_n(x) = \frac{1}{(1+x/n)^n x^{1/n}}$, then

Lebesgue's dominated convergence theorem is applicable to $\{f_n\}$ and we have

$$\begin{aligned}\lim \int \frac{1}{(1+x/n)^n x^{1/n}} dx &= \int \lim \frac{1}{(1+x/n)^n x^{1/n}} \\ &= \int e^{-x} dx = 1.\end{aligned}$$

Example 12.11. Show that $\lim_{n\to\infty} \int_a^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2} dx = 0$ for $a > 0$, but not for $a = 0$.

Solution: If $a > 0$, substitute $u = nx$ to get

$$\int_a^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2} dx = \int_{na}^{\infty} \frac{ue^{-u^2}}{1+u^2/n^2} du = \int_0^{\infty} \chi_{(an,\infty)} \frac{ue^{-u^2}}{1+u^2/n^2} du$$

Let $f_n = \chi_{(na,\infty)} \frac{ue^{-u^2}}{1+u^2/n^2}$. For every u, we can find n large enough so that $u < na$ and hence $\chi_{(na,\infty)}(u) = 0$. So $f_n(u) = 0$ for large n.

Hence $\lim_{n\to 0} f_n = 0$. So, we wish to apply Lebesgue dominated convergence theorem for $\{f_n\}$. Now

$$|f_n(x)| \leq ue^{-u} \qquad (\chi_{(na,\infty)}(u) \leq 1, \quad 1 + u^2/n^2 \geq 1)$$

Also, ue^{-u} is integrable over $(0, \infty)$. Hence, by applying dominated convergence theorem, we get

$$\begin{aligned}\lim_{n\to 0} \int_a^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2} dx &= \lim_{n\to 0} \int_0^{\infty} f_n du \\ &= \int_0^{\infty} \left(\lim_{n\to\infty} f_n\right) du \\ &= 0\end{aligned}$$

If $a = 0$, the same substitution $u = nx$ gives

Since $\frac{ue^{-u^2}}{1+u^2/n^2} \to ue^{-u^2}$ as $n \to \infty$ and $\frac{ue^{-u^2}}{1+u^2/n^2} \leq ue^{-u}$, an integrable function, the Lebesgue's dominated convergence theorem is applicable and we have

$$\begin{aligned}\lim_{n\to\infty} \int_0^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2} dx &= \lim_{n\to\infty} \int_0^{\infty} \frac{ue^{-u^2}}{1+u^2/n^2} du \\ &= \int_0^{\infty} \lim \left(\frac{ue^{-u^2}}{1+u^2/n^2}\right) du \\ &= \int_0^{\infty} ue^{-u^2} du = \frac{1}{2}.\end{aligned}$$

Example 12.12. Let f be non-negative integrable function $[0, 1]$. Then there exist measurable function $\varphi(x)$ such that φf is integrable on $[0, 1]$ and $\varphi(0+) = \infty$.

Solution: By continuous parameter version of Lebesgue's Dominated Convergence Theorem, for every n there exists x_n $(0 < x_n < 1)$ such that $\int_0^{x_n} f dx < \frac{1}{n^3}$.

Let $\{x_n\}$ be a decreasing sequence of numbers as $n \to \infty$.

Define $\varphi(x) = \sum_{k=2}^{\infty}(k-1)\chi_{[x_k, x_{k-1}]}$, So $\varphi(0+) = \lim_{x \to 0+} \varphi(x) = \infty$.

It remains to prove that, φf is integrable.

For if, consider

$$\begin{aligned} \int_{x_k}^{x_{k-1}} \varphi f dx &= \int_{x_k}^{x_{k-1}} (k-1) f dx \\ &< \frac{k-1}{(k-1)^3} \\ &< \frac{1}{(k-1)^2} \\ &< \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \end{aligned}$$

$$\int \varphi f dx < \infty \quad \Rightarrow \quad \varphi f \text{ is measurable.}$$

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of simple function and measurable function.
- derive Fatou's lemma and Lebesgue monotone convergence theorem.

Check Your Progress:

1. Let $f(x) = 0$ at each point $x \in P$, the Cantor set in $[0, 1]$, $f(x) = p$ in each of complementary interval of length 3^{-p}. Show that f is measurable and that $\int_0^1 f dx = 3$.

Choose the correct or more suitable answer:

1. $\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2} dx =$
   (a) 0
   (b) $\frac{1}{3}$
   (c) $\frac{1}{2}$
   (d) $\frac{1}{4}$

2. $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2} =$
   (a) -1
   (b) 0
   (c) $\frac{1}{2}$
   (d) $\frac{1}{3}$

Answer:

(1) *a* (2) *b*

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.
2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.
3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-IV

UNIT-13

RIEMANN AND LEBESGUE INTEGRALS

Objectives

After completion of this unit, students will be able to

- ★ understand the concept of Upper Riemann and Lower Riemann sum.
- ★ explain the difference between Riemann integral and Lebesgue integral.

Overview

In this unit, we will discuss the ideas about Upper and Lower Sum.

13.1. Riemann and Lebesgue Integrals:

If f is Riemann integrable over the finite interval $[a, b]$, then we write its integral by $R\int_a^b f dx$ to distinguish from its Lebesgue integral $\int_a^b f dx$.

Upper Riemann Sum and Lower Riemann Sum: Let f be a bounded function defined over the finite interval $[a, b]$.

Let $a = \xi_0 < \xi_1 < \ldots < \xi_n$ be a partition D of $[a, b]$. Write

$$\begin{aligned} S_D &= \sum_{i=1}^{n} M_i(\xi_i - \xi_{i-1}) \\ \text{and } s_D &= \sum_{i=1}^{n} m_i(\xi_i - \xi_{i-1}) \\ \text{where } M_i &= \sup\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n \\ m_i &= \inf\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n \end{aligned}$$

Here S_D and s_D are respectively called *Upper Riemann Sum* and *Lower Riemann sum*.

Riemann Integrable Function: A function $f(x)$ is said to be *Riemann integrable* over $[a, b]$, if given $\epsilon > 0$ there exists a partition D such that $S_D - s_D < \epsilon$.

Theorem 13.1. *If f is Riemann integrable and bounded over the finite interval $[a, b]$, then f is integrable and $R\int_a^b f dx = \int_a^b f dx$.*

Proof. Let $\{D_n\}$ be a sequence of partition of $[a, b]$ such that

$$S_{D_n} - s_{D_n} < \frac{1}{n} \tag{13.1}$$

for every partition D_n.

$$\begin{aligned} \text{Define } u_n(x) &= \sum_{i=1}^{n} M_i \chi_{[\xi_{i-1},\, \xi_i]}(x) \\ l_n(x) &= \sum_{i=1}^{n} m_i \chi_{[\xi_{i-1},\, \xi_i]}(x) \\ \text{where } M_i &= \sup\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n \\ m_i &= \inf\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n \end{aligned}$$

for all $x \in [a, b]$. Then, for all n, u_n and l_n are measurable simple functions and

$$\begin{aligned} \int_a^b u_n dx &= \sum_{i=1}^{n} M_i \int_{\xi_{i-1}}^{\xi_i} dx = \sum_{i=1}^{n} M_i (\xi_i - \xi_{i-1}) = S_D \\ \textit{i.e.}, \int_a^b u_n dx &= S_D \\ \text{Similarly, } \int_a^b l_n dx &= s_D \end{aligned}$$

Further,

$$\begin{aligned} m_i &\leq f(x) \leq M_i \\ \Rightarrow l_n(x) &\leq f(x) \leq u_n(x) \end{aligned} \tag{13.2}$$

for all $x \in [a, b]$.

We define $U = \inf u_n$ and $L = \sup I_n$.

Then U and L are measurable functions satisfying

$$L \leq f \leq U \tag{13.3}$$

Now,

$$\begin{aligned} \{x : U(x) - L(x) \neq 0\} &= \bigcup_{k=1}^{\infty} \left\{x : U(x) - L(x) > \frac{1}{k}\right\} \\ \text{But } \; U(x) - L(x) &> \frac{1}{k} \\ \Rightarrow \inf u_n(x) - \sup l_n(x) &> \frac{1}{k} \\ \Rightarrow \inf u_n(x) + \inf(-l_n(x)) &> \frac{1}{k} \\ \Rightarrow \inf(u_n(x) - l_n(x)) &> \frac{1}{k} \\ \Rightarrow u_n(x) - l_n(x) &> \frac{1}{k} \quad \forall n \end{aligned}$$

Thus, we have $\left\{x : U(x) - L(x) > \frac{1}{k}\right\} \subseteq \left\{x : u_n(x) - l_n(x) > \frac{1}{k}\right\}$.

Therefore $m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right) \leq m\left(\left\{x : u_n(x) - l_n(x) > \frac{1}{k}\right\}\right)$.

Let $a = m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right)$. Then, we have $m\left(\left\{x : u_n(x) - l_n(x) > \frac{1}{k}\right\}\right) > a$. So,

$$\begin{aligned}
\int_a^b (u_n(x) - l_n(x))dx &> \frac{a}{k} \\
\Rightarrow \int_a^b u_n(x)dx - \int_a^b l_n(x))dx &> \frac{a}{k} \\
\Rightarrow S_{D_n} - s_{D_n} &> \frac{a}{k} \\
\Rightarrow \frac{1}{n} &> \frac{a}{k} \quad \forall n \\
\Rightarrow a &= 0 \\
i.e., \quad m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right) &= 0 \quad \text{for each } k \\
\Rightarrow m(\{x : U(x) - L(x) > 0\}) &= 0
\end{aligned}$$

Since $\{x : U(x) - L(x) \geq 0\}$ which implies $U(x) = L(x)$ $a.e.$

Thus, equation (13.3) gives

$$U = f = L \;\; a.e.$$

Hence f is measurable. So the boundedness of f implies f is integrable.

Thus, equation (13.2) gives

$$\begin{aligned}
\int_a^b l_n dx &\leq \int_a^b f dx \leq \int_a^b u_n dx \\
i.e., \quad s_{D_n} &\leq \int_a^b f dx \leq S_{D_n} \qquad (13.4)
\end{aligned}$$

Since f is Riemann integrable and hence we have

$$\lim_{n\to\infty} s_{D_n} = \lim_{n\to\infty} S_{D_n} = R\int_a^b f dx \qquad (13.5)$$

So, letting $n \to \infty$ in (13.4) and using (13.5), we get

$$R\int_a^b f dx = \int_a^b f dx$$ ■

Note 13.1. The converse of the above theorem does not hold. That is, a Lebesgue integrable function need not be a Riemann integrable.

Consider for example the function f on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Then $f = 1$ $a.e.$, and f is measurable since for any $\alpha \in R$

$$\begin{aligned}
[x : f(x) > \alpha] &= \emptyset, \text{ if } \alpha \geq 1 \\
&= [x : x \text{ is irrational}], \text{ if } 0 \leq \alpha < 1 \\
&= R, \text{ if } \alpha < 0
\end{aligned}$$

So, f is integrable and $\int_0^1 f dx = 1$. But for each partition D of $[0, 1]$, we have

$S_D = 1$ and $s_D = 0$.

Hence f is not Riemann integrable.

Theorem 13.2. *Let f be bounded function defined on the finite interval $[a, b]$, then f is Riemann integrable over $[a, b]$ if, and only if, it is continuous $a.e.$*

Proof. Let $\{D_n\}$ be a sequence of partition of $[a, b]$ such that

$$S_{D_n} - s_{D_n} < \frac{1}{n} \tag{13.6}$$

for every partition D_n.
Define

$$\begin{aligned}
u_n(x) &= \sum_{i=1}^{n} M_i \chi_{[\xi_{i-1}\xi_i]}(x) \\
l_n(x) &= \sum_{i=1}^{n} m_i \chi_{[\xi_{i-1}\xi_i]}(x) \\
\text{where } M_i &= \sup\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n \\
m_i &= \inf\{f(x) : x \in [\xi_i - \xi_{i-1}]\}; \quad i = 1, 2, \ldots, n
\end{aligned}$$

for all $x \in [a, b]$. Then, for all n, u_n and l_n are measurable simple functions and

$$\begin{aligned}
\int_a^b u_n dx &= \sum_{i=1}^{n} M_i \int_{\xi_{i-1}}^{\xi_i} dx = \sum_{i=1}^{n} M_i (\xi_i - \xi_{i-1}) = S_D \\
i.e., \int_a^b u_n dx &= S_D \\
\text{Similarly, } \int_a^b l_n dx &= s_D
\end{aligned}$$

Further,

$$\begin{aligned}
m_i &\leq f(x) \leq M_i \\
\Rightarrow l_n(x) &\leq f(x) \leq u_n(x)
\end{aligned} \tag{13.7}$$

for all $x \in [a, b]$.

