MASTER OF SCIENCES

(M.Sc.)

MMT-104

DIFFERENTIAL GEOMETRY

Semester-I

Author-Dr. Harsh Verdhan Harsh

SURESH GYAN VIHAR UNIVERSITY

Centre for Distance and Online Education,

Mahal Road, Jagatpura, Jaipur-302017

Published by:

S. B. Prakashan Pvt. Ltd.

WZ-6, Lajwanti Garden, New Delhi: 110046

Tel.: (011) 28520627 | Ph.: 9205476295

Email: info@sbprakashan.com | Web.: www.sbprakashan.com

Designed & Graphic by : S. B. Prakashan Pvt. Ltd.

Printed at :

Suresh Gyanvihar University
Department of Mathematics
School of Science,

M.Sc., Mathematics - Syllabus – I year – I Semester (Distance Mode)

COURSE TITLE	**:**	**DIFFERENTIAL GEOMETRY**
COURSE CODE	**:**	**MMT-104**
COURSE CREDIT	**:**	**3**

COURSE OBJECTIVES

While studying the **DIFFERENTIAL GEOMETRY**, the Learner shall be able to:

CO 1: Find the spherical indicatrix of the tangent, principal normal and binormal.

CO 2: Represent the parametric curves in the theory of surfaces.

CO 3: Predict Special intrinsic curves which are related to straight line in Euclidean space.

CO 4: Review the concept of geometric interpretation of the second fundamental form.

CO 5: Describe the concept of compact surfaces

COURSE LEARNING OUTCOMES

After completion of the **DIFFERENTIAL GEOMETRY,** the Learner will be able to:

CLO 1: Empower the knowledge to calculate the curvature and torsion of any space curve interms of parameters.

CLO 2: Describe the relationship between the fundamental coefficients.

CLO 3: Enable to derive on a general surface, the necessary and sufficient condition for the parametric curve to be geodesic.

CLO 4: Evaluate the first and the second fundamental forms of surface.

CLO 5: Demonstrate an understanding tocalculate the Gaussian curvature, the mean curvature, the curvature lines, the asymptotic lines, the geodesics of a surface

BLOCKI : SPACE CURVES

Definition of a space curve – Arc length – Tangent – Normal and binormal – Curvature and torsion – Contact between curves and surfaces – Tangent surface – Involutes and evolutes – Intrinsic equations – Fundamental existence theorem for space curves – Helics.

BLOCKII: INTRINSIC PROPERTIES OF A SURFACE

Definition of a surface – Curves on a surface – Surface of revolution – Helicoids – Metric – Direction coefficients – Families of curves – Isometric correspondence – Intrinsic properties.

BLOCKIII: GEODESICS

Geodesics – Canonical geodesic equations – Normal property of geodesics – Existence theorems – Geodesic parallels – Geodesics curvature- Gauss-Bonnet Theorem – Gaussian curvature – Surface of constant curvature.

BLOCKIV: NON INTRINSIC PROPERTIES OF A SURFACE

The second fundamental form – Principal curvature – Lines of curvature – Developable - Developable associated with space curves and with curves on surface – Minimal surfaces – Ruled surfaces.

BLOCKV: DIFFERENTIAL GEOMETRY OF SURFACES

Compact surfaces whose points are umblics – Hilbert's lemma – Compact surface of constant curvature –Complete surface and their Characterization – Hilbert's Theorem – Conjugate points on geodesics.

REFERENCE BookS :

1. T.J. Willmore, *"An Introduction to Differential Geometry"*, Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print)

UNIT	Chapter(s)	Sections
I	I	1 – 9
II	II	1 – 9
III	II	10 – 18
IV	III	1 – 8
V	IV	1 – 8

2. D.T. Struik, *"Lectures on Classical Differential Geometry"*, Addition –Wesley, Mass, 1950.

3. S. Kobayashi and K. Nomizu, *"Foundations of Differential Geometry"*, Interscience Publishers, 1963.

4. W. Klingenberg, *"A Course in Differential Geometry"*, Graduate Texts in Mathematics,Springer – Verlag 1979.

5. C.E.Weatherburn, *"Differential Geometry of Three Dimensions"*,University Press, Cambridge, 1930.

6. Polynomial, Newton Interpolation Polynomial,Divided differencetable, Interpolation with equidistance points, Spline interpolation

Contents

1 SPACE CURVES 1

1.1 Introduction: 2

1.2 De nitions: 4

1.3 Arc length: 6

1.4 Tangent, normal and binormal: 10

1.5 Curvature and Torsion:.. 19

2 INVOLUTES AND EVOLUTES 39

2.1 Curvature and torsion of a curve given as the intersection of two surfaces: 40

2.2 Contact between curves and surfaces:...................................... 45

2.2.1 Osculating circle:.. 45

2.2.2 Osculating sphere: 47

2.3 Tangent surfaces, Involutes and Evolutes: 52

3 SPHERICAL INDICATRIX 61

3.1 The Spherical Indicatrices or Spherical Images:...................... 62

3.1.1 The Spherical Indicatrix (or spherical image) of the tangent: 63

3.1.2 The Spherical Indicatrix (or spherical image) of the principal normal:.. 64

3.1.3 The Spherical Indicatrix (or spherical image) of the binormal:... 66

3.1.4 Bertrand Curves:... 67

3.2 Intrinisic equations, fundamental existence theorem for space curves: 69

3.2.1 Fundamental theorem for space curves: 69

3.2.2 Intrinsic Equations: 73

3.3 Helices: 74

4 THEORY OF SURFACES 79

4.1 De nition of a surface 80

4.1.1 Regular (or Ordinary) point and Singularities on a surface: 82

4.2 Curves on a surface: 84

4.2.1 Parametric Curves: 84

4.2.2 Tangent plane and Normal: 85

4.3 Surface of Revolution: 87

4.3.1 The Sphere: 87

4.3.2 The general surface of revolution: 88

4.3.3 The anchor ring: 89

5 METRIC 91

5.1 Helicoids: 92

5.1.1 Right helicoid: 92

5.1.2 The general helicoid: 93

5.2 Metric: 94

5.2.1 Geometrical Interpretation of metric: 94

6 FAMILIES OF CURVES 101

6.1 Direction coe cients: 102

6.1.1 Angle between the directions: 103

6.2 Relation between direction coe cients and direction ratios: 103

6.3 Families of Curves: 104

6.3.1 Orthogonal Trajectories: 106

6.3.2 Double family of curves: 107

6.4 Isometric correspondence: 113

6.5 Intrinsic properties: 119

7 GEODESICS-I 123

7.1 Geodesics: 124

7.2 Canonical geodesic equations: 130

7.3 Normal property of Geodesics: 134

8 GEODESICS-II 139

8.1 Existing Theorems: 140

8.2 Geodesic Parallels: 141

8.3 Geodesic Curvature: 142

9 GEODESICS-III 151

9.1 Gauss-Bonnet Theorem: 152

9.2 Gaussian Curvature: 155

9.3 Surfaces of constant curvature: 159

10 THE SECOND FUNDAMENTAL FORM 163

10.1 The second fundamental form: 164

10.2 Principal curvatures: 167

10.3 Lines of curvature: 170

10.3.1 Dupin indicatrix: 172

11 DEVELOPABLE SURFACES-I 177

11.1 Developables: 178

12 DEVELOPABLE SURFACES-II 183

12.1 Developable associated with space curves: 184

12.1.1 Osculating developable: 184

12.1.2 Polar developable: 185

12.2 Rectifying developable: 186

12.3 Developables associated with curves on surfaces: 189

12.4 Minimal surfaces: 191

12.5 Ruled surfaces: 193

13 COMPACT SURFACES 203

13.1 Introduction: 204

13.2 Compact surfaces whose points are umblics:.......... 204

13.3 Hilbert's lemma: 206

14 COMPLETE SURFACES 209

14.1 Compact surfaces of constant Gaussian or mean curvature: 210

14.2 Complete Surfaces: 211

14.3 Characterization of complete surfaces: 212

15 HILBERT'S THEOREM 219

15.1 Hilbert's theorem: 220

15.2 Conjugate points on geodesics: 228

Block-I

Unit-1: Space Curves.

Unit-2: Involutes and Evolutes.

Unit-3: Spherical Indicatrix.

Block-I

UNIT-1

SPACE CURVES

Structure

Objective

Overview

1. 1 Introduction

1. 2 De nitions

1. 3 Arc length

1. 4 Tangent, normal and binormal

1. 5 Curvature and Torsion

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the concept of class m; regular function and equivalent paths.
- de ne the concept of tangent, normal and binormal at any point on a space curve.
- derive Serret-Frenet formulae.
- calculate the curvature and torsion of any space curve in terms of the parameter.

Overview

In this unit, we will explain the concept of tangent, normal and binormal. The necessary and su cient condition for the curve to be plane is established.

1.1. Introduction:

Di erential Geometry is that part of geometry which is treated with the help of Di erential Calculus.

In the theory of plane curves a curve is represented by means of a single equation or by a parametric representation. For example, the circle with centre at the origin and radius a is given by the equation $x^2 + y^2 = a^2$: The parametric representation of the circle is given by $x = a \cos$ and $y = a \sin$; where 0 2 : Similarly, the space curves are represented by three dimensional Euclidean space E_3: Already we are familiarize that two straight lines intersect at a point, two planes are intersect along a straight line and two surfaces intersect along a space curve.

Intersection of two surfaces:

Let $f_1(x; y; z) = 0;$ $f_2(x; y; z) = 0$ represent two surfaces then these two equations together represent the curve of intersection of these surfaces. This curve will be called a plane curve. If it lies on a plane, otherwise it is said to be skew, twisted or tortuous.

For example, we know that if $f_1(x; y; z) = 0$ represents a sphere and $f_2(x; y; z) = 0$ represent a plane then these two equations together represents a circle which is the section of the given sphere by the given plane. In this case, the curve is called a plane curve.

Parametric representation of a space curve:

If the coordinates of a point on a space curve be represented by the equations of the form

$$x = \ _1(t); \quad y = \ _2(t); \quad z = \ _3(t); \tag{1.1}$$

where $_1;$ $_2;$ $_3$ are real valued functions of a single variable t ranging over a set of values $a \quad t \quad b:$

The equations in (1.1) are called the parametric equations of the space curve. Thus we can say that a curve in space is the locus of a point where Cartesian coordinates are functions of a single variable $t:$

Transformation of one representation to another representation:

Let the parametric equations of a space curve be

$$x = t; \quad y = t^2; \quad z = t^3 \tag{1.2}$$

Eliminating the parameter t in the above three equations, we get

$$x^2 = y; \quad y^3 = z^2 \tag{1.3}$$

which is of the form

$$f_1(x; y; z) = 0; \quad f_2(x; y; z) = 0 \tag{1.4}$$

Thus the space curve whose parametric equations are given can be expressed as the intersection of two surfaces given by $x^2 = y;$ $y^3 = z^2:$

Similarly, if the equation of the curve is given by equation (1.4) then eliminating x we get $y = g_1(z)$ and on eliminating y, we get $x = g_2(z)$. Thus, x and y are represented as a functions of z. Now, if the coordinate z is a function of some parameter t say i.e., $z = F_3(t)$ then x and y will be functions of t so that

$$x = F_1(t)\,; \quad y = F_2(t); \quad z = F_3(t) \tag{1.5}$$

are the parametric representations of the space curve whose equations are given by (1.4) as the curve of intersection of two surfaces.

Vector representation of a space curve:

If $\tilde{r}$ be the position vector of a point P on the space curve whose Cartesian coordinates be (x, y, z) then we have

$$\tilde{r} = x\tilde{i} + y\tilde{j} + z\tilde{k} \tag{1.6}$$

$$\text{or} \quad \tilde{r} = f_1(t)\tilde{i} + f_2(t)\tilde{j} + f_3(t)\tilde{k} \tag{1.7}$$

$$\text{or} \quad \tilde{r} = f(t) \quad \text{or} \quad \tilde{r} = (f_1(t), f_2(t), f_3(t)) \tag{1.8}$$

where f is a vector valued function of a single variable t. Thus, we may define vector representation of a space curve as follows:

A space curve is the locus of a point where position vector $\tilde{r}$ with respect to a fixed origin may be expressed as a function of a single parameter.

1.2. Definitions:

Definition 1.1 (Functions of class m).

Let I be a real interval and m a positive integer. A real-valued function f defined on I is said to be of class m or to be a C^m - function, if f has a m^{th} derivative at every point of I and this derivative is continuous on I. Simply, we can say that C^m - function has a continuous m^{th} derivative.

The function f is said to be of class ∞ or C^∞ function when it is differentiable infinite number of times.

Definition 1.2 (Analytic function).

The function f defined over an interval I is said to be analytic, if f is single valued and possesses continuous derivatives of all orders at every point of the interval. This type of functions is said to be of class ω or C^ω function.

Note 1.1. The extension of the concept of class of real valued functions of several variables is quite obvious.

i.e., We can say that a C^m - function of several variables admits all continuous partial derivatives of m^{th} order.

Definition 1.3 (Class of a vector valued function).

A vector-valued function $\tilde{R} = (X, Y, Z)$ defined on I is said to be of class m if it has an m^{th} derivative at every point and if this derivative is continuous on I. This in turn means that each of its components X, Y, Z are of class m. Such a function is given by the vector equation $\tilde{R} = (X, Y, Z)$ or by Cartesian equations $x = X(u)$, $y = Y(u)$, $z = Z(u)$.

Definition 1.4 (Regular).

A vector valued function is said to be regular if $\frac{d\tilde{R}}{du} \neq 0$ on I. i.e., if x, y, z never vanishes simultaneously.

Definition 1.5 (path).

A regular vector valued function of class m is called a path of class m.

Definition 1.6 (Equivalent paths).

Let $\tilde{R}_1$ and $\tilde{R}_2$ be the two paths of same class m defined on intervals I_1 and I_2 respectively. These two paths are said to be equivalent if there exists a strictly increasing function g of class m which maps I_1 onto I_2 and is such that $\tilde{R}_1 = \tilde{R}_2 \circ g$. i.e., This is equivalent to three conditions.

$$X_1 = X_2(g(u))\,;\; Y_1 = Y_2(g(u))\,;\; Z_1 = Z_2(g(u))$$

Any equivalent class of path m determines a unique curve of class m. Any path $\tilde{R}$ determines a unique curve and is called a parametric representation of the curve. $x = X(u)$, $y = Y(u)$, $z = Z(u)$, here u is the parameter.

The mapping g which relates two equivalent paths is called a change of parameter.

Examples of space curves with different parameters:

$$\text{(i)}\quad \tilde{r} = (a\cos u, a\sin u, bu) \qquad 0 \leq u < 2\pi$$

$$\text{(ii)}\quad \tilde{r} = \left(a\frac{1-v^2}{1+v^2}, a\frac{2v}{1+v^2}, 2b\tan^{-1} v\right) \qquad -\infty < v < \infty$$

Both equations represent the same curve (circular helix) in different parameters u and v: In this case, the change of parameter is $v = g(u) = \tan\frac{u}{2}$:

Definition 1.7 (Curve of class m).

A curve of class m in E_3 is a set of points in E_3 associated with an equivalence class of regular parametric representation of class m involving one parameter.

1.3. Arc length:

The distance between two points $\tilde{r}_1 = (x_1; y_1; z_1)$; $\tilde{r}_2 = (x_2; y_2; z_2)$ in E_3 is the number

$$|\tilde{r}_1 - \tilde{r}_2| = \sqrt{(\tilde{r}_1 - \tilde{r}_2)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This distance in space will be used to different distance along a curve of class $m \geq 1$:

Bookwork 1.1. To find an expression for arc length of a curve between two points.

Figure 1.1: Curve length

Let us consider a curve C of class $m \geq 1$ and $\tilde{r} = \tilde{R}(u)$ be the equation of the curve C: Now, our aim is to determine the arc length between the two points A and B on the given curve corresponding to the values a and b of the parameter u:

Now corresponding to any subdivision Δ of the interval $[a; b]$ by points

$$a = u_0 < u_1 < u_2 < \cdots < u_n = b$$

we have the length

$$L_\Delta = \sum_{i=1}^{n} \left| \tilde{R}(u_i) - \tilde{R}(u_{i-1}) \right| \qquad (1.9)$$

of the polygon inscribed to the arc by joining the successive points on it.