We define $U = \inf u_n$ and $L = \sup I_n$.

Then U and L are measurable functions satisfying

$$L \leq f \leq U \tag{13.8}$$

Now,

$$\begin{aligned}
\{x : U(x) - L(x) \neq 0\} &= \bigcup_{k=1}^{\infty} \left\{x : U(x) - L(x) > \frac{1}{k}\right\} \\
\text{But} \quad U(x) - L(x) &> \frac{1}{k} \\
\Rightarrow \quad \inf u_n(x) - \sup l_n(x) &> \frac{1}{k} \\
\Rightarrow \quad \inf u_n(x) + \inf(-l_n(x)) &> \frac{1}{k} \\
\Rightarrow \quad \inf(u_n(x) - l_n(x)) &> \frac{1}{k} \\
\Rightarrow \quad u_n(x) - l_n(x) &> \frac{1}{k} \quad \forall n
\end{aligned}$$

Thus, we have $\{x : U(x) - L(x) > \frac{1}{k}\} \subseteq \{x : u_n(x) - l_n(x) > \frac{1}{k}\}$.

Therefore $m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right) \leq m\left(\left\{x : u_n(x) - l_n(x) > \frac{1}{k}\right\}\right)$.

Let $a = m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right)$. Then, we have $m\left(\left\{x : u_n(x) - l_n(x) > \frac{1}{k}\right\}\right) > a$. So,

$$\begin{aligned}
\int_a^b (u_n(x) - l_n(x))dx &> \frac{a}{k} \\
\Rightarrow \quad \int_a^b u_n(x)dx - \int_a^b l_n(x))dx &> \frac{a}{k} \\
\Rightarrow \quad S_{D_n} - s_{D_n} &> \frac{a}{k} \\
\Rightarrow \quad \frac{1}{n} &> \frac{a}{k} \quad \forall n \\
\Rightarrow \quad a &= 0 \\
\text{i.e.,} \quad m\left(\left\{x : U(x) - L(x) > \frac{1}{k}\right\}\right) &= 0 \quad \text{for each } k \\
\Rightarrow \quad m(\{x : U(x) - L(x) > 0\}) &= 0
\end{aligned}$$

Since $\{x : U(x) - L(x) \geq 0\}$ which implies $U(x) = L(x)$ *a.e.*

Thus, equation (13.8) gives

$$U = f = L \ \ a.e.$$

Suppose that f is Riemann integrable over $[a, b]$. Let $x \in [a, b]$ be such that $x \neq x_i \in D_n$ for every n.

If $U(x) = f(x) = L(x)$, we claim that f is continuous at x. Assume the contrary. f is continuous at x if and only if

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(x)$$

So, $\exists\ \epsilon > 0$ and a sequence $\{x_k\}$ with $\lim_{n\to\infty} x_n = x$ such that

$$\begin{aligned} |f(x_k) - f(x)| &\geq \epsilon \quad \forall k \\ i.e., \quad f(x_k) &\geq f(x) + \epsilon \quad \forall k \\ i.e., \quad U(x_k) &\geq L(x) + \epsilon \quad \forall k \end{aligned}$$

Since $\lim_{n\to\infty} x_n = x$, all x_n but for a finite number of x_n lie in every neighbourhood of x. So

$$U(x) \geq L(x) + \epsilon$$

This contradicts that $U(x) = f(x) = L(x)$. Hence f is continuous at x for $x \neq x_i \in D_n$ for any n and $U(x) = f(x) = L(x)\ a.e.$ Hence f is continuous $a.e.$

Conversely, suppose that f is continuous $a.e.$ Choose a sequence $\{D_n\}$ of partitions of $[a,b]$ such that, for each n, $D_{n+1} \supset D_n$ and $\|D_n\| \to 0$. Suppose that u_n and l_n are defined corresponding to D_n. Then $u_{n+1} \leq u_n$ and $l_{n+1} \geq l_n$ for each n. Now suppose that f is continuous at x. Thus, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup f(x) - \inf f(x) < \epsilon$$

where the sup and inf are taken over $(x-\delta, x+\delta)$. For all n sufficiently large, an interval of D_n containing x will lie in $(x-\delta, x+\delta)$ and so,

$$u_n(x) - l_n(x) < \epsilon$$

But, ϵ is arbitrary, so $U(x) = L(x)$. Since f is continuous $a.e.$ and thus we have $U = L\ a.e.$

By Lebesgue Dominated Convergence theorem, we have

$$\begin{aligned} \lim \int u_n dx &= \int U dx = \int L dx = \lim \int I_n dx \\ \text{so} \quad \lim \int u_n dx &= \int L dx = \lim \int I_n dx \end{aligned}$$

and hence

$$\lim S_{D_n} = \lim s_{D_n}$$

Thus, f is Riemann integrable and hence the theorem. ∎

Definition 13.1. If, for each a and b, f is bounded and Riemann integrable on $[a,b]$ and

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f dx \tag{13.9}$$

exists, then f is said to be *Riemann integrable* on $(-\infty,\infty)$, and and the integral written as $R\int_{-\infty}^{\infty} f dx$.

Theorem 13.3. *Let f be bounded and let f and $|f|$ be Riemann integrable on $(-\infty,\infty)$. Then f is integrable and*

$$\int_{-\infty}^{\infty} f dx = R\int_{-\infty}^{\infty} f dx$$

Proof. Suppose $|f|$ is Riemann integrable on $(-\infty,\infty)$. Thus, $|f|$ is Riemann integrable on $(-a,b)$ for every a,b. So

$$\begin{aligned} \int_a^b |f|dx & \text{ is finite} \\ \Rightarrow\ |f| & \text{ is integrable} \\ \Rightarrow\ f & \text{ is integrable} \end{aligned}$$

Also, f is is Riemann integrable on $(-\infty,\infty)$. Thus, $|f|$ is Riemann integrable on $(-a,b)$ for every a,b. So

$$\int_a^b f dx = R\int_a^b f dx$$

Letting $a\to -\infty$ and $b\to\infty$, we have

$$\int_{-\infty}^{\infty} f dx = R\int_{-\infty}^{\infty} f dx \qquad \blacksquare$$

Theorem 13.4. *Let f be bounded and measurable on a finite interval $[a,b]$ and let $\epsilon>0$. Then there exist*

(i) a step function h such that $\int_a^b |f-h|dx<\epsilon$.

(ii) a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f-g|dx<\epsilon$.

Proof. (i) We have $f=f^+-f^-$ and

$$\int_a^b f dx = \int_a^b f^+dx+\int_a^b f^-dx$$

Since f^+ and f^- are non-negative, so we can assume that $f(x)\geq 0$ for all $x\in[a,b]$.

Now,

$$\int_a^b f dx = \sup\int_a^b \varphi dx$$

where $\varphi\leq f,\quad \varphi$ is simple and measurable. So, we may assume that f is a simple measurable function with $f=0$ outside $[a,b]$. Hence,

$$f = \sum_{i=1}^n a_i\chi_{E_i}$$

where $E_i=[x:f(x)=a_i]$ and $\bigcup_{i=1}^n E_i=[a,b]$.

Let $M=\sup[f(x):x\in[a,b]]$. We may assume that $M>0$.

Suppose that $\epsilon'=\dfrac{\epsilon}{nM}$. For each of the measurable sets E_i there exists open intervals $I_1,I_2,\ldots,I_k$ such that, if $G=\bigcup_{r=1}^k I_r$, then $m(E_i\triangle G)<\epsilon'$. But χ_G is a step function such that

$$\int \left|\chi_{E_i}-\chi_G\right|dx = m(E_i\triangle G)<\epsilon'$$

Construct such step function h_i, say, for each E_i. Then

$$\begin{aligned}\int_a^b \left| f - \sum_{i=1}^{n} a_i h_i \right| dx &< \sum_{i=1}^{n} a_i \epsilon' \\ &\leq nM\epsilon' = \epsilon.\end{aligned}$$

Let $h = \sum_{i=1}^{n} a_i h_i$, then

$$\begin{aligned} h &= \sum_{i=1}^{n} a_i h_i \\ &= \sum_{i=1}^{n} a_i \chi_G \\ \Rightarrow \int_a^b |f - h| dx &< \epsilon.\end{aligned}$$

where h is a step function.

(ii) From (i) there exists a step function h vanishing outside a finite interval such that

$$\int_a^b |f - h| dx < \frac{\epsilon}{2}$$

So to prove (ii), we construct a continuous function g such that

$$\int_a^b |h - g| dx < \frac{\epsilon}{2} \tag{13.10}$$

and such that $g(x) = 0$ whenever $h(x) = 0$.

Let

$$h(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where E_i is the finite interval (c_i, d_i), $i = 1, 2, \ldots, n$. As in (i), it is sufficient to show that (13.10) holds for each χ_{E_i}.

Suppose that $\epsilon < 2(d_i - c_i)$ and define g by

$$g(x) = \begin{cases} 1 & \text{if } x \in \left(c_i + \frac{\epsilon}{4}.d_i - \frac{\epsilon}{4}\right) \\ 0 & \text{if } x \in (c_i, d_i)^c \end{cases}$$

Extend g by linearity to $\left[c_i, c_i + \frac{\epsilon}{4}\right]$ and $\left[d_i - \frac{\epsilon}{4}, d_i\right]$ as shown in the following figure 13.1, so that g is continuous.

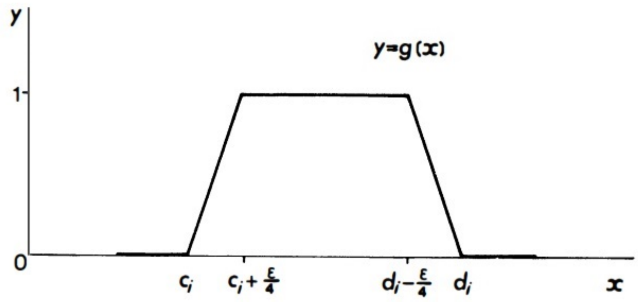

Figure 13.1

Then,

$$\begin{aligned}\int \left|\chi_{E_i} - g\right| dx &< 1\left(c_i, c_i + \frac{\epsilon}{4}\right) + 1\left(d_i - \frac{\epsilon}{4}, d_i\right)\\ &= \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}\end{aligned}$$

Let $h = \chi_{E_i}$, $\int |h - g| dx < \frac{\epsilon}{2}$.

Also, g vanishes outside (c_i, d_i).