Again, we know that sum of the sides of a triangle is greater than the third side so that if we increase the number of points of the subdivision the length of the polygon would be increased. Hence, the length of the arc is defined to be an upper bound of L_Δ; taken over all possible subdivisions of $[a; b]$.

Hence, from (1.9) we have

$$L_\Delta = \sum_{i=1}^{n} \left| \int_{u(i-1)}^{u(i)} \dot{R}(u)\,du \right| \leq \sum_{i=1}^{n} \int_{u(i-1)}^{u(i)} \left|\dot{\tilde{R}}(u)\right| du = \int_a^b \left|\dot{R}(u)\right| du \qquad (1.10)$$

Now (1.10) shows that the right side member of (1.10) is finite and independent of Δ and hence upper bound of L_Δ is always finite.

Now we shall show that the upper bound of L_Δ is actually equal to the right hand side of (1.10).

Let $s = s(u)$ denote the arc length from a to any point u; then the arc length from $u_0 = a$ to u i.e.; $s(u) - s(u_0)$ where $a = u_0 < u < b$.

Therefore from equation (1.10), we have

$$s(u) - s(u_0) = \int_{u_0}^{u} \left|\dot{R}(u)\right| du \qquad (1.11)$$

Also, from the definition of length we have

$$\left|\tilde{R}(u) - \tilde{R}(u_0)\right| \leq s(u) - s(u_0) \qquad (1.12)$$

From equations (1.11) and (1.12), we have

$$\frac{\left|\tilde{R}(u) - \tilde{R}(u_0)\right|}{u - u_0} \leq \frac{s(u) - s(u_0)}{u - u_0} = \frac{1}{u - u_0} \int_{u_0}^{u} \left|\dot{\tilde{R}}(u)\right| du \qquad (1.13)$$

Taking limit as $u \to u_0$; we get

$$\left|\dot{R}(u)\right| \leq \dot{s}(u) \leq \left|\dot{\tilde{R}}(u)\right|$$

$$\Rightarrow \dot{s}(u) = \left|\dot{R}(u)\right|$$

Since this is true for any value of u_0 in the range of u; hence we have

$$s = s(u) = \int_a^u \dot{R}(u)\ du \qquad (1.14)$$

The formula (1.14) is used as formula to determine the arc length from a point a to any point u on the curve.

In terms of Cartesian parametric representation, we have

$$s = \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du$$

Also, the equation (1.14) can be rewritten as

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

In terms of differentials, we have

$$ds^2 = dx^2 + dy^2 + dz^2$$

where ds is called the linear element of the curve C.

Note 1.2. We shall use notation dashes to denote differentiation with respect to arc length s and dots to denote differentiation with respect to any other parameter u. Thus, we have

$$\frac{d\tilde{R}}{ds} = \tilde{R}' ; \quad \frac{d^2\tilde{R}}{ds^2} = \tilde{R}''$$

$$\frac{d\tilde{R}}{du} = \dot{\tilde{R}} ; \quad \frac{d^2\tilde{R}}{du^2} = \ddot{\tilde{R}}$$

Example 1.1. Find the equation of the circular helix $\tilde{r}(u) = a\cos u\,\tilde{i} + a\sin u\,\tilde{j} + bu\tilde{k}$, $-\infty < u < \infty$ from where $a > 0$ referred to s as parameter, and also find the length of one complete turn of the helix.

Solution:

$$\text{Given } \tilde{r}(u) = a\cos u\,\tilde{i} + a\sin u\,\tilde{j} + bu\tilde{k}$$

$$\Rightarrow \quad x = a\cos u ; \quad y = a\sin u ; \quad z = bu$$

$$\dot{x} = -a\sin u ; \quad \dot{y} = a\cos u ; \quad \dot{z} = b$$

$$\text{Arc length } s = \int_0^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,du$$

$$= \int_0^u \sqrt{a^2\left(\sin^2 u + \cos^2 u\right) + b^2}\,du$$

$$= \int_0^u \sqrt{a^2 + b^2}\,du = \sqrt{a^2 + b^2}\,u$$

$$i.e., \quad u = \frac{s}{\sqrt{a^2 + b^2}}$$

Thus, the required equation of circular helix is

$$\tilde{r} = \left(a\cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right);\ a\sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right);\ \frac{bs}{\sqrt{a^2 + b^2}} \right)$$

The range of parameter u to one complete turn of the helix is

$u_0 \le u \le u_0 + 2\pi$

$$\Rightarrow \text{Required length} = \int_{u_0}^{u_0+2\pi} \sqrt{a^2 + b^2}\,du = 2\pi\sqrt{a^2 + b^2}$$

Example 1.2. Find the length of the curve given as the intersection of the surfaces $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;\quad x = a\cosh\left(\frac{z}{a}\right)$ from the point $(a; 0; 0)$ to the point $(x; y; z)$.

Solution: Given equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

The parametric equations of this curve are given by

$x = a\cosh\theta\ ;\quad y = b\sinh\theta\ .$

$$\text{Also } x = a\cosh\left(\frac{z}{a}\right)$$

$$\Rightarrow \cosh\left(\frac{z}{a}\right) = \frac{x}{a}$$

$$\Rightarrow \frac{z}{a} = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$i.e., \quad z = a\cosh^{-1}\left(\frac{x}{a}\right)$$

$$= a\cosh^{-1}\left(\frac{a\cosh\theta}{a}\right)$$

$$= a\cosh^{-1}(\cosh\theta) = a\theta$$

Thus, parametric forms of given curve are

$x = a\cosh\theta\ ;\quad y = b\sinh\theta\ ;\quad z = a\theta\ .$

$$\text{limit } z = 0 \text{ to } z = z$$
$$\Rightarrow a = 0 \text{ to } a = a$$
$$\Rightarrow \theta = 0; \theta =$$

$$\text{Arc length } s = \int_0^{\theta} \sqrt{x'^2 + y'^2 + z'^2}\, d\theta$$
$$= \int_0^{\theta} \sqrt{(a\sinh\theta)^2 + (b\cosh\theta)^2 + a^2}\, d\theta$$
$$= \int_0^{\theta} \sqrt{a^2\left(1 + \sinh^2\theta\right) + b^2\cosh^2\theta}\, d\theta$$
$$= \int_0^{\theta} \sqrt{a^2 + b^2}\cosh\theta\, d\theta$$
$$= \sqrt{a^2 + b^2}\sinh\theta$$
$$\text{i.e., } s = \frac{y}{b}\sqrt{a^2 + b^2}$$

Example 1.3. Prove that the length of the curve $x = 2a\left(\sin^{-1} t + t\sqrt{1 - t^2}\right)$; $y = 2at^2$; $z = 4at$ between the points where $t = t_1$ and $t = t_2$ is $4\sqrt{2}a(t_2 - t_1)$

Solution:

$$\text{Given } x = 2a\left[\sin^{-1} t + t\sqrt{1 - t^2}\right]; \quad y = 2at^2$$
$$x' = 2a\left[\frac{1}{\sqrt{1 - t^2}} + t\frac{1}{2\sqrt{1 - t^2}}(-2t) + \sqrt{1 - t^2}\right] = 4a\sqrt{1 - t^2}$$
$$y = 2at^2 \Rightarrow y' = 4at$$
$$z = 4at \Rightarrow z' = 4a$$
$$s = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2 + z'^2}\, dt$$
$$= \int_{t_1}^{t_2} \sqrt{16a^2\left(1 - t^2\right) + 16a^2t^2 + 16a^2}\, dt$$
$$= \int_{t_1}^{t_2} \sqrt{32a^2}\, dt = 4\sqrt{2}a\,[t]_{t_1}^{t_2}$$
$$= 4\sqrt{2}a(t_2 - t_1)$$

1.4. Tangent, normal and binormal:

Definition 1.8 (Tangent Line).

The tangent line to a curve C at a point $P(t)$ of C is defined as the limiting position of a straight line L through $P(t)$ and neighbouring point $Q(t + \delta t)$ on

C as Q approaches P along the curve.

Bookwork 1.2. Find the unit tangent vector to a curve.

Let be a curve of class 1 and let P; Q be two neighbouring points on the curve. Let be represented by the equation $\tilde{r} = \tilde{r}(u)$ and let P and Q have parameters u_0 and u: * has class 1:

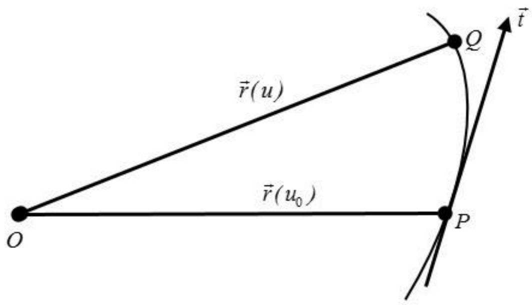


Figure 1.2: Unit tangent vector

By Taylor's theorem

$$\tilde{r} = \tilde{r}(u_0) + (u \quad u_0)\tilde{r}\,(u_0) + O\,(u \quad u_0) \qquad (1.15)$$

Hence

$$\lim_{u \to u_0} \frac{\tilde{r}(u) \quad \tilde{r}(u_0)}{\tilde{r}(u) \quad \tilde{r}(u_0)} = \frac{\tilde{r}(u_0)}{\tilde{r}(u_0)}$$

i:e:; the unit vector along the chord PQ tends to a unit vector at P as Q ! P: This is called the unit tangent vector to at P and it is denoted by $\tilde{t}$:

From (1.13), we have

$$\tilde{t} = \frac{\tilde{r}(u_0)}{\tilde{r}(u_0)} = \frac{\tilde{r}}{s} = \frac{dr}{ds}$$

It is convenient to denote di erentiation with respect to arc length s by prime. Thus, the unit tangent vector becomes $\tilde{t} = \tilde{r}\,^{o}$:

Definition 1.9 (Osculating plane).

Let γ be a curve of class ≥ 2 and let P, Q be two neighboring points on γ. Then the limiting position as $Q \to P$ of that plane which contains the tangent line at P and the point Q is called the osculating plane of γ at P.

Bookwork 1.3. Equation of the osculating plane at a point P.

Let $\tilde{r} = \tilde{r}(s)$ be the given curve γ of class ≥ 2.

The parameters of P and Q be 0 and s respectively.

Position vectors of P and Q are $\tilde{r}(0)$ and $\tilde{r}(s)$ respectively.

Let $\tilde{R}$ be the position vector of the current point T on the plane which contains the tangent line at P and the point Q.

$$\overrightarrow{PT} = \overrightarrow{OT} - \overrightarrow{OP} = \tilde{R} - \tilde{r}(0)$$
$$\tilde{t} = \tilde{r}\,'(0); \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \tilde{r}(s) - \tilde{r}(0)$$

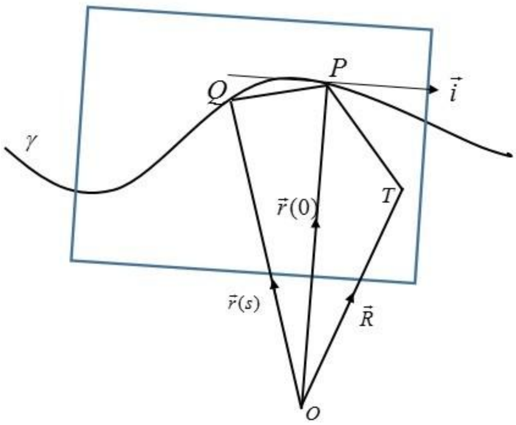


Figure 1.3: Osculating plane

The vectors $\overrightarrow{PT}, \tilde{t}, \overrightarrow{PQ}$ lying in the same plane and therefore their scalar product must be zero, i.e., the equation of the plane is given by

$$\left[\tilde{R} - \tilde{r}(0), \tilde{r}\,'(0), \tilde{r}(s) - \tilde{r}(0)\right] = 0 \tag{1.16}$$

$$\text{Now,} \quad \tilde{r}(s) = \tilde{r}(0) + s\tilde{r}\,'(0) + \frac{s^2}{2!}\tilde{r}\,''(0) + O(s^3)$$

$$\text{i.e.,} \quad \tilde{r}(s) - \tilde{r}(0) = s\tilde{r}(0) + \frac{s^2}{2!}\tilde{r}\,''(0) + O(s^3) \tag{1.17}$$

Using (1.17) in (1.16), as $s \to 0$; we get

$$\left[\tilde{R} - \tilde{r}(0), \tilde{r}'(0), s\tilde{r}'(0) + \frac{s^2}{2}\tilde{r}''(0)\right] = 0$$

$$s\left[\tilde{R} - \tilde{r}(0), \tilde{r}'(0), \tilde{r}'(0)\right] + \frac{s^2}{2}\left[\tilde{R} - \tilde{r}(0), \tilde{r}'(0), \tilde{r}''(0)\right] = 0$$

$$i.e., \left[\tilde{R} - \tilde{r}(0), \tilde{r}'(0), \tilde{r}''(0)\right] = 0 \text{ [* by using(1.16)]}$$

is the equation of osculating plane.

Note 1.3. If the curve is given in terms of an arbitrary parameter u; i.e., $\tilde{r} = \tilde{r}(u)$; then we get

$$\left[\tilde{R} - \tilde{r}(u), \tilde{r}'(u), \tilde{r}''(u)\right] = 0$$

This is the equation of osculating plane if the curve is given in any parameter u.

Remark 1.1.

Equation of osculating plane in Cartesian form:

Let the equation of given curve be $\tilde{r} = x(u)\tilde{i} + y(u)\tilde{j} + z(u)\tilde{k}$ and $\tilde{R} = X\tilde{i} + Y\tilde{j} + Z\tilde{k}$; then the equation of osculating plane is $\left[\tilde{R} - \tilde{r}, \tilde{\dot{r}}, \tilde{\ddot{r}}\right] = 0$.

$$i.e., \begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Definition 1.10 (Point of inflexion).

$$\tilde{t}^2 = 1$$

$$i.e., \frac{d\tilde{r}}{ds} \cdot \frac{d\tilde{r}}{ds} = 1$$

Differentiate with respect to s; we get

$$2\tilde{r}' \cdot \tilde{r}'' = 0$$

$$i.e., \tilde{r}' \cdot \tilde{r}'' = 0$$

If follows that the vectors $\tilde{r}', \tilde{r}''$ are linearly independent unless $\tilde{r}'' = 0$. At a point P where $\tilde{r}'' = 0$ is called a point of inflexion and the tangent line at P is called inflexional.

Equation of the osculating plane at a point of inflexion:

Now we can obtain an osculating plane at a point of inflexion on P unless the curve is a straight line. For this, we consider the relation $\tilde{r}' \cdot \tilde{r}'' = 0$.

Differentiate with respect to s; we get $\tilde{r}' \cdot \tilde{r}''' + \tilde{r}'' \cdot \tilde{r}'' = 0$.

At the point of inflexion $\tilde{r}'' = 0$ and thus we get $\tilde{r}' \cdot \tilde{r}''' = 0$.

Hence $\tilde{r}'$ is linearly independent to $\tilde{r}'''$ except when $\tilde{r}''' = 0$. Repetition of this process, we get $\tilde{r}' \cdot \tilde{r}^{(k)} = 0$ where $\tilde{r}^{(k)}$ is the first non-zero derivative of $\tilde{r}$ at P $(k \geq 2)$. If $\tilde{r}^{(k)} = 0$ for all $k \geq 2$, then since the curve is analytic and we conclude that $\tilde{t}$ is constant and the curve is a straight line. If $\tilde{r}^{(k)} \neq 0$ then we have

$$\tilde{r}(s) - \tilde{r}(0) = s\tilde{t} + \frac{s^k}{k!}\tilde{r}^{(k)}(0) + O\left(s^k\right)$$

as $s \to 0$ and the equation of osculating plane is

$$\left[\tilde{R} - \tilde{r}(0), \tilde{r}'(0), \tilde{r}^{(k)}(0)\right] = 0$$

Example 1.4. Find the equation of the osculating plane at a general point on the curve given by $\tilde{r} = \left(u, u^2, u^3\right)$.

Solution:

$$\text{Given } \tilde{r} = u\tilde{i} + u^2\tilde{j} + u^3\tilde{k}$$

$$\Rightarrow \quad x = u; \quad y = u^2; \quad z = u^3$$

$$\dot{x} = 1; \quad \dot{y} = 2u; \quad \dot{z} = 3u^2$$

$$\ddot{x} = 0; \quad \ddot{y} = 2; \quad \ddot{z} = 6u$$

Let (X, Y, Z) be any point on the osculating plane, then the equation of osculating plane is

$$\begin{vmatrix} X - u & Y - u^2 & Z - u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 0$$

On expanding the determinant, we get

$$6u^2X - 6uY + 2Z - 2u^3 = 0$$

$$\text{i.e.,} \quad 3u^2X - 3uY + Z - u^3 = 0$$

Example 1.5. Find the osculating plane at the point u on the helix
$x = a\cos u; \;\; y = a\sin u; \;\; z = cu.$

Solution:

$$\text{Given} \quad x = a\cos u \, ; \qquad y = a\sin u \, ; \qquad z = cu$$

$$x = \;\; a\sin u \, ; \qquad y = a\cos u \, ; \qquad z = c$$

$$x = \;\; a\cos u \, ; \qquad y = \;\; a\sin u \, ; \qquad z = 0$$

Let (X, Y, Z) be any point on the osculating plane, then the equation of osculating plane is

$$\begin{matrix} X \quad a\cos u & Y \quad a\sin u & Z \quad cu \\ a\sin u & a\cos u & c \\ a\cos u & a\sin u & 0 \end{matrix} = 0$$

On expanding the determinant, we get

$$ac\sin uX \quad ac\cos uY + a^2Z \quad a^2cu = 0$$

$$i.e., \quad c\sin uX \quad c\cos uY + aZ \quad acu = 0$$

which is the required equation of the osculating plane.

Example 1.6. For the curve $x = 3t \, ; \;\; y = 3t^2; \;\; z = 2t^3;$ show that any plane meets it in three points and deduce the equation of the osculating plane at $t = t_3.$

Solution:

Let the equation of the plane be $Ax + By + Cz + D = 0.$

$$A(3t) + B \;\; 3t^2 \;\; + C \;\; 2t^3 \;\; + D = 0$$

$$i.e., \quad 2ct^3 + 3Bt^3 + 3At + D = 0$$

which is a cubic equation in $t.$ So there will be three values of $t.$ Hence the plane meets the given curve in three points.

<u>To nd the equation of osculating plane:</u>

$$\text{Given} \quad x = 3t; \qquad y = 3t^2; \qquad z = 2t^3$$

$$x = 3; \qquad y = 6t; \qquad z = 6t^2$$

$$x = 0; \qquad y = 6; \qquad z = 12t$$

Let (X, Y, Z) be any point on the osculating plane, then the equation of osculating plane is

$$\begin{vmatrix} X-3t & Y-3t^2 & Z-2t^3 \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix} = 0$$

On expanding the determinant, we get

$$2Xt^2 - 2Yt^2 + Z - 2t^3 = 0$$

which is the required equation of the osculating plane.

Definition 1.11 (Normal plane).

The normal plane at a point P on the curve is that plane through P which is orthogonal to the tangent at $P.$

Note 1.4. Clearly the normal plane is perpendicular to the osculating plane.

Definition 1.12 (Principal normal).

The principal normal at P is the line of intersection of the normal plane and the osculating plane at $P.$ A unit vector along the principal normal is denoted by $\tilde{n}.$

Note 1.5. The normal which lies in osculating plane at any point of a curve is called a principal normal.

Definition 1.13 (Bi-normal).

The normal which is perpendicular to the osculating plane at a point is called the binormal and it is denoted by $\tilde{b}.$

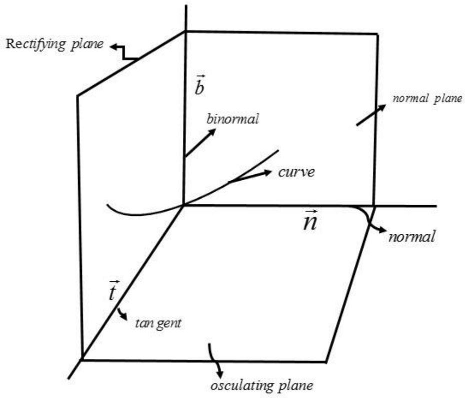
Note 1.6. Clearly binormal is also perpendicular to principal normal.

Fundamental Planes of a space curves:

Through any point on the curve, we have three unit vectors $\tilde{t}, \tilde{n}, \tilde{b}$ forming three mutually perpendicular planes namely osculating plane, rectifying plane and normal plane.

The plane formed by the vectors $\tilde{t}$ and $\tilde{n}$ is called the osculating plane and that of the plane formed by the vectors $\tilde{b}$ and $\tilde{n}$ is called the normal plane. Similarly, the plane formed by the vectors $\tilde{b}$ and $\tilde{t}$ is called the rectifying plane.

The three unit vectors $\tilde{t}; \tilde{b}; \tilde{n}$ form a right handed orthogonal system of axes and satisfying the following relations:



$$\tilde{t}\ \tilde{t} = \tilde{n}\ \tilde{n} = \tilde{b}\ \tilde{b} = 1$$
$$\tilde{t}\ \tilde{n} = \tilde{n}\ \tilde{b} = \tilde{b}\ \tilde{t} = 0$$
$$\tilde{t}\ \tilde{n} = \tilde{b};\ \tilde{n}\ \tilde{b} = \tilde{t};\ \tilde{b}\ \tilde{t} = \tilde{n}$$

Figure 1.4: Planes

Thus at any point on the curve, we have three mutually perpendicular planes. They are

(i) The osculating plane containing $\tilde{t}$ and $\tilde{n}$ and its equation is

$$\tilde{R}\ \ \tilde{r}\ \ \tilde{b} = 0:$$

(ii) The normal plane containing $\tilde{n}$ and $\tilde{b}$ and its equation is

$$\tilde{R}\ \ \tilde{r}\ \ \tilde{t} = 0:$$

(iii) The rectifying plane containing $\tilde{b}$ and $\tilde{t}$ and its equation is

$$\tilde{R}\ \ \tilde{r}\ \ \tilde{n} = 0:$$

Equation of Tangent line and Normal Plane:

Tangent line: Equation of tangent line interms of parameter u is given by $\tilde{R} = \tilde{r} + \ \tilde{r}$ where $\tilde{R}$ is the position vector of a current point on the tangent line and is a scalar.

If we write $\tilde{R} = X\tilde{i} + Y\tilde{j} + Z\tilde{k}$; $\tilde{r} = x\tilde{i} + y\tilde{j} + z\tilde{k}$ and $\tilde{r} = x\tilde{i} + y\tilde{j} + z\tilde{k}$ in the above equation, we get the Cartesian form of equation of tangent line as

$$\frac{X\ \ x}{x} = \frac{Y\ \ y}{y} = \frac{Z\ \ z}{z} =$$

Note 1.7. Instead of the parameter u; if we use parameter s (arc length), then we get the equation of tangent line as

(i) $\tilde{R} = \tilde{r} + \ \tilde{r}^{o}$ where is a scalar. (vector form).

(ii) $\frac{X\ \ x}{x^{o}} = \frac{Y\ \ y}{y^{o}} = \frac{Z\ \ z}{z^{o}} =$ (Cartesian form)

Normal Plane: The equation of normal plane in general parameter u is given by

$$(\tilde{R} - \tilde{r}) \cdot \dot{\tilde{r}} = 0$$

$$\text{or} \quad (\tilde{R} - \tilde{r}) \cdot \tilde{t} = 0 \qquad [\because \dot{\tilde{r}} = \tilde{t}\,]$$

where $\tilde{R}$ is the position vector of current point on the normal plane.

If we take $\tilde{R} = X\tilde{i} + Y\tilde{j} + Z\tilde{k}$; $\tilde{r} = x\tilde{i} + y\tilde{j} + z\tilde{k}$ and $\dot{\tilde{r}} = \dot{x}\tilde{i} + \dot{y}\tilde{j} + \dot{z}\tilde{k}$.

Then the equation of normal plane becomes

$$(X - x)\,\dot{x} + (Y - y)\,\dot{y} + (Z - z)\,\dot{z} = 0$$

Note 1.8. Instead of the parameter u, if the parameter s (arc length) is given, then equation of normal plane is

(i) $(\tilde{R} - \tilde{r}) \cdot \tilde{r}' = 0$ (vector form)

(ii) $(X - x)\,x' + (Y - y)\,y' + (Z - z)\,z' = 0$ (Cartesian form)

Example 1.7. For the curve $x = 3u$; $y = 3u^2$; $z = 2u^3$. Find

(i) Unit tangent vector

(ii) Equation of tangent line

(iii) Equation of normal plane

Solution:

$$\text{Given } x = 3u; \quad y = 3u^2; \quad z = 2u^3$$

$$\tilde{r} = x\tilde{i} + y\tilde{j} + z\tilde{k}$$

$$\Rightarrow \tilde{r} = 3u\tilde{i} + 3u^2\tilde{j} + 2u^3\tilde{k}$$

$$\dot{\tilde{r}} = 3\tilde{i} + 6u\tilde{j} + 6u^2\tilde{k}$$

(i) tangent vector

$$\tilde{t} = \frac{\dot{\tilde{r}}}{|\dot{\tilde{r}}|} = \frac{3\tilde{i} + 6u\tilde{j} + 6u^2\tilde{k}}{\sqrt{3^2 + (6u)^2 + (6u^2)^2}}$$

$$= \frac{\tilde{i} + 2u\tilde{j} + 2u^2\tilde{k}}{\sqrt{(1 + 2u^2)^2}} = \frac{\tilde{i} + 2u\tilde{j} + 2u^2\tilde{k}}{1 + 2u^2}$$

(ii) Equation of tangent line (Cartesian form):

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}$$

$$\frac{X - 3u}{3} = \frac{Y - 3u^2}{6u} = \frac{Z - 2u^3}{6u^2}$$

$$i.e., \quad \frac{X - 3u}{1} = \frac{Y - 3u^2}{2u} = \frac{Z - 2u^3}{2u^2}$$

(iii) Equation of normal plane (Cartesian form):

$$(X - x)\,x' + (Y - y)\,y' + (Z - z)\,z' = 0$$

$$i.e., \quad (X - 3u)\,3 + \left(Y - 3u^2\right)6u + \left(Z - 2u^3\right)6u^2 = 0$$

$$i.e., \quad X + 2uY + 2u^2Z = 3u + 6u^3 + 4u^5$$

(on simplification)

Example 1.8. Find the equation of tangent and normal plane at the point u on the circular helix $x = a\cos u; \quad y = a\sin u; \quad z = bu.$

Solution:

Given $x = a\cos u; \quad y = a\sin u; \; z = bu$

Equation of tangent is

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}$$

$$\frac{X - a\cos u}{-a\sin u} = \frac{Y - a\sin u}{a\cos u} = \frac{Z - bu}{b}$$

Equation of normal plane(Cartesian form):

$$(X - x)\,x' + (Y - y)\,y' + (Z - z)\,z' = 0$$

$$(X - a\cos u)(-a\sin u) + (Y - a\sin u)\,a\cos u + (Z - bu)\,b = 0$$

$$Xa\sin u \quad Ya\cos u = 0$$

1.5. Curvature and Torsion:

Definition 1.14 (Curvature).

The rate of change of the direction of tangent with respect to arc lengths is called the curvature, it is denoted by κ.

Note 1.9. By definition, $|\kappa| = \tilde{t}'$. where κ is the curvature vector. In order to determine the sign of ; we recall that $\tilde{t}^{o} = \tilde{r}^{oo}$ lies in the osculating plane and it is also normal to $\tilde{t}$ and hence $\tilde{t}^{o}$ is proportional to $\tilde{n}$ i:e:; $\tilde{t}^{o} = \tilde{n}$: But we choose the direction of $\tilde{n}$ such that the curvature is always positive. i:e:; $\tilde{t}^{o} = \tilde{n}$:

De nition 1.15 (Radius of curvature).

The reciprocal of the curvature is called the radius of curvature and it is denoted by : i:e:; $= \frac{1}{}$:

De nition 1.16 (Torsion).

The rate at which the osculating plane turns about the tangent at the point P moved, is called the torsion of the curve at P and it is denoted by :

Note 1.10. The torsion may have positive as well as negative direction. Therefore is determined both in magnitude and direction.

De nition 1.17 (Radius of Torsion).

The reciprocal of the torsion is called the radius of torsion and is denoted by : i:e:; $= \frac{1}{}$:

De nition 1.18 (Screw curvature).

The rate of change of the direction of principal normal with respect to arc length as the point P moves along the curve is called the screw curvature vector and its magnitude is $\sqrt{ + }$: Hence $\sqrt{{}^2 + {}^2} = \left|\frac{d\tilde{n}}{ds}\right|$:

Bookwork 1.4 (Serret-Frenet Formulae:).

The following three relations are known as Serret-Frenet Formulae.

(i) $\tilde{t}^{o} = \tilde{n}$

(ii) $\tilde{n}^{o} = \tilde{t} + \tilde{b}$

(iii) $\tilde{b}^{o} = \tilde{n}$

Proof. (i) We know that $\tilde{t}^{2} = 1$ i:e:; $\tilde{t}\,\tilde{t} = 1$

Di erentiating both sides with respect to arc length s; we get

$$\begin{aligned} \tilde{t}\ \tilde{t}^{o} + \tilde{t}^{o}\ \tilde{t} &= 0 \\ \Rightarrow\ 2\tilde{t}\ \tilde{t}^{o} &= 0 \\ \text{i:e:;}\ \tilde{t}\ \tilde{t}^{o} &= 0 \end{aligned}$$

which shows that $\tilde{t}^{o}$ is perpendicular to $\tilde{t}$:

The equation of the osculating plane at a point $P(\tilde{r})$ of the curve is

$$\left[\tilde{R} - \tilde{r},\ \tilde{r}',\ \tilde{r}''\right] = 0$$

$$\Rightarrow \left[\tilde{R} - \tilde{r},\ \tilde{t},\ \tilde{r}''\right] = 0$$

The last equation shows that $\tilde{t}'$ lies in the osculating plane and hence $\tilde{t}'$ is perpendicular to the binormal $\tilde{b}$.

Thus $\tilde{t}'$ is parallel to $\tilde{b} \times \tilde{t}$, which implies that $\tilde{t}'$ is parallel to $\tilde{n}$.

Hence, $\tilde{t}' = \pm\kappa\tilde{n}$.

Therefore, $\tilde{t}' = \kappa\tilde{n}$ [Taking positive sign only]

(iii) We know that $\tilde{b}^2 = 1$, i.e., $\tilde{b}\cdot\tilde{b} = 1$.

Differentiating with respect to s, we get

$$\tilde{b}\cdot\tilde{b}' + \tilde{b}'\cdot\tilde{b} = 0 \Rightarrow \tilde{b}\cdot\tilde{b}' = 0$$

Therefore, $\tilde{b}'$ is perpendicular to $\tilde{b}$ and thus $\tilde{b}'$ lies in the osculating plane.

Also, we know that $\tilde{b}\cdot\tilde{t} = 0$.

Differentiating with respect to arc length s, we get

$$\begin{aligned}
\tilde{b}\cdot\tilde{t}' + \tilde{b}'\cdot\tilde{t} &= 0\\
\Rightarrow \tilde{b}\cdot\kappa\tilde{n} + \tilde{b}'\cdot\tilde{t} &= 0 \quad \text{(using (i))}\\
\Rightarrow \kappa\,\tilde{b}\cdot\tilde{n} + \tilde{b}'\cdot\tilde{t} &= 0 \Rightarrow \tilde{b}'\cdot\tilde{t} = 0
\end{aligned}$$

Therefore, $\tilde{b}'$ is perpendicular to $\tilde{t}$ and hence we get $\tilde{b}'$ must be parallel to $\tilde{n}$.

Thus, we can write $\tilde{b}' = \pm\tau\tilde{n}$.

By convention, we can take $\tilde{b}' = -\tau\tilde{n}$

(ii) We know that $\tilde{n} = \tilde{b} \times \tilde{t}$.