Now,

$$\begin{aligned}\int_a^b |f - g| dx &= \int_a^b |f - h + h - g| dx\\ &\leq \int_a^b (|f - h| + |h - g|) dx\\ &\leq \int_a^b |f - h| dx + \int_a^b |h - g| dx\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\\ i.e., \int_a^b |f - g| dx &< \epsilon\end{aligned}$$

Hence the proof. ■

Example 13.1. Let f be a bounded measurable function defined on the finite interval (a, b). Show that

$$\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x dx = 0$$

Solution: Let $\epsilon > 0$ be given. We show that there is β_0 such that for $\beta > \beta_0$.

$$\left| \int_a^b f(x) \sin \beta x dx \right| < \epsilon$$

By theorem (13.4), there exists a step function

$$h = \sum_{i=1}^{n} a_i \chi_{I_i}$$

where $I_i, i = 1, 2, \ldots, n$ denote disjoint intervals such that $[a, b] = \bigcup I_i$ with

$$\int_a^b |f - h| dx < \frac{\epsilon}{2}$$

Then

$$\begin{aligned}
\left|\int_a^b f(x) \sin \beta x dx\right| &= \left|\int_a^b (f(x) - h(x)) \sin \beta x dx + \int_a^b h(x) \sin \beta x dx\right| \\
&\leq \int_a^b |f - h| |\sin \beta x dx| + \left|\int_a^b h(x) \sin \beta x dx\right| \\
&\leq \int_a^b |f - h| dx + \left|\int_a^b h(x) \sin \beta x dx\right| \\
&< \frac{\epsilon}{2} + \left|\int_a^b h(x) \sin \beta x dx\right|
\end{aligned}$$

Now, if c_i, d_i $(c_i < d_i)$ are end points of the interval I_i, then

$$\begin{aligned}
\left|\int_a^b \chi_{I_i} \sin \beta x dx\right| &= \left|\int_{c_i}^{d_i} \chi_{I_i} \sin \beta x dx\right| \\
&= \left|\frac{1}{\beta} \int_{\beta c_i}^{\beta d_i} \sin y dy\right| \\
&= \frac{|\cos \beta c_i - \cos \beta d_i|}{\beta} \\
&\leq \frac{2}{\beta}
\end{aligned}$$

Let $M = \max[a_i; = 1, 2, \ldots, n]$. For the given $\epsilon > 0$, choose β_0 so that

$$\frac{2}{\beta} < \frac{\epsilon}{2nM} \text{ for } \beta > \beta_0$$

Then, given $\epsilon > 0$ there exists β_0 such that $\beta > \beta_0$ implies

$$\begin{aligned}
\left|\int_a^b f(x) \sin \beta x dx\right| &< \frac{\epsilon}{2} + \left|\int_a^b h(x) \sin \beta x dx\right| \\
&= \frac{\epsilon}{2} + \left|\int_a^b \sum_{i=1}^{n} a_i \chi_{I_i} \sin \beta x dx\right| \\
&\leq \frac{\epsilon}{2} + \sum_{i=1}^{n} \left|a_i \int_a^b \chi_{I_i} \sin \beta x dx\right| \\
&\leq \frac{\epsilon}{2} + \sum_{i=1}^{n} M \frac{2}{\beta} \\
&= \frac{\epsilon}{2} + \frac{2}{\beta} nM \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

$$\begin{aligned} i.e., \quad \left| \int_a^b f(x) \sin \beta x dx \right| &< \epsilon \\ \Rightarrow \lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x dx &= 0 \end{aligned}$$

Example 13.2. Show that if $f \in L(a+h, b+h)$ and $f_h(x) \equiv f(x+h)$, then $f_h \in L(a,b)$ and $\int_{a+h}^{b+h} f dx = \int_a^b f_h dx$.

Solution: Since $f \in L(a+h, b+h)$ and f is integrable and hence $f = f^+ - f^-$.

Consider $f_h(x) = f(x+h)$

$\Rightarrow \quad f_h^+ = f^+(x+h), \quad f_h^- = f^-(x+h)$

Hence, it is sufficient to prove that result for $f \geq 0$.

We know that if f is a non-negative measurable function, then there exists a $\{\varphi_n\}$ of measurable simple functions such that for each x, $\varphi_n(x) \uparrow f(x)$. But $(\varphi_n)_h \uparrow f_h$. So by applying Lebesgue convergence theorem with function $f = \lim \varphi_n$, we have

$$\begin{aligned} \int_{a+h}^{b+h} f dx &= \int_{a+h}^{b+h} \lim \varphi_n dx \\ &= \lim \int_{a+h}^{b+h} \varphi_n dx \\ &= \lim \int_a^b (\varphi_n)_h dx \\ &= \int_a^b \lim (\varphi_n)_h dx \\ &= \int_a^b f_h dx \end{aligned}$$

13.2. Integration with respect to a measure:

We now consider the generalization of the definition and results of Units 11, 12 and 13. Much of the works of Units 11 and 12 holds for a general measure space. Where proofs need only a variation of the notion we refer to the version given for the real line.

Definition 13.2. A *measurable simple function* ϕ is one taking a finite number of non-negative values, each on a measurable set; if $a_1, a_2, \ldots, a_n$ are the distinct values of ϕ, we have $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = [x : \phi(x) = a_i]$. Then the *integral* of ϕ with respect to μ is given by

$$\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Definition 13.3. Let f be measurable, $f : X \to [0, \infty]$. Then the integral of f is

$$\int f d\mu = \sup \left[\int \phi d\mu : \phi \leq f, \ \phi \text{ is a measurable function} \right]$$

Definition 13.4. Let $E \in \mathfrak{S}$, and let f be a measurable function $f : E \to [0, \infty]$; then the integral of f over E is

$$\int_E f d\mu = \int f \chi_E d\mu$$

Theorem 13.5 (Fatou's lemma). *Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$. Then*

$$\liminf \int d\mu \leq \int \liminf f_n d\mu$$

Proof. See Theorem 11.3 ■

Theorem 13.6 (Lebesgue's Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$, such that $f_n(x) \uparrow$ for each x, and let $f = \lim f_n$. Then*

$$\int f dx = \lim \int f_n d\mu$$

Proof. See Theorem 11.4 ■

Theorem 13.7. *Let f be a measurable function, $f : X \to [0, \infty]$. Then there exists a sequence $\{\phi_n$ of measurable simple functions such that , for each x, $\phi_n(x) \uparrow f(x)$.*

Proof. See Theorem 11.5 ■

Theorem 13.8. *Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$; then*

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$$

Proof. See Theorem 11.7 ■

Theorem 13.9. *Let $[[X, \mathfrak{S}, \mu]]$ be a measure space and f a non-negative measurable function. Then $\phi(E) = \int_E f d\mu < \infty$ then $\forall \epsilon > 0, \exists \ \delta > 0$ such that, if $A \in \mathfrak{S}$ and $\mu(A) < \delta$, then $\phi(A) < \epsilon$.*

Proof. Suppose f is a non-negative measurable function and $[[X, \mathfrak{S}, \mu]]$ be a measure space.

The function ϕ is countably additive, if $\{E_n\}$ is a sequence of disjoint sets in $\mathscr{S}$.

Put $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\begin{aligned}
\phi(E) &= \phi\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \int \chi_{\bigcup_{n=1}^{\infty} E_n} f d\mu \\
&= \sum_{n=1}^{\infty} \int \chi_{\bigcup_{n=1}^{\infty} E_n} f d\mu \quad \text{(by using (13.8))} \\
&= \int \sum_{n=1}^{\infty} \chi_{\bigcup_{n=1}^{\infty} E_n} f d\mu \\
&= \int \chi_E f d\mu = \int_E f d\mu \\
i.e., \quad \phi(E) &= \int_E f d\mu
\end{aligned}$$

ϕ is a measure on $[[X, \mathscr{S}]]$. Write $f_n = \max(f, n)$. Then for each n, f_n is measurable, $f_n \uparrow f$, and then by applying Lebesgue's Monotone Convergence theorem, we have

$$\lim \int f_n dx = \int f d\mu$$

Since $\int f d\mu < \infty$, then given $\epsilon > 0$, there exists N such that $\int f d\mu < \int f_N d\mu + \frac{\epsilon}{2}$. If $A \in \mathscr{S}$ and $\mu(A) < \frac{\epsilon}{2N}$, then we have $\int_A f_N d\mu < \frac{\epsilon}{2}$. Take $\delta = \frac{\epsilon}{2N}$, so that

$$\begin{aligned}
\int_A f d\mu &= \int_A (f - f_N + f_N)\, d\mu \\
&\leq \int_A (f - f_N)\, d\mu + \int_A f_N d\mu \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \\
i.e., \quad \phi(A) &< \epsilon
\end{aligned}$$

■

Definition 13.5. If f is measurable and both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite , then f is said to be *integrable*, and the integral of f is $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$.

So, f is integrable, if and only if $|f|$ is integrable.

Notation: The notation $f \in L(X, \mu)$ used to indicate that f belongs to the class of functions integrable with respect to μ. The notation $\int_E f d\mu$ means $\int f \chi_E d\mu$, where $f \in L(X, \mu)$ and $E \in \mathfrak{S}$.

If $f\chi_E$ is integrable, we write $f \in L(E < \mu)$ or simply $f \in L(E)$.

Definition 13.6. If f is a measurable function such that atleast one of $\int f^+ dx$, $\int f^- dx$ is finite, then $\int f dx = \int f^+ dx - \int f^- dx$.

Theorem 13.10. *Let f and g be integrable functions and let a and b are constants. Then $af + bg$ is integrable and*

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$$

If $f = g$ *a.e. then* $\int f d\mu = \int g d\mu$.

Proof. See Theorem 12.2 ■

Theorem 13.11. *Let* f *be integrable, then* $\left|\int f d\mu\right| \leq \int |f| d\mu$ *with equality, if and only if* $f \geq 0$ *a.e. or* $f \leq 0$ *a.e.*

Proof. See Example 12.2 ■

Theorem 13.12 (Lebesgue's Dominated Convergence Theorem)**.** *Let* $\{f_n\}$ *be a sequence of measurable functions such that* $|f_n| \leq g$ *where* g *is an integrable function, and* $\lim f_n = f$ *a.e. Then* f *is integrable,* $\lim \int f_n d\mu = \int f d\mu$, *and* $\lim \int |f_n - f| d\mu = 0$.

Proof. See Theorem 12.3 and Example 12.6 ■

Theorem 13.13. *Let* $\{f_n\}$ *be a sequence of integrable functions such that*

$$\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ *converges a.e., its sum* f, *is integrable and*

$$\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$$

Proof. See Theorem 12.4 ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- find the difference between Riemann and Lebesgue integral.
- integration with respect to a measure.

Check Your Progress:

1. Derive Lebesgue Dominated Convergence theorem.
2. Show that the function $x^{-1} \sin x$ is Riemann integrable on $(-\infty, \infty)$ but its Lebesgue integral does not exist.

Suggested Readings:

1. G. de Barra, “Measure Theory and Integration”, New Age International Pvt. Ltd, Second Edition, 2013.

2. Rana I. K., “An Introduction to Measure and Integration”, Narosa Publishing House Pvt. Ltd., Second Edition, 2007.

3. Royden H. L., “Real Analysis”, Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-V

Block-V

UNIT-14

LEBESGUE DECOMPOSITION

Objectives

After completion of this unit, students will be able to

- ★ define signed measure, positive set and negative set.
- ★ the concept of Hahn decomposition and Jordan decomposition.

Overview

In this unit, we will illustrate the basic concepts of signed measures, positive set and negative set.

14.1. Signed Measures and the Hahn Decomposition:

Definition 14.1. A set function ν defined on a measurable space $[[X, \mathscr{S}]]$ is said to be *signed measure* if the values of ν are extended real numbers and

(i) ν takes at most one of the values ∞, $-\infty$,

(ii) $\nu(\emptyset) = 0$,

(iii) $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$ if $E_i \cap E_j = \emptyset$ for $i \neq j$, where if the left-hand side is infinite, the series on the right hand side has sum ∞ or $-\infty$ as the case may be.

Note 14.1. Every measure is a signed measure.

Example 14.1. Show that $\phi(E) = \int_E f d\mu$ where $\int f d\mu$ is defined, then ϕ is a signed measure.

Solution: Suppose $\phi(E) = \int_E f d\mu$. Suppose $\int f d\mu$ is defined then either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$.

Thus, ϕ takes at most one of the values ∞, $-\infty$ and hence the condition (i) of signed measure is verified.

Clearly, $\phi(\emptyset) = 0$ and hence (ii) is satisfied.

Let $\{E_i\}$ be a sequence of disjoint sets of $\mathscr{S}$ and for $E \in \mathscr{S}$. Write

$$\begin{aligned} \phi^+(E) &= \int_E f^+ d\mu \\ \phi^-(E) &= \int_E f^- d\mu \end{aligned}$$

So, by theorem (13.9), ϕ^+ and ϕ^- are measures. Then

$$\begin{aligned} \phi\left(\bigcup_{i=1}^{\infty} E_i\right) &= \phi^+\left(\bigcup_{i=1}^{\infty} E_i\right) - \phi^-\left(\bigcup_{i=1}^{\infty} E_i\right) \\ &= \sum_{i=1}^{\infty} \phi^+(E_i) - \sum_{i=1}^{\infty} \phi^-(E_i) \qquad (\because \phi^+, \phi^- \text{ are measures}) \\ &= \sum_{i=1}^{\infty} \phi(E_i) \end{aligned}$$

Thus, condition (iii) is verified and hence ϕ is a signed measure.

Definition 14.2. A is a *positive set* with respect to the signed measured ν on $[[X, \mathscr{S}]]$, if $A \in \mathscr{S}$ and $\nu(E) \geq 0$ for each measurable subset E of A. We will omit *with respect to* ν, if the signed measure is obvious from the context.

The next example shows an important way of constructing a new measure from a given signed measure.

Example 14.2. If A is a positive set with respect to ν and if, for $E \in \mathscr{S}$, $\mu(E) = \nu(E \cap A)$, then μ is a measure.

Definition 14.3. A is a *negative set* with respect to ν if it is a positive set with respect to $-\nu$.

Definition 14.4. A is a *null set* with respect to ν or a ν-null set, if it is both positive and negative set with respect to ν.

Note 14.2. If A is a ν-null set, if $A \in \mathscr{S}$ and $\nu(E) = 0$ for all $E \in \mathscr{S}$, $E \subseteq A$.

Example 14.3.

(i) If A is a positive set with respect to ν then every measurable subset of A is a positive set.

(ii) If A is a negative set with respect to ν then every measurable subset of A is a negative set.

(iii) If A is a null set with respect to ν then every measurable subset of A is a null set.

Theorem 14.1. *A countable union of sets positive with respect to a signed measure ν is a positive set.*

Proof. Let $\{A_n\}$ be a sequence of positive sets. Then by theorem (10.2), there is a sequence $\{B_i\}$ of disjoint sets of $\mathscr{S}$ such that $B_n \subseteq A_n$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

Let $E \subseteq \bigcup_{n=1}^{\infty} A_n$. Then

$$\begin{aligned}
E &\subseteq \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \\
\Rightarrow E &= \left(\bigcup_{n=1}^{\infty} B_n\right) \cap E \\
\Rightarrow E &= \bigcup_{n=1}^{\infty} (E \cap B_n) \\
\text{i.e.,} \quad \nu(E) &= \sum_{n=1}^{\infty} \nu(E \cap B_n) \geq 0 \quad (\because E \cap B_n \text{ is a positive set for each } n)
\end{aligned}$$

Thus, if $A_n \in \mathscr{S}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{S}$ and $\nu(E) \geq 0$ for each measurable subset of $\bigcup_{n=1}^{\infty} A_n$.

Hence, $\bigcup_{n=1}^{\infty} A_n$ is a positive set. ■

Corollary 14.1. *A countable union of negative sets with respect to a signed measure* ν *is a negative set.*

Corollary 14.2. *A countable union of null sets with respect to a signed measure* ν *is a null set.*

Theorem 14.2. *Let* ν *be a signed measure on* $[[X, \mathscr{S}]]$*. Let* $E \in \mathscr{S}$ *and* $\nu(E) > 0$*. Then there exists* A*, a set positive with respect to* ν*, such that* $A \subseteq E$ *and* $\nu(A) > 0$*.*

Proof. Suppose ν be a signed measure on $[[X, \mathscr{S}]]$. Let $E \in \mathscr{S}$ and $\nu(E) > 0$.

Case (i)**:** If E contains no set of negative ν-measure, then E is a positive set $A = E$. Thus, there exists a positive set A with respect to measure ν such that $A \subseteq E$ and $\nu(A) = \nu(E) > 0$.

Case (ii)**:** Suppose if E contains negative set with respect to ν measure. *i.e.*, there exists $n \in N$ such that there exists $B \in \mathscr{S}$, $B \subseteq E$ and $\nu(B) < -\frac{1}{n}$.

Let $n_1 \in N$ such that there exists an $E_1 \subseteq E$ with

$$\nu(E_1) < -\frac{1}{n_1}$$

Let $n_2 \in N$ such that there exists an $E_2 \subseteq E - E_1$ with

$$\nu(E_2) < -\frac{1}{n_2}$$

and so on.,

Let $n_k \in N$ such that there exists an $E_k \subseteq E - \bigcup_{i=1}^{k-1} E_i$ with

$$\nu(E_k) < -\frac{1}{n_k}$$

From the construction, $n_1 \leq n_2 \leq n_3 \ldots$ and we have a corresponding sequence $\{E_i\}$ of disjoint subsets of E.

Now, either this process terminates or continues. If the process is stop, say at n_m and

$$C = E - \bigcup_{n=1}^{m} E_i \tag{14.1}$$

Clearly $C \in \mathscr{S}$ and $\nu(E) > 0$. Thus C is a positive set. Now, our aim is to prove that $\nu(C) > 0$. Suppose, if $\nu(C) = 0$, then from (14.1)we have

$$\nu(E) = \nu(C) + \sum_{n=1}^{m} \nu(E_i)$$

$$\Rightarrow \nu(E) = \sum_{n=1}^{m} \nu(E_i) < 0$$

which is a contradiction to the fact that $\nu(E) > 0$. Thus, $\nu(C) > 0$. which gives the required result.

If the process is not terminate, then we are able to inductively construct a sequence $\{E_k\}$ as above.

Put $A = E - \bigcup_{k=1}^{\infty} E_k.$

Now, our aim is to prove that A is a positive set.

Since $\{E_k\}$ is pairwise disjoint and each E_k is disjoint from A, we have

$$\nu(E) = \nu(A) + \nu\left(\bigcup_{n=1}^{\infty} E_k\right) \tag{14.2}$$

But ν cannot take both the values ∞, $-\infty$.

Since $\nu(E)$ is finite and $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) < 0$. Thus, the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges.

Since, the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges, then the series $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges, and in particular $n_k \to \infty$ and $n_k > 1$ for $k > k_0$ say. So, let $B \in \mathscr{B}$, $B \subseteq A$ and $k > k_0$. Then

$$B \subseteq E - \bigcup_{i=1}^{k} E_i$$

So,

$$\nu(B) \geq -\frac{1}{n_k - 1} \tag{14.3}$$

by the definition of n_k. But (14.3) holds good for all $k \geq k_0$, so letting $k \to \infty$, we have $\nu(B) \geq 0$ and so A is a positive set.

It remains to prove that $\nu(A) > 0$. Suppose if $\nu(A) = 0$, then we have $\nu(E) < 0$ which is a contradiction. Thus, $\nu(A) > 0$, which gives the desired result.

Hence the proof. ■

Theorem 14.3. *Let ν be a signed measure on $[[X, \mathscr{S}]]$. Then there exists a positive set A and a negative set B such that $A \cup B = A \cap B = \emptyset$. The pair A, B is said to be a Hahn decomposition of X with respect to ν. It is unique to be extent that if A_1, B_1 and A_2, B_2 are Hahn decompositions of X with respect to ν then $A_1 \triangle A_2$ is a ν-null set.*

Proof. Since ν cannot both the values $-\infty$, ∞. Without loss of generality we may that ν never takes the value ∞ on $\mathscr{S}$, for otherwise we consider $-\nu$, the result of the theorem $-\nu$ implying the result for ν.

Let
$$\lambda = \sup[\nu(C) : C \text{ is a positive set}]$$

Since $\emptyset$ is a positive set and thus, we have $\lambda \geq 0$. So, we can find a sequence of positive sets $\{A_i\}$ such that $\lambda = \lim \nu(A_i)$.

We know that countable union of positive sets is positive and hence $A = \bigcup_{n=1}^{\infty} A_n$ is positive set.

By definition of λ, we have

$$\lambda \geq \nu(A) \tag{14.4}$$

But $A - A_i \subseteq A$ and so $\nu(A - A_i) \geq 0$. Hence it is a positive set. So far each i

$$\nu(A) = \nu(A_i) + \nu(A - A_i) \geq \nu(A_i)$$

$$\text{So, } \nu(A) \geq \lim \nu(A_i) = \lambda \tag{14.5}$$

From, (14.4) and (14.5), we have

$$\nu(A) = \lambda$$

Let $B = A^c$. Then if B contains a positive set D of ν measure.

So, we have $0 < \nu(D) < \infty$.

So, by theorem (14.2), D contains a positive set E such that $0 < \nu(E) < \infty$. But

$$\nu(A \cup E) = \nu(A) + \nu(E) > \lambda$$

which is contradicting the definition of λ, So $\nu(D) \leq 0$ and hence B is negative.

Thus, A and B form a Hahn decomposition.

Further, if A_1, B_1 and A_2, B_2 are Hahn decomposition of X, then

$$A_1 - A_2 = A_1 \cap A_2^c$$
$$= A_1 \cap B_2$$

Thus, $A_1 - A_2$ is a positive set as well as negative set and hence it is a null set.

Similarly, we can prove that $A_2 - A_2$ is a null set.

So,

$$A_1 \triangle A_2 = (A_1 - A_2) \cup (A_2 - A_1)$$

Thus, $A_1 \triangle A_2$ is a union of null sets and hence $A_1 \triangle A_2$ is a null set.

This completes the proof of the theorem. ■

14.2. The Jordan Decomposition:

Definition 14.5. Let ν_1 and ν_2 be measures on $[[X, \mathscr{S}]]$. Then ν_1 and ν_2 are said to be *mutually singular* if, for some $A \in \mathscr{S}$, $\nu_2(A) = \nu_1(A^c) = 0$, and we write $\nu_1 \perp \nu_2$.

Example 14.4. Let μ be a measure and let the measures ν_1, ν_2 be given by $\nu_1(E) = \mu(A \cap E)$, $\nu_2(E) = \mu(B \cap E)$, where $\mu(A \cap B) = 0$ and $E, A, B \in \mathscr{S}$, show that $\nu_1 \perp \nu_2$.

Solution:

$$\begin{aligned} \nu_1(B) &= \mu(A \cap B) = 0 \\ \nu_2(B^c) &= \mu(B \cap B^c) = \mu(\emptyset) = 0 \end{aligned}$$

Thus, $\nu_1(B) = 0$ and $\nu_2(B^c) = 0$.

Therefore, $\nu_1 \perp \nu_2$.

Theorem 14.4. *Let ν be a signed measure on $[[X, \mathscr{S}]]$. Then there exists a measures ν^+ and ν^- on $[[X, \mathscr{S}]]$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. The measures ν^+ and ν^- are uniquely defined by ν and $\nu = \nu^+ - \nu^-$ is said to be the Jordan decomposition of ν.*

Proof. Let A and B be Hahn decomposition of X with respect to ν, then $X = A \cup B$ and $A \cap B = \emptyset$.

Define ν^+ and ν^- by

$$\left.\begin{aligned} \nu^+(E) &= \nu(E \cap A) \\ \nu^-(E) &= -\nu(E \cap B) \end{aligned}\right\} \tag{14.6}$$

for $E \in \mathscr{S}$.

By example (14.2), we have ν^+ and ν^- are measures and

$$\begin{aligned} \nu^+(B) &= \nu(B \cap A) = \nu(\emptyset) = 0 \\ \nu^-(A) &= -\nu(A \cap B) = \nu(\emptyset) = 0 \end{aligned}$$

So, $\nu^+(B) = \nu^-(A) = 0$ and hence $\nu^+ \perp \nu^-$.

Also, for $E \in \mathscr{S}$,

$$\begin{aligned} \nu(E) &= \nu(E \cap A) + \nu(E \cap A^c) \\ &= \nu(E \cap A) + \nu(E \cap B) \\ &= \nu^+(E) - \nu^-(E) \end{aligned}$$

So, $\nu = \nu^+ - \nu^-$.

It remains to prove that the decomposition is unique.

Let $\nu = \nu_1 - \nu_2$ be any other decomposition of ν into mutually singular measures.