Differentiating both sides with respect to s, we get

$$\begin{aligned}
\tilde{n}' &= \tilde{b} \times \tilde{t}' + \tilde{b}' \times \tilde{t}\\
&= \tilde{b} \times \kappa\tilde{n} + (-\tau\tilde{n}) \times \tilde{t} \quad \text{(using (i) and (iii))}\\
&= \kappa(\tilde{b} \times \tilde{n}) + \tau\tilde{b}
\end{aligned}$$

i.e., $\tilde{n}' = -\kappa\tilde{t} + \tau\tilde{b}$

Note 1.11. Serret-Frenet formulae can also be written in the matrix form:

$$\begin{matrix} \tilde{t}' & & 0 & & 0 & \tilde{t} \\ \tilde{n}' & = & & 0 & & \tilde{n} \\ \tilde{b}' & 0 & & & 0 & \tilde{b} \end{matrix}$$

Theorem 1.1. A necessary and su cient condition that a curve be a straight line is that $= 0$ at all points.

Proof. Necessary part: Assume that curve is a straight line.

Any straight line has equation of the form $\tilde{r} = \tilde{a}s + \tilde{b}$; where $\tilde{a}$ and $\tilde{b}$ are constant vectors.

Thus, $\tilde{r}' = \tilde{t} = \tilde{a}$ and $\tilde{r}'' = \tilde{t}' = 0$: i.e.; $\tilde{n} = 0$ and hence $= 0$:

Su cient part: If $= 0$; then $\tilde{r}'' = 0$:

Integrating twice, we get $\tilde{r} = \tilde{a}s + \tilde{b}$ which is the equation of a straight line.

Theorem 1.2. A curve is a plane curve if and only if $= 0$ at all points.

Proof. Necessary part: Let the curve lie in a plane. Then the plane curve lie on the osculating plane.

Therefore, the plane must be xed and so $\tilde{b}$ does not change, which means that $\tilde{b}$ is a constant vector.

$$\begin{aligned} \tilde{b}' &= 0 \quad \tilde{n} = 0 \quad \tilde{n} = 0 \\ \tilde{n}^2 &= 0 \\ \tilde{n} \cdot \tilde{n} &= 0 \\ {}^2\, 1 &= 0 \\ \text{i.e.;} \quad &= 0 \end{aligned}$$

Su cient part: Assume that $= 0$:

Now, our aim is to prove that the curve is a plane curve.

$$\begin{aligned} \tilde{b}' &= \tilde{n} \\ \tilde{b}' &= 0 \quad (* = 0) \end{aligned}$$

$\tilde{b}$ is a constant vector:

$$\text{Now } (\tilde{r}\cdot\tilde{b})' = \tilde{r}\cdot\tilde{b}' + \tilde{r}'\cdot\tilde{b}$$
$$= 0 + \tilde{r}'\cdot\tilde{b} = \tilde{t}\cdot\tilde{b} = 0$$
$$\Rightarrow \tilde{r}\cdot\tilde{b} = \text{constant} = C \text{ (say)}$$
$$\Rightarrow (x\tilde{i} + y\tilde{j} + z\tilde{k})\cdot(b_1\tilde{i} + b_2\tilde{j} + b_3\tilde{k}) = C$$
$$\Rightarrow b_1x + b_2y + b_3z = C \quad \text{which is a plane equation.}$$

Thus the point (x, y, z) satisfies the plane equation for all values of x, y, z and hence the curve is a plane curve.

This completes the proof of the theorem.

Bookwork 1.5.
The necessary and sufficient condition for the curve to be a plane curve is $[\tilde{r}', \tilde{r}'', \tilde{r}'''] = 0$

Proof. We know that $\tilde{r}' = \tilde{t}$.

$$\text{Thus, we have } \tilde{r}'' = \tilde{t}' = \kappa\tilde{n} \quad \text{(i) (by Serret Frenet formulae)}$$
$$\text{Now, } \tilde{r}'\times\tilde{r}'' = \tilde{t}\times\kappa\tilde{n}$$
$$\text{i.e., } \tilde{r}'\times\tilde{r}'' = \kappa\tilde{b} \quad (\because \tilde{t}\times\tilde{n} = \tilde{b})$$

Differentiating both sides with respect to s, we get

$$\tilde{r}'\times\tilde{r}''' + \tilde{r}''\times\tilde{r}'' = \kappa\tilde{b}' + \kappa'\tilde{b}$$
$$\Rightarrow \tilde{r}'\times\tilde{r}''' + 0 = \kappa(-\tau\tilde{n}) + \kappa'\tilde{b} \quad \text{(by Serret Frenet formulae)}$$
$$\Rightarrow \tilde{r}'\times\tilde{r}''' = -\kappa\tau\tilde{n} + \kappa'\tilde{b} \quad \text{(ii)}$$

Taking scalar products of (i) and (ii), we get

$$\tilde{r}''\cdot(\tilde{r}'\times\tilde{r}''') = \kappa\tilde{n}\cdot(-\kappa\tau\tilde{n} + \kappa'\tilde{b})$$
$$\Rightarrow [\tilde{r}'', \tilde{r}', \tilde{r}'''] = -\kappa^2\tau\,\tilde{n}\cdot\tilde{n} + \kappa'\,\tilde{n}\cdot\tilde{b}$$
$$= -\kappa^2\tau + \kappa'(0)$$
$$\text{i.e., } [\tilde{r}', \tilde{r}'', \tilde{r}'''] = \kappa^2\tau \quad \text{(iii)}$$

If the left hand member of (iii) is zero, then either $\kappa = 0$ or $\tau = 0$.

Now, let $\tau \neq 0$ at some point of the curve, then in this neighbourhood of this point $\tau \neq 0$. Hence $\kappa = 0$ in this neighbourhood and hence the curve is a straight line and therefore $\tau = 0$ on this line and this is a contradiction to our assumption. Hence $\tau = 0$ at all points and the curve is a plane curve.

Conversely, if the curve is a plane curve then = 0

Therefore from (iii); we get

$$\left[\tilde{r}'; \tilde{r}''; \tilde{r}'''\right] = {}^2(0) = 0$$

Note 1.12. The above theorem can also be stated as the necessary and sufficient condition for the curve to be plane is $\left[\dot{\tilde{r}}; \ddot{\tilde{r}}; \dddot{\tilde{r}}\right] = 0$

Proof.

$$\left[\tilde{r}'; \tilde{r}''; \tilde{r}'''\right] = \left[\dot{\tilde{r}}u'; \ddot{\tilde{r}}u'^2 + \dot{\tilde{r}}u''; \dddot{\tilde{r}}u'^3 + \dot{\tilde{r}}u''' + 3\ddot{\tilde{r}}u'u''\right]$$

$$= u'^6\left[\dot{\tilde{r}}; \ddot{\tilde{r}}; \dddot{\tilde{r}}\right]$$

$$\left[\dot{\tilde{r}}; \ddot{\tilde{r}}; \dddot{\tilde{r}}\right] = s^6\left[\tilde{r}'; \tilde{r}''; \tilde{r}'''\right] \quad * \; u' = \frac{du}{ds} = \frac{1}{ds/du} = \frac{1}{s}$$

Hence when $\left[\tilde{r}'; \tilde{r}''; \tilde{r}'''\right] = 0$ is the necessary and sufficient condition for the curve to be a plane it follows that $\left[\dot{\tilde{r}}; \ddot{\tilde{r}}; \dddot{\tilde{r}}\right]$ is also a necessary and sufficient condition for the curve to be a plane.

Example 1.9. Show that Serret-Frenet formulae can be written in the form $\tilde{t}' = \tilde{w} \times \tilde{t}$; $\tilde{n}' = \tilde{w} \times \tilde{n}$; $\tilde{b}' = \tilde{w} \times \tilde{b}$ and also determine $\tilde{w}$. ($\tilde{w}$ is called Darbouxe vector of the curve)

Solution: We know that from Serret-Frenet formulae

$$\tilde{t}' = \tilde{n} = \tilde{t} \times \tilde{t} + \tilde{b} \times \tilde{t} \qquad [* \; \tilde{t} \times \tilde{t} = 0; \; \tilde{b} \times \tilde{t} = \tilde{n}]$$

$$= (\tilde{t} + \tilde{b}) \times \tilde{t}$$

$$= \tilde{w} \times \tilde{t}; \quad \text{where } w = \tilde{t} + \tilde{b}$$

$$\tilde{n}' = \tilde{b} - \tilde{t} = \tilde{t} \times \tilde{n} + \tilde{b} \times \tilde{n}$$

$$= (\tilde{t} + \tilde{b}) \times \tilde{n} = \tilde{w} \times \tilde{n}$$

$$\tilde{b}' = \tilde{n} = \tilde{t} \times \tilde{b} + \tilde{b} \times \tilde{b} \qquad [* \; \tilde{b} \times \tilde{b} = 0; \; \tilde{n} = \tilde{t} \times \tilde{b}]$$

$$= (\tilde{t} + \tilde{b}) \times \tilde{b} = \tilde{w} \times \tilde{n}$$

Example 1.10. Prove that for any curve

(i) $\left[\tilde{t}'; \tilde{t}''; \tilde{t}'''\right] = {}^5\frac{d}{ds}\left(\frac{}{}\right)$

(ii) $\left[\tilde{b}'; \tilde{b}''; \tilde{b}'''\right] = {}^3[\;' \;\; ' \;] = {}^5\frac{d}{ds}\left(\frac{}{}\right)$

Solution:

(i) We know that $\tilde{r}^{o} = \dfrac{d\tilde{r}}{ds} = \tilde{t}$

Di erentiating both sides with respect to arc length $s;$ we get

$$\tilde{t}^{o} = \tilde{r}^{oo} = \frac{d\tilde{t}}{ds} = \tilde{n}$$

Again di erentiating both sides with respect to arc length $s;$ we get

$$\tilde{r}^{ooo} = \frac{d}{ds}\ \tilde{n}$$

$$= {}^{o}\tilde{n} + \tilde{n}^{o}$$

$$= {}^{o}\tilde{n} + \quad \tilde{t} + \tilde{b} \quad \text{(by Serret Frenet formulae)}$$

$$\tilde{t}^{oo} = \tilde{r}^{ooo} = {}^{o}\tilde{n} \quad {}^{2}\tilde{t} + \quad \tilde{b}$$

$$\tilde{t}^{ooo} = \tilde{r}^{(iv)} = \quad {}^{oo} \quad {}^{3} \quad {}^{2}\ \tilde{n} \quad 3 \quad {}^{o}\tilde{t} + 2 \quad {}^{o} \quad + \quad {}^{o} \quad \tilde{b}$$

$$\left[\tilde{t}^{o}\ \tilde{t}^{oo}\ \tilde{t}^{ooo}\right] = \begin{vmatrix} 0 & & 0 \\ {}^{2} & {}^{o} & \\ 3 \ {}^{o} \ {}^{oo} \ {}^{3} & {}^{2} & 2 \ {}^{o} + \ {}^{o} \end{vmatrix}$$

On expanding the determinant, we get

$$= {}^{5}\left(\frac{{}^{o} \quad {}^{o}}{{}^{2}}\right)$$

$$= {}^{5}\frac{d}{ds} -$$

(ii) We know that $\tilde{b}^{o} = \tilde{n}$

$$\tilde{b}^{oo} = {}^{o}\tilde{n} \quad \tilde{n}^{o} = \tilde{t} \quad {}^{o}\tilde{n} \quad {}^{2}\tilde{b}$$

$$\tilde{b}^{ooo} = \tilde{t}^{o} + \quad {}^{o} + \quad {}^{o}\ \tilde{t} \quad {}^{o}\tilde{n}^{o} \quad {}^{oo}\tilde{n} \quad {}^{2}\tilde{b}^{o} \quad 2 \quad {}^{o}\tilde{b}$$

$$= 2 \quad {}^{o} + \ {}^{o}\ \tilde{t} + \ {}^{2} \quad {}^{oo} + \ {}^{3} \quad 3 \ {}^{o}\tilde{b}$$

$$) \ \left[\tilde{b}^{o}\ \tilde{b}^{oo}\ \tilde{b}^{ooo}\right] = \begin{vmatrix} 0 & & 0 \\ {}^{2} & {}^{o} & {}^{2} \\ 2 \ {}^{o} + \ {}^{o} & {}^{2} & {}^{oo} + \ {}^{3} \quad 3 \ {}^{o}\tilde{b} \end{vmatrix}$$

On expanding the determinant, we get

$$= {}^{5}\frac{(\ {}^{o} \quad {}^{o})}{{}^{2}}$$

$$= {}^{5}\frac{d}{ds} -$$

Example 1.11. Show that the principal normals at consecutive points do not intersect unless $\tau = 0$

Solution:

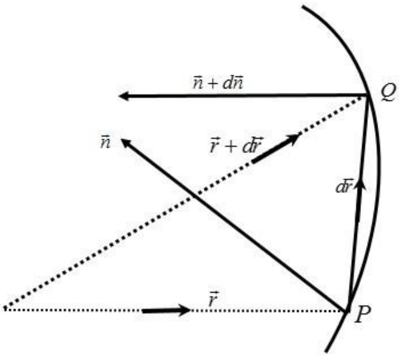


Figure 1.5: Curve length

Let P and Q be two consecutive points with position vectors $\tilde{r}$ and $\tilde{r} + d\tilde{r}$ on the curve C.

Let the principal normals at these points be $\tilde{n}$ and $\tilde{n} + d\tilde{n}$.

The principal normals will intersect if the three vectors $\tilde{n}, \tilde{n} + d\tilde{n}$ and $d\tilde{r}$ are coplanar.

$$\begin{aligned}
i.e., \quad [\tilde{n}, \tilde{n} + d\tilde{n}, d\tilde{r}] &= 0 \\
i.e., \quad [\tilde{n}, \tilde{n}, d\tilde{r}] + [\tilde{n}, d\tilde{n}, d\tilde{r}] &= 0 \\
\left[\tilde{n}, \frac{d\tilde{n}}{ds}, \frac{d\tilde{r}}{ds}\right] &= 0 \\
i.e., \quad [\tilde{n}, \tau\tilde{b} - \kappa\tilde{t}, \tilde{t}] &= 0 \\
i.e., \quad [\tilde{n}, \tau\tilde{b}, \tilde{t}] &= 0 \\
i.e., \quad \tau[\tilde{n}, \tilde{b}, \tilde{t}] &= 0 \\
\Rightarrow \tau &= 0 \quad (\because [\tilde{n}, \tilde{b}, \tilde{t}] = 1)
\end{aligned}$$

Hence the principal normals at consecutive points do not intersect unless $\tau = 0$.