Then $X = A \cup B$ and $A \cap B = \emptyset$, $\quad i.e., B = A^c$ and $\nu_1(B) = \nu_2(A) = 0$.

Let $D \subseteq A$, then

$$\begin{aligned} \nu(D) &= \nu_1(D) - \nu_2(D) \\ &= \nu_1(D) \geq 0 \qquad (\because D \subseteq A) \end{aligned}$$

So, A is positive set with respect to ν.

In a similar way, we can prove that B is a positive set with respect to ν.

For each $E \in \mathscr{S}$, we have

$$\begin{aligned} \nu_1(E) &= \nu_1(E \cap A) = \nu(E \cap A) \\ \nu_2(E) &= -\nu(E \cap B) \end{aligned}$$

So every such decomposition of ν is obtained from a Hahn decomposition of X as in (14.6).

So, it is enough to prove that if A, B and A', B' are two Hahn decomposition's then the measures obtained in (14.6) are the same. We have

$$\begin{aligned} \nu(A \cup A') &= \nu(A \cap A') + \nu(A \triangle A') \\ &= \nu(A \cap A' \quad (\because \nu(A \triangle A') = 0) \end{aligned}$$

For each $E \in \mathscr{S}$, as $A \cup A'$ is a positive set, we have

$$\begin{aligned} \nu(E \cap (A \cap A')) &\leq \nu(E \cap A) \leq \nu(E \cap (A \cup A')) \\ \nu(E \cap (A \cap A')) &\leq \nu(E \cap A') \leq \nu(E \cap (A \cup A')) \end{aligned}$$

From, the above inequalities, we have $\nu(E \cap A) = \nu(E \cap A')$ and ν^+ defined in (14.6) is unique.

Then $\nu^- = \nu - \nu^+$ is also unique.

Hence the proof. ■

Example 14.5. Let $[[X, \mathscr{S}, \mu]]$ be a measure space and let $\int f d\mu$ exists. Define $\nu(E) = \int_E f d\mu$, for $E \in \mathscr{S}$. Find a Hahn decomposition with respect to ν and the Jordan decomposition of ν.

Solution: Define $\nu(E) = \int_E f d\mu$, for $E \in \mathscr{S}$.

Then, by example (14.1), we have ν is a signed measure.

Let

$$\begin{aligned} A &= [x : f(x) \geq 0] \\ B &= [x : f(x) < 0] \end{aligned}$$

Clearly A and B form a Hahn decomposition. While ν^+ and ν^- are given by

$$\begin{aligned} \nu^+(E) &= \int_E f+ d\mu \\ \nu^-(E) &= \int_E f- d\mu \end{aligned}$$

Thus, ν^+ and ν^- form the Jordan decomposition.

Definition 14.6. The *total variation* of a signed measure ν is $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

Clearly, $|\nu|$ is a measure on $[[X,\mathscr{S}]]$ and for each $E \in \mathscr{S}$, $|\nu(E)| \leq |\nu|(E)$.

Definition 14.7. A signed measure ν on $[[X,\mathscr{S}]]$ is σ-finite, if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathscr{S}$ and for each n, $|\nu(X_n)| < \infty$.

Example 14.6. Show that the signed measure ν is finite or σ-finite respectively, if and only if $|\nu|$, or if, and only if, both ν^+ and ν^- are σ-finite.

Solution: Suppose $|\nu(E)| < \infty$. Then

$$\nu(E) = \nu^+(E) - \nu^-(E) < \infty$$

Thus, both ν^+ and ν^- are not infinite. Since $\nu^+(E) < \infty$ and $\nu^-(E) < \infty$.

So,

$$|\nu|(E) = \nu^+(E) + \nu^-(E) < \infty$$

Hence, ν is finite if and only if $|\nu|$ is finite.

Similarly, we can prove the result for σ-finiteness.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- signed measures, positive sets and negative sets.
- derive Hahn decomposition theorem.

Check Your Progress:

1. Prove that a countable union of sets positive with respect to a signed measure ν is a positive set.
2. State and Prove Jordan decomposition.

Say True/False:

1. A is a negative set with respect to ν if it is a positive set with respect to ν.
2. Every measure is a signed measure.

Answer:

(1) F (2) T

Suggested Readings:

1. G. de Barra, “Measure Theory and Integration”, New Age International Pvt. Ltd, Second Edition, 2013.
2. Rana I. K., “An Introduction to Measure and Integration”, Narosa Publishing House Pvt. Ltd., Second Edition, 2007.
3. Royden H. L., “Real Analysis”, Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-V

UNIT-15

RADON-NIKODYM THEOREM AND ITS APPLICATIONS

Objectives

After completion of this unit, students will be able to

- ★ derive Radon-Nikodym theorem.
- ★ understand the application of Radon-Nikodym theorem.

Overview

In this unit, we will discuss in detail about the derivation of Radon-Nikodym theorem and its applications.

15.1. The Radon-Nikodym Theorem:

Definition 15.1. If μ, ν are measures on the measurable set $[[X, \mathscr{S}]]$ and $\nu(E) = 0$ whenever $\mu(E) = 0$, then we say that ν is *absolutely continuous* with respect to μ and we write $\nu \ll \mu$.

Definition 15.2. If μ, ν are measures on the measurable set $[[X, \mathscr{S}]]$ and $\nu(E) = 0$ whenever $|\mu|(E) = 0$, then we say that ν is *absolutely continuous* with respect to μ and we write $\nu \ll \mu$.

Example 15.1. Show that the following conditions on the signed measures μ and ν on $[[X, \mathscr{S}]]$ are equivalent: $(i)\nu \ll \mu$, $(ii)\, |\nu| \ll |\mu|$ $\quad (iii)$ $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Solution: From Definition (15.2), we see that ν, μ, if and only if $\nu \ll |\mu|$. So, we may assume that $\mu \geq 0$. As $|\nu| = \nu^+ + \nu^-$, we see that $|\nu| \ll \mu$ implies $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, so $\nu \ll \mu$. For the opposite implications, suppose that $\nu = \nu^+ - \nu^-$ with a Hahn decomposition A, B. Then if $\nu \ll \mu$ and $\mu(E) = 0$ we have $\mu(E \cap A) = 0$ so $\nu^+(E) = 0$ and similarly $\nu^-(E) = 0$. So $|\nu|(E) = 0$.

Theorem 15.1 (Radon-Nikodym Theorem). *If $[[X\mathscr{S}, \mu]]$ is a σ-finite measure space and ν is a σ-finite measure on $\mathscr{S}$ such that $\nu \ll \mu$, then there exists a finite-valued non-negative measurable function f on X such that for each $E \in \mathscr{S}$, $\nu(E) = \int_E f d\mu$. Also, f is unique in the sense that if $\nu(E) = \int_E g d\mu$ for each $E \in \mathscr{S}$, then $f = g$ a.e.(μ).*

Proof. Assume that the result has been proved for finite measures. Then in the general case, we have

$$
\begin{aligned}
X &= \bigcup_{n=1}^{\infty} A_n, \quad \mu(A_n) < \infty \\
\text{and } X &= \bigcup_{m=1}^{\infty} B_m, \quad \nu(B_m) < \infty
\end{aligned}
$$

and $\{A_n\}$, $\{B_m\}$ may be sequence of disjoint sets.

So, Put $\bigcup_{n,m=1}^{\infty} (A_n \cap B_m)$. Then we obtain X as the union of disjoint sets on which both ν and μ are finite, say $X = \bigcup_{n=1}^{\infty} X_n$.

Let $\mathscr{S}_n = [E \cap X_n : E \in \mathscr{S}]$, a σ-algebra over X_n.

Now, consider μ and ν restricted to $\mathscr{S}_n$, then we obtain a non-negative function f_n such that if $E \in \mathscr{S}_n$, $\nu(E) = \int_E f_n d\mu$.

So, if $A \in \mathscr{S}$, $A = \bigcup_{n=1}^{\infty} A_n$, (say), where $A_n \in \mathscr{S}_n$.

Define $f = f_n$ on X_n.

Then by example (10.8), we have f is measurable function on X, and

$$\nu(A) = \sum_{n=1}^{\infty} \int_{A_n} f_n d\mu = \int_A f d\mu$$

Thus, the general case follows.

So, we need to show that for finite measures such a function f exists.

Let **K** be the class of non-negative measurable functions with respect to μ and satisfying

$$\int_E f d\mu \leq \nu(E) \quad \forall E \in \mathscr{S}$$

Since $0 \in$ **K** and hence **K** is non-empty.

Let

$$\alpha = \sup\left[\int f d\mu : f \in \mathbf{K}\right]$$

and let $\{f_n\}$ be a sequence in **K** such that $\lim \int f_n d\mu = \alpha$.

If B is any fixed measurable set, n a fixed integer and

$$g_n = \max\{f_1, f_2, \ldots, f_m\}$$

Now, we can prove by induction that B is the union of disjoint measurable sets B_i, $i = 1, 2, \ldots, n$, such that

$$g_n = f_i \text{ on } B_i, \; i = 1, 2, \ldots, n.$$

For $n = 2$ and let

$$B_1 = [x : x \in B, f_1(x) \geq f_2(x)]$$
$$B_2 = B - B_1$$

Clearly, B_1 and B_2 are disjoint measurable sets and $B = B_1 \cup B_2$.

Assume that the decomposition is possible for n, let

$$g_{n+1} = \max\{f_1, f_2, \ldots, f_{n+1}\}$$
$$= \max\{g_n, f_1\}$$

So,

$$\begin{aligned} B &= F_n \cup B_{n+1}, \\ \text{where} \quad g_{n+1} &= f_{n+1} \text{ on } B_{n+1} \\ g_{n+1} &= g_n \text{ on } F_n \end{aligned}$$

and $F_n \cap B_{n+1} = \emptyset$.

But, then by inductive hypothesis, we have

$$\begin{aligned} F_n &= \bigcup_{n=1}^{\infty} B_n \\ \text{and} \quad g_{n+1}(x) &= f(x) \quad \text{for } x \in B_i, \; i = 1, 2, \ldots, n+1. \end{aligned}$$

Now, since each $f_i \in \mathbf{K}$,

$$\int_B g_n d\mu = \sum_{n=1}^{\infty} \int_{B_i} f_i d\mu \leq \sum_{i=1}^{n} \nu(B_i) = \nu(B) \tag{15.1}$$

Also, we have $g_n \uparrow$, so write

$$f_0 = \lim g_n$$

Then by equation (15.1) and by Lebesgue's Monotone convergence theorem, we have

$$\int_E f_0 d\mu = \lim \int_E g_n d\mu \leq \nu(E)$$

so $f \in \mathbf{K}$. Hence

$$\alpha \geq \int f_0 d\mu \geq \int g_n d\mu \geq \int f_n d\mu$$

So, $\alpha = \int f_0 d\mu$.

Since $\int f_0 d\mu \leq \nu(X) < \infty$. Hence there exists a finite-valued non-negative measurable function, such that $f = f_0 \; a.e.(\mu)$.

Next, we will show that if

$$\nu_0(E) = \nu(E) - \int_E f d\mu$$

then $\nu_0(E) = 0$, for each $E \in \mathscr{S}$.

By the construction of f, ν_0 is non-negative.

Assume that ν_0 is not identically zero on $\mathscr{S}$.

Let $C \in \mathscr{S}$ and $\nu_0(C) > 0$. Then by the suitable choice of ϵ, $0 < \epsilon < 1$, $(\nu_0 - \epsilon\mu)(C) > 0$.

But, by theorem (14.2), we can find A such that $(\nu_0 - \epsilon\mu)(A) > 0$. where A is a positive set with respect to $\nu_0 - \epsilon\mu$.

Also, $\mu(A) > 0$, for otherwise, as $\nu \ll \mu$ we would have $\nu(A) = 0$ and hence

$(\nu_0 - \epsilon\mu)(A) = 0$. So, for $E \in \mathscr{S}$

$$\epsilon\mu(E \cap A) \leq \nu_0(E \cap A) = \nu(E \cap A) - \int_{E \cap A} f d\mu$$

Hence, if $g = f\epsilon\chi_A$, for each $E \in \mathscr{S}$, we have

$$\int_E g d\mu = \int_E f d\mu + \epsilon\mu(E \cap A) \leq \int_{E-A} f d\mu + \nu(E \cap A) \leq \nu(E)$$

So, $g \in \mathbf{K}$.