Example 1.12. Prove that the position vector of current point on a curve satisfies the differential equation

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\tilde{r}}{ds^2}\right)\right] + \frac{d}{ds}\left(\frac{\tau}{\kappa}\frac{d\tilde{r}}{ds}\right) + \frac{\kappa}{\tau}\frac{d^2\tilde{r}}{ds^2} = 0$$

Solution:

$$\text{We know that} \quad = \frac{1}{\ }; \quad = \frac{1}{\ }; \quad \frac{d\tilde{r}}{ds} = \tilde{t}; \quad \frac{d^2\tilde{r}}{ds^2} = \frac{d\tilde{t}}{ds} = \tilde{n}$$

$$\begin{aligned}
L.H.S. &= \frac{d}{ds}\left(\frac{1}{\ }\frac{d}{ds}\left(\frac{1}{\ }\tilde{n}\right)\right) + \frac{d}{ds}\left(-\tilde{t}\right) + -\tilde{n} \\
&= \frac{d}{ds}\left(\frac{1}{\ }\frac{d}{ds}(\tilde{n})\right) + \frac{d}{ds}\left(-\tilde{t}\right) + -\tilde{n} \\
&= \frac{d}{ds}\left(\frac{1}{\ }\left(\tilde{t} + \tilde{b}\right)\right) + \frac{d}{ds}\left(-\tilde{t}\right) + \tilde{n} \quad \text{(using Serret Frenet formulae)} \\
&= \frac{d\tilde{b}}{ds} \quad \frac{d}{ds}\left(-\tilde{t}\right) + \tilde{n} + \frac{d}{ds}\left(-\tilde{t}\right) \\
&= \tilde{n} \quad \frac{d}{ds}\left(-\tilde{t}\right) + \frac{d}{ds}\left(-\tilde{t}\right) + \tilde{n} = 0
\end{aligned}$$

Bookwork 1.6. Find the curvature and Torsion of any curve $\tilde{r} = \tilde{r}(u)$

Proof. Let the equation of a given curve be $\tilde{r} = \tilde{r}(u)$ where u is any general parameter.

$$\dot{\tilde{r}} = \frac{d\tilde{r}}{du} = \frac{d\tilde{r}}{ds}\frac{ds}{du} = \tilde{r}^{\,0}\dot{s}$$

$$\Rightarrow \dot{\tilde{r}} = \dot{s}\tilde{t} \tag{1.18}$$

Differentiating with respect to parameter u; we get

$$\ddot{\tilde{r}} = \dot{\tilde{t}}\dot{s} + \tilde{t}\ddot{s} = \tilde{t}\ddot{s} + \tilde{t}^{\,0}\dot{s}\dot{s} \qquad \left[\dot{\tilde{t}} = \frac{d\tilde{t}}{du} = \frac{d\tilde{t}}{ds}\frac{ds}{du} = \tilde{t}^{\,0}\dot{s}\right]$$

$$= \tilde{t}\ddot{s} + \tilde{n}\dot{s}^2$$

$$\ddot{\tilde{r}} = \ddot{s}\tilde{t} + \dot{s}^2\tilde{n} \tag{1.19}$$

Again differentiating with respect to u; we get

$$\dddot{\tilde{r}} = \ddot{s}\,\tilde{t}^{\,0}\frac{ds}{du} + \dddot{s}\tilde{t} + \dot{s}^2\dot{\tilde{n}} + 2\dot{s}\ddot{s}\tilde{n} + \dot{s}^2\tilde{n}^{\,0}\frac{ds}{du}$$

$$= \left(\dddot{s} - {}^2\dot{s}^3\right)\tilde{t} + \left(\dot{s}\ddot{s} + \dot{s}^2 + 2\dot{s}\ddot{s}\right)\tilde{n} + \dot{s}^3\tilde{b} \tag{1.20}$$

Vector cross multiplying (1.18) and (1.19), we get

$$\dot{\tilde{r}} \times \ddot{\tilde{r}} = \dot{s}\tilde{t} \times \left(\ddot{s}\tilde{t} + \dot{s}^2\tilde{n}\right) = \dot{s}\ddot{s}\tilde{t} \times \tilde{t} + \dot{s}^3\,\tilde{t} \times \tilde{n} = 0 + \dot{s}^3\tilde{b}$$

$$\text{i.e.,} \quad \dot{\tilde{r}} \times \ddot{\tilde{r}} = \dot{s}^3\tilde{b} \tag{1.21}$$

Taking vector dot product of (1.21) and (1.20), we get

$$\tilde{r} \quad \tilde{r} \quad \dddot{\tilde{r}} = \quad s^3 \quad s^3$$

$$\text{i.e.,} \quad \left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right] = \ ^2 \ s^6 \tag{1.22}$$

It remains to nd the value of and :

To Find :

From (1.21), we have

$$\tilde{r} \quad \tilde{r} = \quad s^3 \tilde{b} = \quad s^3 \tag{1.23}$$

$$\Rightarrow \quad = \frac{\tilde{r} \quad \tilde{r}}{s^3}$$

$$\text{i.e.,} \quad = \frac{\tilde{r} \quad \tilde{r}}{\tilde{r}^{\ 3}} \tag{1.24}$$

To Find :

From (1.22), we have

$$= \frac{\left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right]}{^2 s^6}$$

$$= \frac{\left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right]}{\left(s^3\right)^2}$$

$$\text{i.e.,} \quad = \frac{\left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right]}{\tilde{r} \quad \tilde{r}^{\ 2}} \quad \text{(using (1.23))} \tag{1.25}$$

Note 1.13. If the equation of the curve is given in terms of arc length s: i.e., $\tilde{r} = \tilde{r}(s)$; then $s = \frac{ds}{du}$ and $\frac{ds}{ds} = 1$. Then $\tilde{r}, \tilde{r}, \dddot{\tilde{r}}$ becomes $\tilde{r}^{\,\prime}, \tilde{r}^{\,\prime\prime}, \tilde{r}^{\,\prime\prime\prime}$

Thus (1.23) becomes $= \tilde{r}^{\,\prime} \quad \tilde{r}^{\,\prime\prime}$.

Similarly from (1.25), we have $= \dfrac{\left[\tilde{r}^{\,\prime}, \tilde{r}^{\,\prime\prime}, \tilde{r}^{\,\prime\prime\prime}\right]}{\tilde{r}^{\,\prime} \quad \tilde{r}^{\,\prime\prime\ 2}}$

Example 1.13. Show that the curve $x = t$; $y = \dfrac{1 + t}{t}$; $z = \dfrac{1 \quad t^2}{t}$ lies in a plane.

Solution: We know that the necessary and su cient condition for a curve to be a plane curve is $\left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right] = 0$.

Hence, it is enough to prove that $\left[\tilde{r}, \tilde{r}, \dddot{\tilde{r}}\right] = 0$.

$$
\begin{aligned}
\tilde{r} &= x\tilde{i} + y\tilde{j} + z\tilde{k} \\
&= t\,\tilde{i} + \frac{1+t}{t}\tilde{j} + \frac{1-t^2}{t}\tilde{k} \\
\dot{\tilde{r}} &= \tilde{i} - \frac{1}{t^2}\tilde{j} - \left(\frac{1}{t^2}+1\right)\tilde{k} \\
\ddot{\tilde{r}} &= 0\tilde{i} + \frac{2}{t^3}\tilde{j} + \frac{2}{t^3}\tilde{k} \\
\dddot{\tilde{r}} &= 0\tilde{i} - \frac{6}{t^4}\tilde{j} - \frac{6}{t^4}\tilde{k}
\end{aligned}
$$

$$
\left[\dot{\tilde{r}}, \ddot{\tilde{r}}, \dddot{\tilde{r}}\right] = \begin{vmatrix} 1 & -\frac{1}{t^2} & -\left(\frac{1}{t^2}+1\right) \\ 0 & \frac{2}{t^3} & \frac{2}{t^3} \\ 0 & -\frac{6}{t^4} & -\frac{6}{t^4} \end{vmatrix}
= \left(\frac{2}{t^3}\right)\left(-\frac{6}{t^4}\right) \begin{vmatrix} 1 & -\frac{1}{t^2} & -\left(\frac{1}{t^2}+1\right) \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}
= 0
$$

Example 1.14. Find the curvature and torsion of the cubic curve given by $\tilde{r} = (u, u^2, u^3)$.

Solution:

$$
\begin{aligned}
\dot{\tilde{r}} &= (1, 2u, 3u^2) \\
\ddot{\tilde{r}} &= (0, 2, 6u); \quad \dddot{\tilde{r}} = (0, 0, 6)
\end{aligned}
$$

$$
\dot{\tilde{r}} \times \ddot{\tilde{r}} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 6u^2\tilde{i} - 6u\tilde{j} + 2\tilde{k}
$$

$$
\left|\dot{\tilde{r}} \times \ddot{\tilde{r}}\right| = \sqrt{36u^4 + 36u^2 + 4} = 2\sqrt{9u^4 + 9u^2 + 1}
$$

$$
\left[\dot{\tilde{r}}, \ddot{\tilde{r}}, \dddot{\tilde{r}}\right] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix} = 12
$$

$$|\tilde{r}'|^2 = 1 + 4u^2 + 9u^4$$

$$\Rightarrow \kappa = \frac{|\tilde{r}' \times \tilde{r}''|}{|\tilde{r}'|^3}$$

$$= \frac{2\sqrt{9u^4 + 9u^2 + 1}}{\left(1 + 4u^2 + 9u^4\right)^{3/2}}$$

$$\tau = \frac{[\tilde{r}', \tilde{r}'', \tilde{r}''']}{|\tilde{r}' \times \tilde{r}''|^2}$$

$$= \frac{12}{4\left(9u^4 + 9u^2 + 1\right)} = \frac{3}{9u^4 + 9u^2 + 1}$$

Example 1.15. Find the curvature and torsion of the curve
$x = a(3t - t^3)$; $y = 3at^2$; $z = a(3t + t^3)$.

Solution:

$$\tilde{r} = \left(a(3t - t^3),\ 3at^2,\ a(3t + t^3)\right)$$

$$\tilde{r}' = \left(3a - 3at^2,\ 6at,\ 3a + 3at^2\right)$$

$$\tilde{r}'' = (-6at,\ 6a,\ 6at); \quad \tilde{r}''' = (-6a,\ 0,\ 6a)$$

$$\tilde{r}' \times \tilde{r}'' = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 3a - 3at^2 & 6at & 3a + 3at^2 \\ -6at & 6a & 6at \end{vmatrix} = 18a^2\left[(t^2 - 1)\tilde{i} - 2t\tilde{j} + (1 + t^2)\tilde{k}\right]$$

$$|\tilde{r}' \times \tilde{r}''| = 18a^2\sqrt{2}\,(1 + t^2)$$

$$[\tilde{r}', \tilde{r}'', \tilde{r}'''] = (\tilde{r}' \times \tilde{r}'') \cdot \tilde{r}'''$$

$$= 18a^2\left[(t^2 - 1)\tilde{i} - 2t\tilde{j} + (1 + t^2)\tilde{k}\right] \cdot \left[-6a\tilde{i} + o\tilde{j} + 6a\tilde{k}\right]$$

$$= 18a^2\left[-6a(t^2 - 1) + 6a(t^2 + 1)\right] = 216a^3$$

$$|\tilde{r}'| = 3\sqrt{2}\,a(1 + t^2)$$

$$\Rightarrow \kappa = \frac{|\tilde{r}' \times \tilde{r}''|}{|\tilde{r}'|^3}$$

$$= \frac{18a^2\sqrt{2}\,(1 + t^2)}{54\sqrt{2}\,a^3(1 + t^2)^3} = \frac{1}{3a(1 + t^2)^2}$$

$$\tau = \frac{[\tilde{r}', \tilde{r}'', \tilde{r}''']}{|\tilde{r}' \times \tilde{r}''|^2} = \frac{216a^3}{\left[18a^2\sqrt{2}\,(1 + t^2)\right]^2} = \frac{1}{3a(1 + t^2)^2}$$

Thus, $\kappa = \tau = \dfrac{1}{3a(1 + t^2)^2}$

Example 1.16. For the curve $x = 3u$; $y = 3u^2$; $z = 2u^3$. Show that $\kappa = \tau = \frac{3}{2}\left(1 + 2u^2\right)^2$.

Solution

$$\tilde{r} = (3u, 3u^2, 2u^3)$$
$$\dot{\tilde{r}} = (3, 6u, 6u^2)$$
$$\ddot{\tilde{r}} = (0, 6, 12u)$$
$$\dddot{\tilde{r}} = (0, 0, 12)$$
$$\dot{\tilde{r}} \times \ddot{\tilde{r}} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 3 & 6u & 6u^2 \\ 0 & 6 & 12u \end{vmatrix}$$
$$= 36u^2\,\tilde{i} - (36u)\,\tilde{j} + 18\tilde{k}$$
$$|\dot{\tilde{r}} \times \ddot{\tilde{r}}| = 18\sqrt{4u^4 + 4u^2 + 1}$$
$$= 18\,(2u^2 + 1)$$
$$|\dot{\tilde{r}}| = 3\sqrt{1 + 4u^2 + 4u^4}$$
$$= 3\,(1 + 2u^2)$$
$$[\dot{\tilde{r}}, \ddot{\tilde{r}}, \dddot{\tilde{r}}] = \begin{vmatrix} 3 & 2u & 6u^2 \\ 0 & 6 & 12u \\ 0 & 0 & 12 \end{vmatrix} = 216$$
$$\rho = \frac{1}{\kappa} = \frac{|\dot{\tilde{r}}|^3}{|\dot{\tilde{r}} \times \ddot{\tilde{r}}|} = \frac{3}{2}\left(1 + 2u^2\right)^2$$
$$\sigma = \frac{1}{\tau} = \frac{|\dot{\tilde{r}} \times \ddot{\tilde{r}}|^2}{[\dot{\tilde{r}}, \ddot{\tilde{r}}, \dddot{\tilde{r}}]} = \frac{3}{2}\left(1 + 2u^2\right)^2$$

Thus, $\rho = \sigma = \frac{3}{2}\left(1 + 2u^2\right)^2.$

Example 1.17. For the curve $x = a\tan\theta$; $y = a\cot\theta$; $z = a\sqrt{2}\log\tan\theta$: Prove that $\rho = \sigma = \dfrac{2\sqrt{2}a}{\sin^2 2\theta}.$

Solution:

$$\tilde{r} = a\left(\tan\theta, \cot\theta, \sqrt{2}\log\tan\theta\right)$$

Differentiating both sides with respect to arc length s, we get

$$\tilde{r}\,' = \tilde{t} = a\left(\sec^2\theta, -\operatorname{cosec}^2\theta, \frac{\sqrt{2}}{\sin\theta\cos\theta}\right)\frac{d\theta}{ds} \tag{1.26}$$

Squaring on both sides, we get

$$\tilde{t}^2 = 1 = a^2 \; \sec^4 + \operatorname{cosec}^4 + \frac{2}{\sin^2 \cos^2} \left(\frac{d}{ds}\right)^2$$

$$\text{i.e.,} \quad \left(\frac{ds}{d}\right)^2 = a^2 \frac{\sin^4 + \cos^4 + 2\sin^2 \cos^2}{\sin^4 \cos^4} \qquad (1.27)$$

$$= \frac{a^2 \left(\sin^2 + \cos^2\right)^2}{\sin^4 \cos^4}$$

$$= \frac{a^2}{\sin^4 \cos^4}$$

$$\Rightarrow \frac{ds}{d} = \frac{a}{\sin^2 \cos^2} \qquad (1.28)$$

$$\text{Now,} \quad \tilde{t} = \left(a \sec^2 ; \operatorname{cosec}^2 ; \frac{p}{\sin \cos}\right) \frac{\sin^2 \cos^2}{a}$$

$$= \left(\sin^2 ; \cos^2 ; p\, 2\sin \cos\right)$$

Differentiating with respect to s; we get

$$\tilde{t}' = \tilde{n} = \left(2\sin \cos ; 2\cos \sin ; p\, 2\cos 2\right) \frac{d}{ds}$$

$$= \left(\sin 2 ; \sin 2 ; p\, 2\cos 2\right) \frac{\sin^2 \cos^2}{a} \qquad (1.29)$$

Squaring we get

$$^2 = \frac{\sin^4 \cos^4}{a^2} \left[2 \left(\sin^2 2 + \cos^2 2\right)\right] \qquad (1.30)$$

$$= \frac{2\sin^4 \cos^4}{a^2}$$

$$\text{i.e.,} \quad = \frac{p\, 2\sin^2 \cos^2}{a} \qquad (1.31)$$

$$\text{Hence} \quad = \frac{1}{} = \frac{a}{p\, 2\sin^2 \cos^2} = \frac{2p\, 2a}{\sin^2 2}$$

Substitute (1.31) in (1.29), we get

$$\tilde{n} \frac{p\, 2\sin^2 \cos^2}{a} = \frac{\sin^2 \cos^2}{a} \left(\sin 2 ; \sin 2 ; p\, 2\cos 2\right)$$

$$\Rightarrow \tilde{n} = \frac{1}{p\, 2} \left(\sin 2 ; \sin 2 ; p\, 2\cos 2\right)$$

Differentiating with respect to s; we get

$$\tilde{n}' = \tilde{t} + \tilde{b} = \frac{1}{p\, 2} \left(2\cos 2 ; 2\cos 2 ; 2p\, 2\sin 2\right) \frac{d}{ds}$$

(by using Serret Frenet formulae)

$$\text{i.e.,} \quad \tilde{b} \; \tilde{t} = \frac{2\sin^2 \cos^2}{a\, p\, 2} \left(\cos 2 ; \cos 2 ; p\, 2\sin 2\right)$$

Squaring, we get

$$^2 + {}^2 = \frac{2\sin^4 \quad \cos^4}{a^2} \quad 2\cos^2 2 \quad + 2\sin^2 2 \quad = \frac{4\sin^4 \quad \cos^4}{a^2}$$

$$^2 = \frac{4\sin^4 \quad \cos^4}{a^2} \qquad ^2$$

$$= \frac{4\sin^4 \quad \cos^4}{a^2} \qquad \frac{2\sin^4 \quad \cos^4}{a^2}$$

$$\text{i.e.,} \quad ^2 = \frac{2\sin^4 \quad \cos^4}{a^2}$$

$$\Rightarrow \quad = \frac{\sqrt{2}\sin^2 \quad \cos^2}{a} = \frac{\sin 2}{2\sqrt{2a}}$$

(negative sign is taken for a left handed system)

$$\Rightarrow \quad = \frac{1}{\quad} = \frac{2\sqrt{2a}}{\sin^2 2}$$

$$= \quad = \frac{2\sqrt{2a}}{\sin^2 2}$$

Behaviour of a curve in the neighbourhood of one of its points:

At a point P on the curve let axes ox, oy, oz be taken along $\tilde{t}, \tilde{n}$ and $\tilde{b}$, and let X, Y, Z be the coordinates of a neighbouring point Q of the curve relative to these axes.

If the curve is of class ≥ 4 and if s denotes the small arc PQ then using Taylor's theorem, we get

$$\tilde{r}(s) = \tilde{r}(0) + s\tilde{r}'(0) + \frac{s^2}{2!}\tilde{r}''(0) + \frac{s^3}{3!}\tilde{r}'''(0) + \frac{s^4}{4!}\tilde{r}^{iv}(0) + O(s^5) \quad \text{as } s \to 0$$

Now by Serret-Frenet formulae

$$\tilde{r}'(0) = \tilde{t}; \quad \tilde{r}''(0) = \tilde{n}; \quad \tilde{r}'''(0) = {}'\tilde{n} \quad {}^2\tilde{t} + \quad \tilde{b}$$

$$\tilde{r}^{iv}(0) = {}'' \quad {}^3 \quad {}^2 \quad \tilde{n} \quad 3 \quad {}'\tilde{t} + 2 \quad ' \quad + \quad ' \quad \tilde{b}$$

At P, $\tilde{r}(0) = 0$

$$\tilde{r}(s) = s\tilde{t} + \frac{s^2}{2}\tilde{n} + \frac{s^3}{6} \quad {}'\tilde{n} \quad {}^2\tilde{t} + \quad \tilde{b}$$

$$+ \frac{s^4}{24}\Big[\quad '' \quad {}^3 \quad {}^2 \quad \tilde{n} \quad 3 \quad {}'\tilde{t} + 2 \quad ' \quad + \quad ' \quad \tilde{b}\Big]$$

But $\tilde{r}(s) = X\tilde{t} + Y\tilde{n} + Z\tilde{b}$

Equating like wise coe cients, we get

$X = s \; {}^{2}s^{3} \quad {}^{0}\_ \; s^{4} \; +$

$2 \quad {}^{0} \; 3$

$6 \quad 8$

$+ \; 1 \; {}^{00} \quad 3 \quad s^{4} \; +$

$s \quad s$ (1.32)

$Y = \; +$

4

$2 \quad 6 \quad 24$

$s^{3} \quad 1$

$+$

$Z = \; + \; (2\,{}^{0} + {}^{0})\, s$

$6 \quad 24$

—

—

24

–

It follows that as a rst order approximation the chord PQ is along the

9

–

tangent; its projection on the principal normal is a magnitude of the second order, and its projection on the binormal is of the third order.

From the above relations (1.32), two relations can be deducted which are

–

–

analogous to Newton's formula for the curvature of plane curve and these

are

–

2Y

3Y X2 as s ! 0

as s ! 0

XY

–

–

2

Further, we can easily prove that $\left(X^2 + Y^2 + Z^2\right)^{1/2} \approx s\left(1 - \frac{s}{\;}\right)$.

Example 1.18. Show that the projection of the curve near P on the osculating plane is approximately the curve $Z = 0;\ Y = \frac{1}{2} X^2$; its projection on the rectifying plane is approximately $y = 0;\ z = \frac{1}{6} x^3$ and its projection on the normal plane is approximately $x = 0;\ z^2 = \frac{2}{\;} y^3$

–

–

–

$6\,s^3$

o

X = s

–

–

–

–

$$Z = s^3$$

−

6

–

–

From above, retaining only the rst term and then we get X t s;

–

Y t s^2; z t s^3:

–

–

–

–

–

−

–

–

−

plane are respectively $Y = 0$; $Z = \frac{}{6} X^2$ and $X = 0$; $Z^2 = \frac{2}{9} \frac{2}{} Y^3$:

Example 1.19. Show that the length of the common perpendicular d of the tangents at two near points distance s; apart is approximately given by $d = \frac{s^3}{12}$:

Solution:

Let $P; Q$ have parameters 0 and s respectively. The unit tangent vectors at P and Q are $\tilde{r}'(0); \tilde{r}'(s)$; so the unit vector of the common perpendicular is along $\tilde{r}'(s) \quad \tilde{r}'(0)$: The projection of the vector $\tilde{r}(s) \quad \tilde{r}(0)$ in this direction is equal to d; so

$$d = \frac{\tilde{r}(s) \quad \tilde{r}(0); \tilde{r}'(s); \tilde{r}'(0)}{\tilde{r}'(s) \quad \tilde{r}'(0)}$$

Let Us Sum Up:

In this unit, the students acquired knowledge to

nd the equation of osculating plane at a point.

the concept of Normal Plane and Principal Plane .

derive Serret-Frenet formulae.

Check Your Progress:

1. Find the arc length of the curve $\tilde{r} = e^t \cos t; e^t \sin t; e^t$:
2. Find the osculating plane at the point $t = 1$ of a curve $\tilde{r} = 3at; 3bt^2; ct^3$:
3. Find the curvature and torsion at $= \frac{}{4}$ of the curve $\tilde{r} = (a \cos \; ; \; a \sin \; ; \; a \cos 2 \;)$:
4. Find the curvature and torsion of the curve $\tilde{r} = (a(u \quad \sin u); \; a(1 \quad \cos u); \; bu)$:
5. Find the curvature and torsion of the curve $x = a \cos u$; $y = a \sin u$; $z = au \cot$:

6. For the curve $\tilde{r} = \sqrt{6}at^3;\ a\ 1+3t^2\ ;\ \sqrt{6}at\ ;$

   show that $= \ = \dfrac{1}{a\,(1+3t^2)^2}:$

Self Assessment Problems:

1. De ne a curve.

2. De ne class of function m and regular function.

3. De ne arc length.

4. De ne the curvature and torsion.

5. Prove that a necessary and su cient condition for the curve to be a straight line is that $= 0$ and for a plane curve $= 0:$

6. Derive the Serret-Frenet formulae.

7. Derive the formula for curvature and torsion in terms of the parameters s and $u:$

Answer:

1. $s = \sqrt{3}\,(e^u \quad 1)$

2. $\frac{x}{a} \quad \frac{y}{b} + \frac{z}{c} = 1$

3. $^2 = \dfrac{5}{a^2\,(1+4a^2)^3}\ ; \quad = \dfrac{6}{5a}$

4. $= \dfrac{a}{b^2 + 4a^2\sin^4\frac{t}{2}}; \quad = \dfrac{b}{b^2 + 4a^2\sin^4\frac{t}{2}}.$

5. $= \dfrac{\sin^2}{a}; \quad = \dfrac{\sin \quad \cos}{a}$

Choose the correct or more suitable answer:

1. The plane containing the vectors $\tilde{t}$ and $\tilde{n}$ is called the $::::::$

   (a) osculating plane

(b) normal plane

(c) rectifying plane

(d) tangent plane.

2. The parametric equations for the cubic curve is given by $x = u;\ y = u^2;\ z = u^3\ (-1 < u < -1)$, then the equaton of the curve is

   (a) $x = z$ (b) $xz = y^3$ (c) $x^2z = y$ (d) $xz = y^2$:

3. A necessary and su cient condition that a curve be a straight line is that : : : : : :

   (a) > 0 at some points

   (b) > 0 at all points points

   (c) $= 0$ at some points

   (d) $= 0$ at all points.

Answer:

(1) a (2) d (3) d

Glossaries:

Torsion: In the di erential geometry of curves in three dimensions, the torsion of a curve measures how sharply it is twisting out of the osculating plane.

Suggested Readings:

1. T.J. Willmore, An Introduction to Di erential Geometry , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, Di erential Geometry of Three Dimensions , University Press, Cambridge, 1930.

Block-I

UNIT-2

INVOLUTES AND EVOLUTES

Structure

Objective

Overview

2. 1 Curvature and torsion of a curve given as the intersection of two surfaces

2. 2 Contact between curves and surfaces

2. 2. 1 Osculating circle

2. 2. 2 Osculating sphere

2. 3 Tangent surfaces, Involutes and Evolutes

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the conceptof contact between curves and surfaces.
- derive the equation of an involute and evolute.
- nd spherical indicatrix of the tangent, principal normal and binormal.

Overview

In this unit, we will illustrate how to nd the curvature and torsion of a given curve. Also we will explain the concept of osculating plane and osculating sphere.

2.1. Curvature and torsion of a curve given as the intersection of two surfaces:

Let the equation of the curve be given as the intersection of two surfaces $f(x, y, z) = 0$; $g(x, y, z) = 0$ and if a set of parametric equations can be found easily, we may proceed as follows:

We know that ∇f and ∇g are normal vectors to the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ respectively.

Therefore unit tangent vector $\tilde{t}$ is parallel to $\nabla f \times \nabla g$.

Let $\nabla f \times \nabla g = \tilde{h}$. Then $\tilde{t}$ is parallel to $\tilde{h}$.

Therefore $\tilde{h} = \lambda \tilde{t}$, for some constant λ.

$$\Rightarrow \tilde{h} = \lambda \tilde{r}' \quad [* \tilde{r}' = \tilde{t}]$$

$$\Rightarrow h_1 \tilde{i} + h_2 \tilde{j} + h_3 \tilde{k} = \lambda (x' \tilde{i} + y' \tilde{j} + z' \tilde{k})$$

Equating likewise terms, we get

$$h_1 = \lambda x'; \quad h_2 = \lambda y'; \quad h_3 = \lambda z'$$

$$\text{i.e.,} \quad x' = \frac{h_1}{\lambda}; \quad y' = \frac{h_2}{\lambda}; \quad z' = \frac{h_3}{\lambda}$$

Now, by total di erentiation formula

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$\Rightarrow \frac{df}{ds} = x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{df}{ds} = \frac{h_1}{}\frac{\partial f}{\partial x} + \frac{h_2}{}\frac{\partial f}{\partial y} + \frac{h_3}{}\frac{\partial f}{\partial z}$$

Multiplying both sides by ; we get

$$\frac{df}{ds} = h_1\frac{\partial f}{\partial x} + h_2\frac{\partial f}{\partial y} + h_3\frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{d}{ds} = h_1\frac{\partial}{\partial x} + h_2\frac{\partial}{\partial y} + h_3\frac{\partial}{\partial z} = \quad \text{(say)}$$

$$\Rightarrow \frac{d}{ds} = \qquad (2.1)$$

$$\text{Also,} \quad \tilde{t} = \tilde{h} \qquad (2.2)$$

Operating (2.1) in (2.2), we get

$$\frac{d}{ds}\tilde{t} = \tilde{h}$$

$$\Rightarrow \tilde{t}' + '\tilde{t} = \tilde{h} \qquad (2.3)$$

$$\text{i.e.,} \quad {}^2\tilde{n} + '\tilde{t} = \tilde{h} \quad \text{[by Serret-Frenet formulae]} \qquad (2.4)$$

Taking vector product of (2.2) and (2.4), we get

$$\tilde{t} \quad {}^2\tilde{n} + '\tilde{t} = \tilde{h} \quad \tilde{h}$$

$${}^3\tilde{b} + {}^2\,'(0) = \tilde{h} \quad \tilde{h}$$

$$\Rightarrow {}^3\tilde{b} = \tilde{} \quad \text{where } \tilde{} = \tilde{h} \quad \tilde{h} \qquad (2.5)$$

$$\Rightarrow {}^3\,b =$$

$$\text{i.e.,} \quad {}^3 = \tilde{}\ \tilde{}$$

$$\text{i.e.,} \quad = \frac{}{3}; \quad \text{which gives} \quad :$$

Now, operating (2.1) and (2.5), we get

$$\frac{d}{ds}\, {}^3\tilde{b} = \tilde{}$$

$${}^4\tilde{n} + {}^3\,'\tilde{b} = \tilde{} \qquad (2.6)$$

Taking scalar product of (2.4) and (2.6), we get

$$^{2}\tilde{n} + {}^{o}\tilde{t} \quad {}^{4}\tilde{n} + {}^{3}\,{}^{o}\tilde{b} = \tilde{h}$$

$$\Rightarrow \quad {}^{6}\,{}^{2} = \tilde{h}$$

$$\text{i.e.,} \quad = \frac{\tilde{h}}{{}^{6}\,{}^{2}}; \quad \text{which gives} \quad :$$

Example 2.1. Find the curvature and torsion of the curve of intersection of two quadric surfaces $ax^2 + by^2 + cz^2 = 1$ and $a'x^2 + b'y^2 + c'z^2 = 1$.

Solution:

$$\text{Let } f = ax^2 + by^2 + cz^2 - 1$$

$$g = a'x^2 + b'y^2 + c'z^2 - 1$$

We know that ∇f is normal to the surface $f = 0$ and ∇g is normal to the surface $g = 0$.

$$\nabla f = \left(\tilde{i}\frac{\partial}{\partial x} + \tilde{j}\frac{\partial}{\partial y} + \tilde{k}\frac{\partial}{\partial z}\right)\left(ax^2 + by^2 + cz^2 - 1\right)$$

$$\text{i.e.,} \quad \nabla f = 2ax\tilde{i} + 2by\tilde{j} + 2cz\tilde{k}$$

$$\text{Similarly,} \quad \nabla g = 2a'x\tilde{i} + 2b'y\tilde{j} + 2c'z\tilde{k}$$

$$\Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 2ax & 2by & 2cz \\ 2a'x & 2b'y & 2c'z \end{vmatrix}$$

$$\text{i.e.,} \quad \nabla f \times \nabla g = 4\left(bc' - b'c\right)yz\tilde{i} + 4\left(a'c - ac'\right)xz\tilde{j} + 4\left(ab' - a'b\right)xy\tilde{k}$$

$$\text{i.e.,} \quad \nabla f \times \nabla g = 4xyz\left(\frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k}\right) = 4Ayz\tilde{i} + 4Bxz\tilde{j} + Cxyk$$

$$\text{where } A = bc' - b'c;\; B = ca' - c'a;\; C = ab' - a'b$$

Since the unit tangent vector $\tilde{t}$ parallel to $\nabla f \times \nabla g$, we can take

$$\frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k} = \lambda\tilde{t} \tag{2.7}$$

$$\Rightarrow \frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k} = \lambda\tilde{r}' \quad [\because \tilde{t} = \tilde{r}']$$

$$\Rightarrow \frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k} = \lambda\left(\frac{dx}{ds}\tilde{i} + \frac{dy}{ds}\tilde{j} + \frac{dz}{ds}\tilde{k}\right)$$

Equating like-wise coefficients, we get

$$\frac{dx}{ds} = \frac{1}{\lambda}\frac{A}{x}; \quad \frac{dy}{ds} = \frac{1}{\lambda}\frac{B}{y}; \quad \frac{dz}{ds} = \frac{1}{\lambda}\frac{C}{z} \tag{2.8}$$

Now, if F is any scalar or vector function

$$\frac{dF}{ds} = \frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} \qquad (2.9)$$

$$\Rightarrow \frac{dF}{ds} = \frac{A}{x}\frac{\partial F}{\partial x} + \frac{B}{y}\frac{\partial F}{\partial y} + \frac{C}{z}\frac{\partial F}{\partial z} \quad \text{(using (2.8))} \qquad (2.10)$$

This formula converts the derivatives with respect to arc length s in to derivatives with respect to co-ordinates.

$$\text{Equation (2.7)} \Rightarrow \tilde{t} = \frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k} \qquad (2.11)$$