But, $\int g d\mu = \int f d\mu + \epsilon\mu(A) > \alpha$, a contradicting the maximality of α.

Hence $\nu_0 = 0$ on S. *i.e.*, $\int_E f d\mu = \nu(E)$.

Thus, f is the required function has the desired properties.

It remains to prove the uniqueness of f.

If possible, let g be another measurable function have these properties.

Then, for $E \in \mathscr{S}$, $\int_E (f - g) d\mu = 0$ and taking $E = [x : f(x) > g(x)]$, we get $f \leq g$ *a.e.* and similarly $f \geq g$ *a.e.* and hence $f = g$ *a.e.* So, f is unique. ■

Corollary 15.1. *Theorem* (15.1) *can be extended to the case when ν is a σ-finite signed measure.*

Proof. The Jordan decomposition gives $\nu = \nu^+ - \nu^-$ and

$$\nu^+(E) = \int_E f_1 d\mu;$$
$$\nu^-(E) = \int_E f_2 d\mu$$

where f_1 and f_2 are non-negative measurable functions of which at least one is integrable.

So, for $E \in \mathscr{S}$, $\nu(E) = \nu^+(E) - \nu^-(E) = \int_E f d\mu$ where the integral of $f = f_1 - f_2$ is well defined. ■

Corollary 15.2. *Theorem* (15.1) *can be further extended to allow μ to be signed measure, where by $\int_E f d\mu$ we then mean $\int_E f^+ d\mu - \int_E f^- d\mu$ provided this difference is not indeterminate. Any two such functions f and g are equal $a.e.(|\mu|)$.*

Proof. Let A, B be Hahn decomposition with respect to μ, so that

$$\mu^*(E) = \mu(E \cap A)$$
$$\mu(E) = -\mu(E \cap B)$$

Now, $\nu \ll \mu^*$ and μ^* is σ-finite.

By applying theorem (15.1) on μ^*, we get

$$\nu(E \cap A) = \int_{E \cap A} f_1 d\mu^+$$

for an appropriate function f_1 on A.

Similarly, we have

$$\nu(E \cap B) = \int_{E\cap B} f_2 d\mu^-$$

for an appropriate function f_1 on B.

Define

$$f = f_1 \text{ on } A$$
$$f = -f_2 \text{ on } B$$

Then f is measurable function on X and

$$\nu(E) = \int_{A\cap B} f_1 d\mu^+ - \int_{B\cap E} (-f_2) d\mu^-$$

As ν is a signed measure, this will not be of the form $\infty - \infty$. So,

$$\nu(E) = \int_E f d\mu$$

is well defined.

Any two such functions, from the construction agree except on a set of zero μ^+ and μ^- -measure, giving the result. ■

15.2. Some Applications of the Radon-Nikodym Theorem:

Theorem 15.2. *Let μ be a signed measure on $[[X, \mathscr{S}]]$ and let ν be a finite-valued signed measure on $[[X, \mathscr{S}]]$ such that $\nu \ll \mu$; then given $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu|(E) < \epsilon$ whenever $|\mu|(E) < \delta$.*

Proof. Since $\nu \ll \mu$ is equivalent to $|\nu| \ll \mu$ and ν is finite-valued if and only if ν is finite-valued.

Assume that ν and μ are measures.

Suppose, if the result is not true, then there exists a positive ϵ and a sequence $\{E_n\}$ of sets of $\mathscr{S}$ such that $\mu(E_n) < \frac{1}{2^n}$, but $\nu(E_n) \geq \epsilon$.

Consider

$$\limsup E_n = \bigcap_{k=1}^{\infty} F_k, \quad F_k = \bigcup_{m=k}^{\infty} E_m$$

For each k, $\mu(\limsup E_n) = \mu(F_k) \leq \sum_{k=m}^{\infty} \frac{1}{2^m} = \frac{1}{2^{k-1}}$.

So, $\mu(\limsup E_n) = 0$,

But, for each k, $\nu(F_k) \geq \epsilon$ and ν is finite.

Hence, by theorem (10.6), we have

$$\nu(\limsup E_n) = \nu(\lim F_k) = \lim \nu(F_k) \geq \epsilon$$

contradicting to our hypothesis that $\nu \ll \mu$ and hence our assumption is wrong.

i.e., given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu|(E) < \epsilon$ whenever $|\mu|(E) < \delta$. ■

Definition 15.3. Let μ and ν be σ-finite signed measure on $[[X, \mathscr{S}]]$ and suppose that $\nu \ll \mu$. Then the *Radon-Nikodym derivative* $\dfrac{d\nu}{d\mu}$ of ν with respect to μ, is any measurable function f such that $\nu(E) = \int_E f d\mu$ for each $E \in \mathscr{S}$, where if μ is a signed measure $\int f d\mu = \int f d\mu^+ - \int f d\mu^-$.

Notation: In the equation below connecting Radon-Nikodym derivatives, we will indicate the measure, say μ with respect to which the functions are equal $a.e.$ by the notation $[\mu]$. In the case of a signed measure, the functions are equal $a.e.(|\mu|)$.

Theorem 15.3. *If ν_1, ν_2 are σ-finite measures on $[[X, \mathscr{S}]]$ and $\nu_1 \ll \mu$, $\nu_2 \ll \mu$, then*

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}[\mu] \tag{15.2}$$

Proof. Suppose ν_1 and ν_2 are σ-finite measures, then clearly $\nu_1 + \nu_2$ is also σ-finite measures.

Also, $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$.

For $E \in \mathscr{S}$, then

$$\begin{aligned} (\nu_1 + \nu_2)(E) &= \nu_1(E) + \nu_2(E) \\ &= \int_E \frac{d\nu_1}{d\mu} d\mu + \int_E \frac{d\nu_2}{d\mu} d\mu \\ \frac{d(\nu_1 + \nu_2)}{d\mu} &= \frac{d\nu_1}{d\nu} + \frac{d\nu_2}{d\nu} \\ i.e., \quad \frac{d(\nu_1 + \nu_2)}{d\mu} &= \frac{d\nu_1}{d\nu} + \frac{d\nu_2}{d\nu}[\mu] \end{aligned}$$

Hence the proof. ■

Theorem 15.4. *If $\nu_1, \nu_2, \nu_1 + \nu_2$ and μ are σ-finite signed measures on $[[X, \mathscr{S}]]$ and $\nu_1 \ll \mu$, $\nu_2 \ll \mu$, then* (15.2) *holds.*

Proof. Since $\nu_1 + \nu_2$ is σ-finite signed measures, so $\nu_1(E) + \nu_2(E)$ never takes both values $\infty, -\infty$.

Case (i)**:** Suppose μ is a measure.

For $i = 1, 2$, let $\nu_i = \nu_i^+ - \nu_i^-$ with Hahn decompositions A_i, B_i.

Consider the four sets $A_1 \cap B_1,\ A_1 \cap B_2,\ A_2 \cap B_1,\ A_2 \cap B_2$ separately.

Now, consider the subset $A_1 \cap B_2$, so we have

$$\nu_1 + \nu_2 = \nu_1^+ - \nu_2^-$$

So, for $F \subseteq A_1 \cap B_2$,

$$\begin{aligned}
(\nu_1+\nu_2)(F) &= \nu_1^+(F) - \nu_2^-(F)\\
&= \int_F \left(\frac{d\nu_1^+}{d\mu} - \frac{d\nu_2^+}{d\mu}\right) d\mu\\
&= \int_F \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}\right) d\mu\\
i.e.,\quad (\nu_1+\nu_2)(F) &= \int_F \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}\right) d\mu \qquad (15.3)
\end{aligned}$$

Since $\dfrac{d\nu}{d\mu} = -\dfrac{d(-\nu)}{d\mu}[\mu]$.

Since $E \in \mathscr{S}$ can be written as the union of four such sets F.

Similarly, we can consider other three subsets, we get

$$(\nu_1+\nu_2)(F) = \int_F \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_1}{d\mu}\right) d\mu \qquad (15.4)$$

$$(\nu_1+\nu_2)(F) = \int_F \left(\frac{d\nu_2}{d\mu} + \frac{d\nu_1}{d\mu}\right) d\mu \qquad (15.5)$$

$$(\nu_1+\nu_2)(F) = \int_F \left(\frac{d\nu_2}{d\mu} + \frac{d\nu_2}{d\mu}\right) d\mu \qquad (15.6)$$

Adding the equation (15.3), (15.4), (15.5) and (15.6) we have

$$\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}[\mu]$$

Case (ii): Suppose μ is a signed measure.

Let A, B be a corresponding Hahn decomposition. Write $\mathscr{S}' = [E \cap A : E \in \mathscr{S}]$ and let μ', ν_1', ν_2' be the restriction of μ, ν_1, ν_2 to $\mathscr{S}'$.

Similarly, $\mathscr{S}'', \nu'', \nu_1'', \nu_2''$ in the case of B.

Now applying case (i) to A and B, we have

$$\left.\begin{aligned}
\frac{d(\nu_1'+\nu_2')}{d\mu'} &= \frac{d\nu_1'}{d\mu'} + \frac{d\nu_2'}{d\mu'}[\mu]\\
\frac{d(\nu_1''+\nu_2'')}{d(-\mu'')} &= \frac{d\nu_1''}{d(-\mu'')} + \frac{d\nu_2''}{d(-\mu'')}[\mu]
\end{aligned}\right\} \qquad (15.7)$$

Write $f_i = \dfrac{d\nu_i'}{d\mu'}$ on A, $f_i = -\dfrac{d\nu_i''}{d(-\mu'')}$ on B. Then for each $E \in \mathscr{S}$, we have

$$\begin{aligned}\int_E f_i d\mu &= \int_{E\cap A} f_i d\mu' - \int_{E\cap B} f_i d(-\mu'') \\ &= \nu_i(E\cap A) + \nu_i(E\cap B) = \nu_i(E) \quad \text{for } i = 1, 2.\end{aligned}$$

Similarly for $\nu_1 + \nu_2$, since

$$\begin{aligned}(\nu_1+\nu_2)' &= \nu_1' + \nu_2' \\ \text{and} \quad (\nu_1+\nu_2)'' &= \nu_1'' + \nu_2''\end{aligned}$$

To get the required, we may subtract equation (15.7). ■

Example 15.2. Let μ be a σ-finite measure and ν a σ-finite signed measure and let $\nu \ll \mu$;
show that $\dfrac{d|\nu|}{d\mu} = \left|\dfrac{d\nu}{d\mu}\right|$.

Solution: Let $\nu = \nu^+ - \nu^-$ with a corresponding Hahn decomposition A, B. As in the theorem (15.4), we have

$$\begin{aligned}\left|\frac{d\nu}{d\mu}\right| &= \frac{d\nu^+}{d\mu}\ [\mu] \text{ on A} \\ \text{and} \quad \left|\frac{d\nu}{d\mu}\right| &= \frac{d\nu^-}{d\mu}\ [\mu] \text{ on B}\end{aligned}$$

So, by theorem (15.3), we have

$$\left|\frac{d\nu}{d\mu}\right| = \frac{d\nu^+}{d\mu} + \frac{d\nu^-}{d\mu} = \frac{d|\nu|}{d\mu}\ [\mu]$$

Theorem 15.5. *Let ν be a signed measure and let μ, ν be measures on $[[X, \mathscr{S}]]$ such that λ, μ, ν are σ-finite, $\nu \ll \mu$ and $\mu \ll \lambda$; then*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\ [\lambda] \tag{15.8}$$

Proof. Write $\nu = \nu^+ - \nu^-$.

We know that $-\dfrac{d\nu^+}{d\lambda} = \dfrac{d(-\nu^-)}{d\lambda}\ [\lambda]$ and similarly for $\dfrac{d\nu^-}{d\mu}$.