Operating (2.10) on (2.11), we get

$$\frac{d}{ds}\tilde{t} = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\left(\frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k}\right)$$

$$\Rightarrow {}^2\tilde{n} + {}^o\tilde{t} = -\frac{A^2}{x^3}\tilde{i} - \frac{B^2}{y^3}\tilde{j} - \frac{C^2}{z^3}\tilde{k} \qquad (2.12)$$

Vector cross multiplying (2.11) and (2.12), we get

$$\tilde{t} \times \left({}^2\tilde{n} + {}^o\tilde{t}\right) = \left(\frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k}\right) \times \left(-\frac{A^2}{x^3}\tilde{i} - \frac{B^2}{y^3}\tilde{j} - \frac{C^2}{z^3}\tilde{k}\right)$$

$${}^3\tilde{b} = \frac{BC}{y^3z^3}\left(Bz^2 - Cy^2\right)\tilde{i} + \frac{AC}{x^3z^3}\left(Cx^2 - Az^2\right)\tilde{j} + \frac{AB}{x^3y^3}\left(Ay^2 - Bx^2\right)\tilde{k} \qquad (2.13)$$

$$\text{Now, } Bz^2 - Cy^2 = (ca' - c'a)z^2 - (ab' - a'b)y^2$$

$$= a' - a \quad \text{(after simplification)}$$

$$\text{Similarly, } Cx^2 - Az^2 = b' - b; \quad Ay^2 - Bx^2 = c' - c$$

$$\Rightarrow {}^3\tilde{b} = \frac{BC}{y^3z^3}(a' - a)\tilde{i} + \frac{CA}{z^3x^3}(b' - b)\tilde{j} + \frac{AB}{x^3y^3}(c' - c)\tilde{k}$$

$$= \frac{ABC}{x^3y^3z^3}\left[\frac{x^3}{A}(a' - a)\tilde{i} + \frac{y^3}{B}(b' - b)\tilde{j} + \frac{z^3}{C}(c' - c)\tilde{k}\right]$$

Taking modulus and then squaring on both sides, we get

$$^6\,^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\left[\frac{x^6}{A^2}(a' - a)^2 + \frac{y^6}{B^2}(b' - b)^2 + \frac{z^6}{C^2}(c' - c)^2\right]$$

$$\text{i.e.,} \; ^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\left[\sum \frac{x^6}{A^2}(a' - a)^2\right]$$

This gives the value of :

Note 2.1.

$$\tilde{t} = \frac{A}{x}\tilde{i} + \frac{B}{y}\tilde{j} + \frac{C}{z}\tilde{k}$$

$$\text{i:e:;} \quad {}^2 = \frac{A^2}{x^2} + \frac{B^2}{y^2} + \frac{C^2}{z^2}$$

$$\text{i:e:;} \quad {}^6 = \sum \frac{A^2}{x^2}$$

Next, we have to calculate :

$$\frac{{}^3 x^3y^3z^3}{ABC}\tilde{b} = \frac{x^3}{A}\left(a^o - a\right)\tilde{i} + \frac{y^3}{B}\left(b^o - b\right)\tilde{j} + \frac{z^3}{X}\left(c^o - c\right)\tilde{k}$$

$$\tilde{b} = \frac{x^3}{A}\left(a^o - a\right)\tilde{i} + \frac{y^3}{B}\left(b^o - b\right)\tilde{j} + \frac{z^3}{X}\left(c^o - c\right)\tilde{k} \qquad (2.14)$$

$$\text{where} \quad = \frac{{}^3 x^3y^3z^3}{ABC}$$

Again operating (2.10) on (2.14), we get

$$\frac{d}{ds}\tilde{b} = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\left(\frac{x^3}{A}\left(a^o - a\right)\tilde{i} + \frac{y^3}{B}\left(b^o - b\right)\tilde{j} + \frac{z^3}{C}\left(c^o - c\right)\tilde{k}\right)$$

$$\Rightarrow \quad \tilde{b}^o + \tilde{b}^o = \frac{A}{x}\frac{3x^2}{A}\left(a^o - a\right)\tilde{i} + \frac{B}{y}\frac{3y^2}{B}\left(b^o - b\right)\tilde{j} + \frac{C}{z}\frac{3z^2}{B}\left(c^o - c\right)\tilde{k}$$

$$\Rightarrow \quad \tilde{n} + {}^o\tilde{b} = 3x\left(a^o - a\right)\tilde{i} + 3y\left(b^o - b\right)\tilde{j} + 3z\left(c^o - c\right)\tilde{k}$$

$$\text{i:e:;} \quad \tilde{n} + {}^o\tilde{b} = 3x\left(a^o - a\right)\tilde{i} + 3y\left(b^o - b\right)\tilde{j} + 3z\left(c^o - c\right)\tilde{k} \qquad (2.15)$$

Taking scalar product of (2.12) and (2.15), we get

$$\left({}^2\tilde{n} + {}^o\tilde{t}\right)\cdot\left({}^o\tilde{n} + {}^o\tilde{b}\right) = \left(\frac{A}{x^3}\tilde{i} + \frac{B}{y^3}\tilde{j} + \frac{C}{z^3}\tilde{k}\right)\cdot\left(3x\left(a^o - a\right)\tilde{i} + 3y\left(b^o - b\right)\tilde{j} + 3z\left(c^o - c\right)\tilde{k}\right)$$

$$\Rightarrow \quad {}^3(1) + 0 + 0 + 0 = \frac{3xA}{x^3}\left(a^o - a\right) + \frac{3yB}{y^3}\left(b^o - b\right) + \frac{3zC}{z^3}\left(c^o - c\right)$$

$${}^3 = 3\sum \frac{A^2}{x^2}\left(a^o - a\right)$$

$$\text{:e:;} \quad = \frac{3\sum \frac{A}{x^2}\left(a^o - a\right)}{{}^3}$$

$$= \frac{3ABC\sum \frac{A^2}{x^2}\left(a^o - a\right)}{{}^6\,{}^2x^3y^3z^3}$$

$$) = \frac{3x^3y^3z^3}{ABC} \quad \frac{x^2}{x^6} \quad \sum A^2 \quad a^{o} \quad a \quad \frac{}{2} \quad : \quad \sum \quad a^2 \; a \quad A^2$$

This gives the value of :

2.2. Contact between curves and surfaces:

Let $\tilde{r} = \tilde{r}(u)$ be a curve and $F(x; y; z) = 0$ be a surface, then the point of intersection of curve and surface is given by the parameter value u which are the roots of the equation $F(x(u); y(u); z(u)) = 0$ or $F(u) = 0$:

Note 2.2. If u_0 is a root of $F(u) = 0$ then $F(u_0) = 0$:

If $F^{o}(u_0) = 0$ but $F^{oo}(u_0) \neq 0$; then we can say that the curve and surface have two point of contact at $\tilde{r}(u_0)$:

If $F^{o}(u_0) = 0$, $F^{oo}(u_0) = 0$; but $F^{ooo}(u_0)$ 0 then we can say that the curve and surface have three point contact at $\tilde{r}(u_0)$:

In general, if $F^{o}(u_0) = F^{oo}(u_0) = = F^{(n\ 1)}(u_0) = 0$; but $F^{(n)}(u_0) \neq 0$; then we can say that the curve and surface have n-point of contact at $\tilde{r}(u_0)$:

2.2.1. Osculating circle:

De nition 2.1. A curve in the osculating plane which has three point of contact with the curve at P is called osculating circle at P:

Bookwork 2.1. Derive the equation of the osculating circle

Proof. We know that the section of the sphere by a plane is a circle. Let Osculating circle in the osculating plane be given by as the intersection of the plane and the sphere.

$$\tilde{r} \quad \tilde{c} \quad = \qquad (2.16)$$

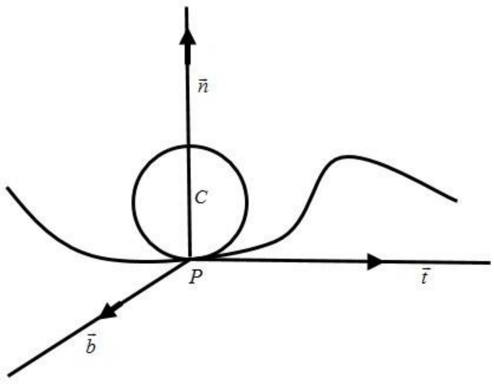


Figure 2.1: Osculating circle

where $\tilde{r}$ is the position vector of the generic point and $\tilde{c}$ is the position vector of the centre C and a is the radius of the sphere.

Let the equation of the curve be $\tilde{r} = \tilde{r}(s)$: The point of intersection of the curve and sphere is given by

$$F(s) \quad = \quad \tilde{r} \quad \tilde{c}^{\,2} \quad a^2 = 0 \qquad (2.17)$$

The condition for three point of contact are $F = F^{o} = F^{oo} = 0$:

Di erentiate (2.17) with respect to s we get

$$\tilde{r} \quad \tilde{c}^{\,2} \quad = \quad a^2$$

$$\tilde{r} \quad \tilde{c} \quad \tilde{r}^{\,o} \quad = \quad 0 \quad (i:e:;) \ \tilde{r} \quad \tilde{c} \quad \tilde{t} = 0 \qquad (* \ \tilde{r}^{o} = \tilde{t}) \qquad (2.18)$$

Again di erentiating with respect to s; we get

$$\tilde{r} \quad \tilde{c} \quad \tilde{r}^{\,oo} + \tilde{r}^{\,o} \quad \tilde{r}^{\,o} \quad = \quad 0$$

$$i:e:; \quad \tilde{r} \quad \tilde{c} \quad \tilde{n} + +\tilde{t} \quad \tilde{t} \quad = \quad 0$$

$$i:e:; \quad \tilde{r} \quad \tilde{c} \ \tilde{n} = \frac{1}{\ } = \qquad (2.19)$$

Equation (2.18) shows that $\tilde{r} \quad \tilde{c}$ lies in the normal plane at P: But by de nition, it also lies in the osculating plane at P: Hence $\tilde{r} \quad \tilde{c}$ must be along the line of intersection of the osculating plane and the normal plane, thus it must lie along $\tilde{n}$:

$$\tilde{r} \quad \tilde{c} \quad = \quad \tilde{n} \ \text{ where } \quad \text{is any scalar:} \qquad (2.20)$$

Substitute (2.20) in (2.17) and (2.19), we get

$$a = \quad ; \quad =$$

Thus, the position vector of the centre of osculating circle is given by

$$\tilde{c} = \tilde{r} \quad \tilde{n} = \tilde{r} + \quad \tilde{n} \qquad \text{(using(2.20))} \tag{2.21}$$

2.2.2. Osculating sphere:

De nition 2.2. The osculating sphere at a point P is de ned as the sphere which has four point of contact with the curve at P.

Bookwork 2.2. Derive the equation of the osculating sphere

Proof. Let $\tilde{c}$ be the position vector and R be the radius of the sphere. Then its equation is given by $\tilde{r} \quad \tilde{c}^{\,2} = R^2$ where $\tilde{R}$ is the position vector of the generic point. The point of intersection of the curve $\tilde{r} = \tilde{r}(s)$ with the sphere is given by

$$F(s) = \tilde{r} \quad \tilde{c}^{\,2} \quad R^2 \tag{2.22}$$

The condition for four point of contact are

$$F(s) = 0; \quad F^{o}(s) = 0; \quad F^{oo}(s) = 0; \quad F^{ooo}(s) = 0$$

Di erentiate (2.22) thrice with respect to s; we get

$$\tilde{r} \quad \tilde{c}^2 = R^2$$
$$\tilde{r} \quad \tilde{c} \quad \tilde{r}^{\,o} = 0$$
$$\tilde{r} \quad \tilde{c} \quad \tilde{r}^{\,oo} + \tilde{r}^{\,o} \quad \tilde{r}^{\,o} = 0$$
$$\tilde{r} \quad \tilde{c} \quad \tilde{r}^{\,ooo} + \tilde{r}^{\,o} \quad \tilde{r}^{\,oo} + 2\tilde{r}^{\,o} \quad \tilde{r}^{\,oo} = 0$$

We know that

$$\tilde{r} = \tilde{t}; \quad \tilde{r}^{\,o} \quad \tilde{r}^{\,o} = 1; \tilde{r}^{\,oo} = \tilde{t}^{\,o} = \quad \tilde{n}$$
$$\tilde{r}^{\,o} \quad \tilde{r}^{\,oo} = \tilde{t} \quad \tilde{n} = 0$$
$$\tilde{r}^{\,ooo} = \tilde{t}^{\,oo} = \quad \tilde{n}^{\,o} = {}^{o}\tilde{n} + \quad \tilde{n}^{\,o} = {}^{o}\tilde{n} + \quad \tilde{t} + \quad \tilde{b}$$

Using the above relations, we get

$$(\tilde{r} - \tilde{c})^2 = R^2 \qquad (2.23)$$

$$(\tilde{r} - \tilde{c}) \cdot \tilde{t} = 0 \qquad (2.24)$$

$$(\tilde{r} - \tilde{c}) \cdot \tilde{n} = \qquad (2.25)$$

$$(\tilde{r} - \tilde{c}) \cdot \tilde{b} \quad \tilde{t} + {}^{0}\tilde{n} = 0 \qquad (2.26)$$

Using (2.24) and (2.25) in (2.26), we get

$$(\tilde{r} - \tilde{c}) \cdot \tilde{b} = \frac{{}^{0}}{} = {}^{0} \qquad (2.27)$$

$$\text{where } \frac{1}{} = \; ; {}^{0} = \frac{{}^{0}}{{}^{2}} ; \; = \frac{1}{} \qquad (2.28)$$

From (2.24), we see that $\tilde{r} - \tilde{c}$ is perpendicular to $\tilde{t}$:

Thus we can express $\tilde{r} - \tilde{c}$ as a linear combination of $\tilde{n}$ and $\tilde{b}$

$$\tilde{r} - \tilde{c} = \tilde{n} + \tilde{b}$$

$$\tilde{c} = \tilde{r} + \tilde{n} + {}^{0}\tilde{b} \qquad (2.29)$$

$$R = {}^{2} + {}^{2} = {}^{2} + {}^{2} \, {}^{0\,2} = \frac{{}^{2}\,{}^{2} + {}^{0\,2}}{{}^{4}\,{}^{2}} \qquad (2.30)$$

Equation (2.30) gives the radius of spherical curvature.

Again ${}^{0} = 0$ then is constant. So (2.30) gives $R =$ and (2.29) gives $\tilde{c} = \tilde{r} + \tilde{n}$:

Centre of osculating sphere coincides with the osculating circle.

Example 2.2. Show that the osculating plane at P has in general three point contact with the curve at P:

Solution:

Let Q be a neighbouring point of P and the arc $PQ = s$: Then $\tilde{r}(s)$ can be expanded in a Taylor series as

$$\tilde{r}(s) = \tilde{r}(0) + \frac{\tilde{r}^{\,0}(0)}{1!} s + \frac{\tilde{r}^{\,00}(0)}{2!} s^2 + \frac{\tilde{r}^{\,000}(0)}{3!} s^3 + \cdots$$

$$\text{i:e:;} \quad \tilde{r}(s) - \tilde{r}(0) = \frac{\tilde{r}^{\,0}(0)s}{1!} + \frac{\tilde{r}^{\,00}(0)s^2}{2!} + \frac{\tilde{r}^{\,000}(0)s^3}{3!}$$

(neglecting higher powers of s)

From the equation of osculating plane, we have

$$F(s) = \tilde{r}(s) \quad \tilde{r}(0); \tilde{r}^{o}(0); \tilde{r}^{oo}(0)$$

$$= \frac{\tilde{r}^{o}(0)s}{1!} + \frac{\tilde{r}^{oo}(0)s^2}{2!} + \frac{\tilde{r}^{ooo}(0)s^3}{3!} \quad ; \tilde{r}^{o}(0); \tilde{r}^{oo}(0)$$

$$= \frac{s^3}{6} \tilde{r}^{o}(0); \tilde{r}^{oo}(0); \tilde{r}^{ooo}(0) = \frac{s^3}{6}$$

$$) \; F(0) + \frac{F^{o}(0)s}{1!} + \frac{F^{oo}(0)s^2}{2!} + \frac{F^{ooo}(0)s^3}{3!} = \frac{s^3}{6} \quad 3$$

Equating likewise coe cients, we get

$$F(s) = 0; \quad F^{o}(0) = 0; \quad F^{oo}(0) = 0 \text{ and } \frac{F^{ooo}(0)}{3!} = \frac{3}{6} \quad 0$$

Thus we have $F(s) = 0; \quad F^{o}(0) = 0; \quad F^{oo}(0) = 0 \quad F^{ooo}(0) \neq 0$:

Hence the osculating plane has three-point contact at P:

Example 2.3. If the radius of spherical curvature is constant, prove that the curve either lies on a sphere or has constant curvature.