So by theorem (15.4), we need to prove for measures only.

So, suppose that ν is a measure and by Radon-Nikodym theorem, take for $\dfrac{d\nu}{d\mu}$ and $\dfrac{d\mu}{d\lambda}$ the non-negative functions f and g respectively.

Now, our aim is to prove that for

$$F \in \mathscr{S}, \quad \nu(F) = \int_F fg\,d\lambda.$$

Let ψ be a measurable simple function,

$$\psi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

then, we have

$$\begin{aligned}
\int_F \psi d\mu &= \sum_{i=1}^{n} a_i \mu(E_i \cap F) \\
&= \sum_{i=1}^{n} a_i \int_{E_i \cap F} g d\lambda \\
&= \int_F \psi g d\lambda
\end{aligned}$$

Let $\{\psi_n\}$ be a sequence of measurable functions such that $\psi_n \uparrow f$. Then

$$\begin{aligned}
\nu(F) &= \int_F f d\mu \\
&= \lim \int_F \psi_n d\mu \\
&= \lim \int_F \psi_n g d\lambda \\
&= \int_F fg d\lambda \qquad (\because \psi_n g \uparrow fg).
\end{aligned}$$

Hence, $\dfrac{d\nu}{d\lambda} = \dfrac{d\nu}{d\mu}\dfrac{d\mu}{d\lambda}$ $[\lambda]$. This completes the proof. ∎

Theorem 15.6. *Let λ, μ, ν be σ-finite signed measures on $[[X, \mathscr{S}]]$ such that $\nu \ll \mu$ and $\mu \ll \lambda$; then* (15.8) *holds.*

Proof. Let A_1, B_1 and A_2, B_2 be the Hahn decomposition with respect to λ and μ respectively.

Consider the four sets $A_i \cap B_j$, $i, j = 1, 2$, separately.

Now. consider the set $A_1 \cap B_2$, we let

$$\mathscr{S}' = [E \cap A_1 \cap B_2 : E \in \mathscr{S}]$$

and also, let λ', μ' be the restriction of λ, μ to $\mathscr{S}'$.

So, λ' and $-\mu'$ are measures. Now, applying theorem (15.5) on $A_1 \cap B_2$, we have

$$\frac{d\mu}{d\lambda'} = \frac{d\nu}{d(-\mu')}\frac{d(-\mu')}{d\lambda'} \, [\mu]$$

As in the proof of Theorem (15.4), we see that $-\dfrac{d\nu}{d(-\mu')}$ is the restriction of $\dfrac{d\nu}{d\mu}$ to $A_1 \cap B_2$ and $-\dfrac{-d(-\mu')}{d\lambda'}$ that of $\dfrac{d\mu}{d\lambda}$ to $A_1 \cap B_2$. So on $A_1 \cap B_1$, we get

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

Adding all such four equations, we get the required results. Hence the proof. ∎

Theorem 15.7 (Lebesgue Decomposition Theorem)**.** *Let $[[X, \mathscr{S}, \mu]]$ be a σ-finite measure space and ν a σ-finite measure on $\mathscr{S}$. Then $\nu = \nu_0 + \nu_1$ where ν_0 and ν_1 are measures on $\mathscr{S}$ such that $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$. This is the Lebesgue decomposition of the measure ν with respect to μ and it is unique.*

Proof. Let $\lambda = \nu + \mu$, then clearly ν is σ-finite and $\mu \ll \lambda$.

By Radon-Nikodym Theorem, there exists a non-negative finite-valued measurable function f such that if $E\mathscr{S}$, then

$$\mu(E) = \int_E f d\lambda$$

Then, clearly $X = A \cup B$ and $A \cap B = \emptyset$ and $\mu(B) = \int_B f d\lambda = 0$.

For each $E \in \mathscr{S}$, Define measures ν_0, ν_1 by

$$\nu_0(E) = \nu(E \cap B)$$
$$\nu_1(E) = \nu(E \cap A)$$

Clearly, $nu = \nu_0 + \nu_1$.

Since $\nu_0(A) = \nu(A \cap B) = \nu(\emptyset) = 0$. Thus, we have $\nu_0 \perp \mu$.

If $\mu(E) = 0$, then $\int_E f d\lambda = 0$. So $f = 0\ a.e.(\lambda)$ on E.

But f is positive on $E \cap A$, so $\lambda(E \cap A) = 0$.

Also, we have $\nu \ll \lambda$, so $\nu_1(E) = \nu(E \cap A) = 0$. $i.e., \nu_1 \ll \mu$.

Next, we have to prove that the decomposition is unique.

Assume that $\nu = \nu_0 + \nu_1 = \nu'_0 + \nu'_1$ where $\nu_0 \perp \mu$, $\nu'_0 \perp \mu$, $\nu_1 \ll \mu$, $\nu'_1 \ll \mu$. So, there exists A, B, A', B' such that $X = A \cup B = A' \cup B'$; $A \cap B = A' \cap B' = \emptyset$ and $\nu_0(B) = \mu(A) = \nu'_0(B') = \mu(A') = 0$.

Let $E \in \mathscr{S}$, then

$$E = (E \cap B \cap B') \cup (E \cap A' \cap B) \cup (E \cap A \cap A') \cup (E \cap A \cap B')$$

Clearly μ is zero on the last three sets in this union and by absolute continuity, we have ν_1 and ν'_1 are zero.

Since $\nu'_1 - \nu_1 = \nu_0 - \nu'_0$, we have

$$(\nu'_1 - \nu_1)(E) = (\nu'_1 - \nu_1)(E \cap B \cap B') = \left(\nu_0 - \nu'_0\right)(E \cap B \cap B') = 0 \quad (\because \nu_0(B) = \nu'_0(B) = 0)$$

So, $\nu_1(E) = \nu'_1(E)$, which implies $\nu_0(E) = \nu'_o(E)$ and hence the proof. ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of Radon-Nikodym theorem and its applications.
- derive Lebesgue decomposition theorem.

Check Your Progress:

1. If ν is a signed measure and $E_1 \subseteq E_2 \subseteq \ldots$, then prove that $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim \nu(E_i)$.

2. If ν is a signed measure and $E_1 \supseteq E_2 \supseteq \ldots$, prove that $|\nu(E_i)| < \infty$, and $\nu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim \nu(E_i)$.

3. Show that if ν_1, ν_2 and μ are measurable and $\nu_1 \perp \mu$, $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.

4. Show that if μ and ν are σ-finite signed measures and $\mu \ll \nu$, $\nu \ll \mu$, then $\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1} [\mu]$.

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.

2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.

3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

Block-V

UNIT-16

BOUNDED LINEAR FUNCTIONALS

Structure

Objectives

After completion of this unit, students will be able to

- under the concept of normed vector space and linear functional.
- ★ derive Riesz representation theorem.

Overview

In this unit, we will illustrate the concepts of normed linear space and linear functional.

16.1. Bounded Linear Functionals on L^P :

Definition 16.1. Let V be a real vector space. Then V is a *normed vector space* if there is a function $||x||$ defined for each $x \in V$ such that

(i) $||x|| \geq 0 \ \forall x$,

(ii) $||x|| = 0$ if and only if $x = 0$,

(iii) $||\alpha x|| = |\alpha| \cdot ||x||$ for any real number α and for each $x \in V$.

(iv) $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in V$.

Definition 16.2. A function G on the normed linear space V to the real numbers is a *linear functional* if $\forall x, y \in V$ and $a, b \in R$ we have $G(ax + by) = aG(x) + bG(y)$.

Definition 16.3. A linear functional G on the normed linear space V is *bounded* if $\exists\, K \geq 0$ such that

$$|G(x)| \quad \leq \quad K||x|| \quad \forall x \in V \tag{16.1}$$

Then the norm of G, denoted by $||G||$, is the infimum of the numbers K for which (16.1) holds.

So, easily $|G(x)| \leq ||G|| \cdot ||x||$. Then dividing by $||G||$, we see that

$$||G|| \quad = \quad \sup[|G(x)| : ||x|| \leq 1]]$$

When dim $V = 0$, we have $||G|| = \sup[|G(x)| : ||x|| = 1]]$

Definition 16.4. If $[[X, \mathscr{S}, \mu]]$ is a measure space and $p > 0$, we define $L^p(X, \mu)$ or more briefly $L^P(\mu)$ to the class of measurable functions $\left[f : \int |f|^p d\mu < \infty\right]$.

Definition 16.5. Let $f \in L^p(\mu)$, then the L^p -norm of f, denoted by $||f||_p$, is given by $\left(\int |f|^p d\mu\right)^{1/p}$.

Now, let us see the important inequality namely Holder's inequality with out proof.

Theorem 16.1 (Holder's inequality)**.** *Let* $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $f \in L^P(\mu)$, $g \in L^q(\mu)$. *Then* $fg \in L^1(\mu)$ *and*

$$\int |fg| d\mu \leq \left(\int |f|^p d\mu\right)^{1/p} \cdot \left(\int |g|^q d\mu\right)^{1/q}$$

Theorem 16.2. *If* $f \in L^1(\mu)$ *and* $g \in L^\infty(\mu)$, *then* $fg \in L^1(\mu)$ *and* $||fg||_1 \leq ||f||_1 ||g||_\infty$.

Example 16.1. The following are equivalent for a linear functional G: (i) G is bounded, (ii) G is continuous at 0, (iii) G is continuous at each $x \in V$.

Solution:

$(i) \Rightarrow (ii)$: Suppose that G is bounded. Then there exists $K \geq 0$ such that $|G(x)| \leq K\|x\|, \quad \forall x \in V.$ Now,we shall prove that G is continuous at 0.

Let $x_n \to 0,$ then

$$|G(x_n)| \leq \|G\| \cdot \|x_n\| \to 0$$

Thus, G is continuous at $0.$

$(ii) \Rightarrow (iii)$: Assume that G is continuous at 0 and now, we have to prove that G is continuous at each point $x \in V.$

Let $x_n \to x,$ then

$$\begin{aligned} \|G(x_n - G(x)\| &= \|G(x_n - x)| \\ &\leq \|G\| \cdot \|x_n - x\| \to 0 \end{aligned}$$

Thus, G is continuous at every point $x \in V.$

$(iii) \Rightarrow (ii)$: Assume that G is continuous at every point $x \in V.$ Now, we shall prove that G is continuous at $0.$

Since, G is continuous at every point $x \in V,$ in particular G is continuous at the origin.

$(ii) \Rightarrow (i)$: Assume that G is continuous at $0.$ Our wish is to prove that G is bounded.

Assume the contrary that G is not bounded, then there exists $\{x_n\}$ such that $\|x_n\| \leq 1,$ but $|G(x_n\| \geq n.$

If $y_n = n^{-1}x_n,$ so that $\|y_n\| \to 0,$ we have $|G(y_n)| \geq 1,$ thus G is not continous at $0,$ which contradicts our assumption.

Hence G is bounded.

Example 16.2. Define G on $L^p(\mu)$ by $G(f) = \int fgd\mu$ for a fixed $g \in L^q(\mu),$ p and q being conjugate indices with $p \geq 1$ and with $q = \infty$ in the case when $p = 1.$ Then G is bounded linear functional and $\|G\| \leq \|g\|_q.$

It will from the main theorem of this section that $\|G\| = \|g\|_q$ for this kind of functional. It is convenient to deal separately with the case $1 < p < \infty$

and $p = 1$. The next theorem shows that L^q is in a sense, the set of bounded linear functionals or dual space of L^p.

Theorem 16.3 (Riesz Representation Theorem for L^p, $p > 1$). *Let G be a bounded linear functional on $L^p(X,\mu)$. Then there exists a unique element g of $L^q(X,\mu)$ such that*

$$G(f) = \int fgd\mu \quad \text{for each } f \in L^p \tag{16.2}$$

where p, q are conjugate indices. Also

$$||G|| = ||g||_q \tag{16.3}$$

Proof. First, we shall prove the uniqueness. For this, we assume that g and g' are two functions satisfies the hypothesis of the theorem.

Let E be any set of finite measures, so that $\chi_E \in L^p$. Then

$$\begin{aligned}\int_E (g - g')d\mu &= \int \chi_E(g - g'd\mu \\ &= 0 \\ \Rightarrow g &= g' \; a.e. \quad (\because [x : g(x) \neq g'(x)] \text{ has } \sigma\text{-finite measure}).\end{aligned}$$

Thus, the uniqueness is proved.

If $||G||= 0$, then $G(f) = 0$ for all f, so $g \equiv 0$ satisfies (16.2) and (16.3). So, assume that $||G||> 0$.

Suppose $\mu(X) < \infty$. For each $E \in \mathscr{S}$, define $\lambda(E) = G(\chi_E)$;

Claim: λ is a signed measure.

Clearly $\lambda(\emptyset) = 0$.

$$\begin{aligned}\lambda(A \cup B) &= G(\chi_{A\cup B}) \\ &= G(\chi_A) + G(\chi_B) \\ &= \lambda(A) + \lambda(B)\end{aligned}$$

for disjoint sets A and B. Thus, λ is finitely additive.

Let $E = \bigcup_{i=1}^{\infty} E_i$ and let $A_n = \bigcup_{i=1}^{n} E_i$. We have

$$||A_n - E||_p = (\mu(E - A_n))^{1/p} \to 0 \text{ as } n \to \quad (\because G \text{ is continuous by example 8.9}).$$

Thus, we have $\lambda(A_n) \to \lambda(E)$, so λ is countably additive. Further, λ takes only finite-value, since G takes only finite values. Hence λ is a signed measure.

Also, if $\mu(E) = 0$, then $||\chi_E||_p= 0$, which implies $\lambda(E) = 0$ that is $\lambda \ll \mu$. So, by Raydon Nikodym theorem, there exists $g \in L^1(\mu)$ such that for each $E \in \mathscr{S}$

$$G(\chi_E) = \int_E gd\mu = \int \chi_E gd\mu$$

Claim: g has the required properties.

By the linearity of integration, we have

$$G(\phi) = \int \phi g d\mu$$

for any measurable simple function ϕ.

But each function $f \in L^{\infty}(\mu)$ is the uniform limit $a.e.$ of a sequence $\{\psi_n\}$ where each ψ_n is the difference of measurable simple functions, and so $\|f - \psi_n\}|_p \to 0$.

Hence, by the continuity of G, we have

$$G(f) = \int f g d\mu \quad \text{for each } f \in L^{\infty}(\mu) \tag{16.4}$$

It remains to show that $\|G\| = \|g\|_g$.

Let the function α on X be defined by

$$\begin{aligned} \alpha &= 1 \quad \text{where } g > 0 \\ \alpha &= -1 \quad \text{where } g < 0 \end{aligned}$$

Clearly α is measurable and $\alpha g = \pm g = |g|$.

Let

$$E_n = [x : |g(x)| \leq n]]$$

and put $f = \alpha \chi_{E_n} |g|^{q-1}$, where p, q are conjugate indices. Then $|f|^p = |g|^q$ on E_n, $f \in L^{\infty}(\mu)$.

Hence, by equation (16.4), we have

$$\begin{aligned} \int_{E_n} |g|^q d\mu &= \int f g d\mu \\ &= G(f) \\ &\leq \|G\| \cdot \|f\|_p \\ &= \|G\| \left(\int_{E_n} |g|^q d\mu \right)^{1/p} \end{aligned}$$

$$i.e., \quad \int_{E_n} |g|^q d\mu = \|G\| \left(\int_{E_n} |g|^q d\mu \right)^{1/p} \tag{16.5}$$

If, $\left(\int_{E_n} |g|^q d\mu\right)^{1/p} = 0$ then it is obvisou. So, we assume that $\left(\int_{E_n} |g|^q d\mu\right)^{1/p} \neq 0$.

Divide (16.5) both sides by $\left(\int_{E_n} |g|^q d\mu\right)^{1/p}$, we get

$$\begin{aligned}
\left(\int_{E_n}|g|^q d\mu\right)^{1-\frac{1}{p}} d\mu &\leq ||G|| \\
\Rightarrow \left(\int_{E_n}|g|^q d\mu\right)^{1/q} d\mu &\leq ||G|| \\
i.e., \int_{E_n}|g|^q d\mu &\leq ||G||^q \\
\Rightarrow \int \chi_{E_n}|g|^q d\mu &\leq ||G||^q
\end{aligned}$$

Since, $\chi_{E_n} \uparrow 1$ and by Lebesgue's Monotone convergence theorem, we have

$$||g||_q \leq ||G|| \tag{16.6}$$

and in particular $g \in L^q(\mu)$.

By example 8.10, we have

$$||G|| \leq ||g||_q \tag{16.7}$$

Thus, from (16.6) and (16.7), we have

$$||G|| = ||g||_q$$

So, (16.2) holds for $f \in L^\infty(X,\mu)$.

But the bounded functions are dense in L^p. For it is sufficient to show that every non-negative function $f \in L^p$ is the limit in the mean of order p, of a sequence $\{f_n\}$ of bounded functions.

Put $f_n = \min(f, n)$. Then $0 \leq (f - f_n)^p \leq f^p$ and $f - f_n \to 0$ $a.e.$ So by Lebesgue Dominated convergence theorem, we have $||f - f_n||_p \to 0$.

By Continuity of G, we have $G(f_n) \to G(f)$.

Also, by holder's inequality, we have $\int f_n g d\mu \to \int f g d\mu$.

Hence, $G(f) = \int f g d\mu$. Thus, the theorem is proved for finite measure space.

Now, we extend the result to the case when $X = \bigcup_{i=1}^{\infty} X_i$, where the X_i are disjoint measurable set of finite μ-measure.

Any function f_i on X_i, measurable with respect to the σ-algebra of sets $E \cap X_i$, $E \in \mathscr{S}$, can be extended to f on X by putting $f = 0$ on X_i^c. Then G has the restriction G_i on $L(X_i, \mu)$, where $G_i(f_i) = G(f)$, and we have $||G_1|| \leq ||G||$.

By the first part, we have

$$G_i(f_i) = G(\chi_{X_i} f) = \int_{X_i} f g_i d\mu$$

for each $f \in L^p(X,\mu)$, for each i, and for a suitable $g_i \in L^q(X,\mu)$.

Extend g_i to X by putting $g_i = 0$ on X_i^c and write $g = \sum g_i$.

If $Y_n = \bigcup_{i=1}^{n} X_i$, then

$$G(\chi_{Y_n} f) = \int_{Y_n} f(g_1 + g_2 + \ldots + g_n)d\mu, \quad \forall f \in L^p(X, \mu)$$

As in the first part, since $\mu(Y_n) < \infty$ and $\|g_1 + g_2 + \ldots + g_n\| \leq G$ for each n. So

$$\begin{aligned}
\left(\|g\|_q\right)^q &= \int \left|\sum g_i\right|^q d\mu \\
&= \int \lim \left|\sum_{i=1}^{n} g_i\right|^q d\mu \\
&\leq \liminf \int \left|\sum_{i=1}^{n} g_i\right| d\mu \\
&\leq \|G\|^q \quad \text{(By Fatou's lemma)} \\
i.e., \left(\|g\|_q\right)^q &\leq \|G\|^q \\
i.e., \|g\| &\leq \|G\|
\end{aligned}$$

Also, by example 8.10, we have
$$\|G\| \leq \|g\|$$

Thus, we have $\|G\| = \|g\|_q$.

Also, $\chi_{Y_n} f \to f$ in the mean of order p so $G(\chi_{Y_n} f) \to G(f)$.

But $\sum_{i=1}^{n} g_i \to g$ in the mean of order q, so by Holder's inequality we have

$$\int \chi_{Y_n} f \sum_{i=1}^{n} g_i d\mu \to \int f g d\mu$$

Now, consider the general case where μ need not be σ-finite. We show that there exists a $X_0 \in \mathscr{S}$ which is of σ-finite measure that is X_0 is the union of a sequence of sets of finite measure, and such that if $f = 0$ on X_0 then $G(f) = 0$.

Let $\{f_n\}$ be such that $\|f_n\|_p = 1$ and $G(f_n) \geq \|G\|(1 - 1/n)$ and $X_0 = \bigcup_{n=1}^{\infty}[x : f(x) \neq 0]$ has σ-finite measure.

Let $E \in \mathscr{S}$ with $E \subseteq X_0^c$, then
$$\|f_n + t\chi_E\|_p = (1 + t^p \mu(E))^{1/p} \quad \text{for } t \geq 0$$

Also,
$$G(f_n) - G(\pm t\chi_E) \leq |G(f \mp t\chi_E)| \leq \|G\|(1 + t^p \mu(E))^{1/p}$$

and
$$\|G(t\chi_E)| \leq \|G\||(1 + t^p \mu(E))^{1/p} - 1 + n^{-1}|$$

for every n.

Let $n \to \infty$ and then divide by $t(>0)$ to get

$$||G(t\chi_E)|\leq ||G||\frac{(1+t^p\mu(E))^{1/p}-1}{t}$$

Since $p > 1$, we may apply l'Hospital rule as $t \to 0$ to get $G(\chi_E) = 0$. So G vanishes for simple functions and hence for measurable functions which equal zero on X_0.

Hence, by the proof for the σ-finite case we can find $g \in L^q(X_0)$ such that

$$G(\chi_{X_0}f) \quad = \quad \int_{X_0} fgd\mu$$

Define g to be zero on X_0^c, to get the required function g of the theorem.

i.e., $G(f) = \int fgd\mu$. Hence the proof. ■

Theorem 16.4 (Riesz Representation Theorem for $L^1(\mu)$)**.** *Let* $[[X, \mathscr{S}, \mu]]$ *be a* σ*-finite measure space and let* G *be a bounded linear functional on* $L^1(X,\mu)$*. Then there exists a unique* $G \in L^\infty(X,\mu)$ *such that*

$$G(f) \quad = \quad \int fgd\mu \quad \text{for each} \quad f \in L^1(\mu) \tag{16.8}$$

Also, $||G||= ||g||_\infty$.

Proof. Assume that $[[X, \mathscr{S}, \mu]]$ is a finite measure space. As in the previous theorem, we can construct a unique function g such that (16.8) holds for $f \in L^\infty(X,\mu)$. Now, our aim is to prove that $g \in L^\infty$.

We have

$$\left|\int_E gd\mu\right| \quad \leq \quad ||G||||\chi_E||_1 = ||G||\mu(E), \quad \forall E \in \mathscr{S} \tag{16.9}$$

Suppose that $|g(x)|> ||G||$ on a set A of positive measure and write

$$E_n = [x : |g(x)|> (1 + 1/n)||G||]$$

So $A = \cup E_n$.

Hence for some n, we have $\mu(E_n) > 0$ and $|g(x)|> (1 + 1/n)||G||$ on E_n. Then

$$\int_{E_n} gd\mu \quad \geq \quad ||G||(1 + 1/n)\mu(E_n)$$

contradicting (16.9) as we may suppose $||G||\geq 0$. So $||g||_\infty\leq ||G||$ and hence $||g||_\infty= ||G||$.

Now, we extend (16.8), as in the previous theorem to all functions $f \in L^1(\mu)$. Extend as before, to the σ-finite case; we now have $||g_1 + g_2 + \ldots + g_n||\leq ||G||$ for each n, So $||g||_\infty\leq ||G||$. For the last part of the σ-finite case in the previous theorem, Holder's inequality is replaced by theorem 8.13. ■

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of bounded linear functionals.
- derive Riesz representation thoerem.

Check Your Progress:

1. Derive Riesz representation theorem for L^p $(p > 1)$.
2. Derive Riesz representation theorem for L^1.

Suggested Readings:

1. G. de Barra, "Measure Theory and Integration", New Age International Pvt. Ltd, Second Edition, 2013.
2. Rana I. K., "An Introduction to Measure and Integration", Narosa Publishing House Pvt. Ltd., Second Edition, 2007.
3. Royden H. L., "Real Analysis", Prentice Hall of India Pvt. Ltd., Third Edition, 1995.

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