Solution:

The radius of spherical curvature at R is given by

$$R^2 = \quad + \quad ^2 \qquad (2.31)$$

Di erentiating both sides with respect to s; we get

$$0 = 2^{o} + 2 \; ^{o} \; \frac{d}{ds} \; ^{o}$$

$$i.e.; \quad + \frac{d}{ds} \; ^{o} \; ^{o} = 0$$

$$) \text{ Either}^{o} = 0 \text{ or } + \frac{d}{ds} \; ^{o} = 0$$

Case 1:

$$^{o} = 0 \;) = \text{constant}$$

$$) \; \frac{1}{} = \text{constant}$$

$$i.e.; \quad = \text{constant}$$

Thus, the curvature is constant.

Case 2:

$$+ \frac{d}{ds} \; ^{o} = 0 \qquad (2.32)$$

Centre of curvature $\tilde{C}$ is given by

$$\tilde{C} = \tilde{r} + \tilde{n} + {}^{0} \tilde{b}$$

$$\frac{d\tilde{C}}{ds} = \tilde{r}^{0} + {}^{0}\tilde{n} + {}^{0}\tilde{n} + {}^{0} \tilde{b}^{0} + \frac{d}{ds} \tilde{b}^{0} \tilde{}$$

$$= \tilde{t} + \tilde{t} + \tilde{b} + {}^{0}\tilde{n} + {}^{0} \tilde{n} + \frac{d}{ds} {}^{0} \tilde{b}$$

(using Serret Frenet formulae)

$$= -\tilde{b} + \frac{d}{ds} {}^{0} \tilde{b} = - + \frac{d}{ds} {}^{0} \tilde{b} = 0 \tilde{b} = 0$$

$$i.e., \quad \frac{d\tilde{C}}{ds} = 0$$

Therefore, $\tilde{C}$ is a constant vector.

i.e., the centre of the osculating sphere is a xed point. Also by given the radius is constant.

Hence the osculating sphere is a xed sphere and the given curve lies on this sphere.

Example 2.4. Prove that the necessary and su cient condition that a curve lies on a sphere is that $- + \frac{d}{ds}({}^{0}) = 0$ at every point on the curve.

Proof. Necessary part:If the curve lies on a sphere, then the sphere will be the osculating sphere for every point on the curve, so that radius of osculating sphere R is constant. We have,

$$R^2 = + {}^{2} \qquad (2.33)$$

Di erentiating both sides with respect to s; we get

$$\Rightarrow 0 = 2 {}^{0} + 2 {}^{0} \frac{d}{ds} {}^{0} \qquad \text{[* R is a constant]}$$

$$2 {}^{0} + \frac{d}{ds} {}^{0} = 0 \Rightarrow - + \frac{d}{ds} {}^{0} = 0$$

Thus, the condition is necessary.

Su cient Part: Assume that the condition $- + \frac{d}{ds}({}^{0}) = 0$ is satis ed at every point on the curve.

$$- + \frac{d}{ds}\ {}^{0} = 0$$

$$\text{i.e.,} \quad + \frac{d}{ds}\ {}^{0} = 0$$

$$\text{i.e.,} \quad 2\ {}^{0} + 2\ {}^{0} \frac{d}{ds}\ {}^{0} = 0 \quad \text{(Multiplying both sides by } 2\ {}^{0}\text{)}$$

$$\Rightarrow d\ {}^{2} + d\ {}^{0\ 2} = 0$$

$$\text{Integrating,} \quad {}^{2} + {}^{0\ 2} = \text{constant}$$

$$\Rightarrow R^2 = \text{constant} \Rightarrow R = \text{constant}$$

Also, we have the centre of the osculating sphere $\tilde{C}$ as $\tilde{C} = \tilde{r} + \tilde{n} + ({}^{0})\tilde{b}$

$$\text{i.e.,} \quad \frac{d\tilde{C}}{ds} = 0\ \tilde{b} = 0$$

Therefore $\tilde{C}$ is a constant vector i.e., the centre of the osculating sphere is a xed point, already we have proved that $R =$ constant.

i.e., The given curve must lie on a sphere. Hence, the condition is su cient.

Example 2.5. Find the equation of the osculating sphere and osculating circle at $(1; 2; 3)$ on the curve $x = 2t + 1;\ y = 3t^2 + 2;\ z = 4t^3 + 3.$

Solution: Given that $\tilde{r} = 2t + 1;\ 3t^2 + 2;\ 4t^3 + 3$:

At $t = 0$ $(1;\ 2;\ 3)$ is a point on the curve.

Di erentiating both sides with respect to $s;$ we get

$$\tilde{r} = 2; 6t; 12t^2 = (2; 0; 0) \quad \text{at } t = 0$$

$$\tilde{r} = (0; 6; 24t) = (0; 6; 0) \quad \text{at } t = 0$$

$$\dddot{\tilde{r}} = (0; 0; 24) = (0; 0; 24) \quad \text{at } t = 0$$

Let the equation of the osculating sphere be $\quad \tilde{r} \quad \tilde{C}^{\ 2} = R^2$ (2.34)

Where $\tilde{c}$ is the position vector of the centre ; R is the radius and $\tilde{c} = a\tilde{i} + b\tilde{j} + c\tilde{k}$

Now for a four point contact at $\tilde{r};$ we have di erentiate (2.34) with respect to $t;$ we get

$$\tilde{r} \quad \tilde{c} \quad \tilde{r} = 0 \qquad (2.35)$$

$$\tilde{r} \quad \tilde{c} \quad \tilde{r} + \tilde{r}^{\ 2} = 0$$

$$\tilde{r} \quad \tilde{c} \quad \dddot{\tilde{r}} + 3\tilde{r}\,\tilde{r} = 0$$

At $t = 0$; the (2.35) reduces to

$$\left[\left(\tilde{i} + 2\tilde{j} + 3\tilde{k}\right) - \left(a\tilde{i} + b\tilde{j} + c\tilde{k}\right)\right] \cdot 2\tilde{i} = 0$$

$$\text{i.e.,} \quad (1 - a)\,2 = 0 \Rightarrow a = 1$$

Similarly, $b = \dfrac{8}{3}; \quad c = 3.$

Osculating sphere (2.34) passes through $(1, 2, 3)$ is

$$\left[\left(\tilde{i} + 2\tilde{j} + 3\tilde{k}\right) - \left(\tilde{i} + \frac{8}{3}\tilde{j} + 3\tilde{k}\right)\right]^2 = R^2$$

$$\text{i.e.,} \quad R^2 = \frac{4}{9} \Rightarrow R = \frac{2}{3}$$

Hence the equation of the osculating sphere is

$$(x-1)^2 + \left(y - \frac{8}{3}\right)^2 + (z-3)^2 = \frac{4}{9}$$

$$\text{i.e.,} \quad 3x^2 + 3y^2 + 3z^2 - 6x - 16y - 18z + 50 = 0$$

The osculating circle is the intersection of the osculating plane and the osculating sphere.

$$\left[\tilde{R} - \tilde{r},\ \tilde{r}',\ \tilde{r}''\right] = 0$$

At $t = 0$; we have

$$\left[(x-1)\tilde{i} + (y-2)\tilde{j} + (z-3)\tilde{k}\right] \cdot 12\tilde{k} = 0$$

$$\text{i.e.,} \quad z - 3 = 0$$

Hence the equation of the osculating circle is

$3x^2 + 3y^2 + 3z^2 - 6x - 16y - 18z + 50 = 0; \quad z - 3 = 0.$

2.3. Tangent surfaces, Involutes and Evolutes:

Definition 2.3. If there is a one-one correspondence between points of two curves C and C_1 such that the tangent at any point to C is a normal to the corresponding point of C_1 is called an involute of C and C is called an evolute of C_1.

Bookwork 2.3. Find involute of a given curve

Let $\tilde{r} = \tilde{r}(s)$ be the given space curve C; C_1 be an involute of C. The quantities belonging to curve C_1 will be denoted by the suffix. Then the position vector $\tilde{r}_1$ of any point P_1 on C_1 is given by

$$\tilde{r}_1 = \tilde{r} + \tilde{t} \qquad (2.36)$$

where is to be determined.

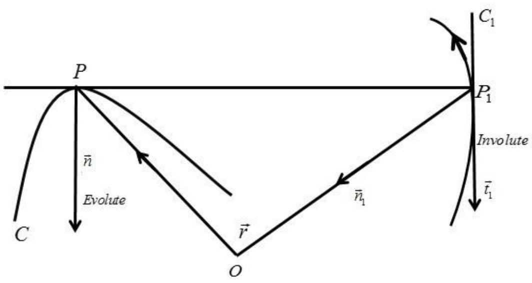


Figure 2.2: Involute and Evolute

Di erentiate (2.36) with respect to $s;$ we get

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{r}}{ds} + \frac{d\tilde{t}}{ds} + {}^{o}\tilde{t}$$

$$i.e.; \quad \tilde{t}_1\frac{ds_1}{ds} = \tilde{t} + \tilde{n} + {}^{o}\tilde{t}$$

$$\Rightarrow \tilde{t}_1\frac{ds_1}{ds} = 1 + {}^{o}\ \tilde{t} + \tilde{n} \qquad (2.37)$$

Taking dot product on both sides with $\tilde{t};$ we get

$$1 + {}^{o}\frac{ds}{ds_1} = 0 \quad \text{using } \tilde{t}\ \tilde{t}_1 = 0$$

$$i.e.; \quad 1 + {}^{o} = 0$$

Integrating, we get

$$s + \ = c \quad \text{where } s \text{ is an arbitrary constant}$$

$$\Rightarrow \tilde{r}_1 = \tilde{r} + (c \quad s)\tilde{t}$$

This is the required equation of involute C_1 of C:

Substitute the value of in (2.37), the unit tangent vector $\tilde{t}_1$ is given by

$$\tilde{t}_1 = (c \quad s)\frac{ds}{ds_1}\tilde{n} \qquad (* \ {}^{o} = 1) \qquad (2.38)$$

From above, we see that $\tilde{t};$ is parallel to $\tilde{n}:$ Taking the positive direction along the involute such that $\tilde{t}_1 = \tilde{n}$; we get

$$\frac{ds_1}{ds} = (c \quad s)$$

Bookwork 2.4. Find the equation of an evolute of a given curve C

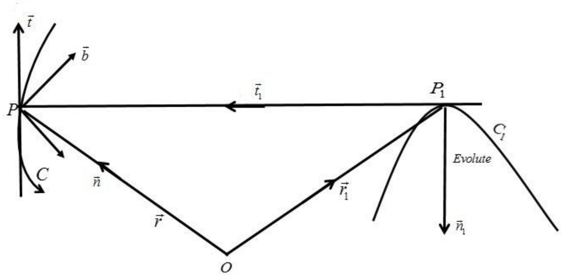


Figure 2.3: Evolute

Let $\tilde{r} = \tilde{r}(s)$ be the curve. Here, we shall use the notation su x to denote the quantities belonging to the curve C_1: Let $\tilde{r}_1$ be the position vector of P_1 on C_1: Let $\tilde{r}$ be the position vector of P on C: Since the tangents to curve C_1 are normals to the curve C; the point P_1 must lie in the normal plane to the curve at P:

$$\tilde{r}_1 = \tilde{r} + \tilde{n} + \tilde{b} \quad (2.39)$$

$$\text{i:e:;} \quad \tilde{r}_1 \quad \tilde{r} = \tilde{n} + \tilde{b}$$

Where and are to be determined.

Di erentiate with respect to s; we get

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{r}}{ds} + {}^0\tilde{n} + \frac{d\tilde{n}}{ds} + {}^0\tilde{b} + \frac{d\tilde{b}}{ds}$$

$$\tilde{t}_1 = (1 \quad)\tilde{t} + {}^0 \quad \tilde{t} + {}^0 \quad \tilde{n} + {}^0 + \quad \tilde{b} \quad \frac{ds}{ds_1} \quad (2.40)$$

Since $\tilde{t}_1$ lies in the normal plane at P to the curve C; so it must be parallel to $\tilde{n} + \tilde{b}$:

Comparing like-wise coe cients of equation (2.39), we get

$$1 = 0 \Rightarrow = \frac{1}{\ } =$$

$$\frac{{}^{0}}{\ } = \frac{{}^{0} +}{\ }$$

$$\Rightarrow = \frac{{}^{0} \quad {}^{0}}{{}^{2} + {}^{2}} = \frac{d}{ds}\tan^{1}\left(\frac{\ }{\ }\right)$$

Upon integration, we get

$$a + \int ds = \tan^{1}\frac{\ }{\ } \quad \text{where } a \text{ is a constant}$$

$$= \tan\left(\int ds + a\right)$$

i:e:; $= \cot\left(\int ds + a\right)$ or $= \cot\left(\int ds + a\right)$ (* =)

Thus, equation (2.39) becomes,

$$\tilde{r}_1 = \tilde{r} + \tilde{n} + \cot\left(\int ds + a\right)\tilde{b}$$

which is the required equation of evolute C_1 of C: Bookwork

2.5. Find the curvature $_1$ and torsion $_1$ of the involute.

Solution:

The equation of the involute is $\tilde{r}_1 = \tilde{r} + (c \quad s)\tilde{t}$

Differentiating both sides with respect to s; we get

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \tilde{r}^{0} \quad \tilde{t} + (c \quad s)\tilde{t}^{0}$$

i:e:; $\tilde{t}_1\dfrac{ds_1}{ds} = \tilde{t} \quad \tilde{t} + (c \quad s)\ \tilde{n} \qquad \left(\tilde{t}_1 = \dfrac{d\tilde{r}_1}{ds_1} = \text{unit tangent of the involute at } P\right)$

$$\Rightarrow \tilde{t}_1\frac{ds_1}{ds} = (c \quad s)\ \tilde{n} \tag{2.41}$$

This shows that the unit tangent $\tilde{t}_1$ of the involute is parallel to the unit normal $\tilde{n}$ of the given curve.

Taking the positive direction along the involute, we get

$$\tilde{t}_1 = \tilde{n} \quad \text{and} \tag{2.42}$$

$$\frac{ds_1}{ds} = (c \quad s) \tag{2.43}$$

Now, Di erentiating equation (2.42) with respect to s; we get

$$\frac{d\tilde{t}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{n}}{ds}$$

$$i.e.; \quad {}_1\tilde{n}_1 = \tilde{t} + \tilde{b} \quad \frac{ds}{ds_1}$$

$$i.e.; \quad {}_1\tilde{n}_1 = \frac{\tilde{b} \quad \tilde{t}}{(c \quad s)} \qquad \text{(using(2.43))} \qquad (2.44)$$

Squaring both sides of equation (2.44), we get

$$\,^2_1 = \frac{^2 + ^2}{^2(c \quad s)^2}$$

$$\Rightarrow \quad _1 = \frac{p\overline{^2 + ^2}}{(c \quad s)} \qquad (2.45)$$

From equation (2.44), we have

$$\tilde{n}_1 = \frac{\tilde{b} \quad \tilde{t}}{_1(c \quad s)^2} = \frac{\tilde{b} \quad \tilde{t} \quad (c \quad s)}{p\overline{^2 + ^2}\,(c \quad s)} = \frac{\tilde{b} \quad \tilde{t}}{p\overline{^2 + ^2}} \qquad (2.46)$$

Di erentiating both sides with respect to s; we get

$$_1\tilde{n}_1\frac{ds_1}{ds} = \frac{(\ ^0 \quad ^0)\ \tilde{t} \quad \tilde{b}}{^2 + ^2\ ^{3=2}} \qquad * \tilde{b}^0 = \tilde{n} \text{ and } \tilde{t}^0 = \tilde{n} \qquad (2.47)$$

Squaring both sides of equation (2.47) and using equation(2.43), we get

$$^2_1\ ^2(c \quad s)^2 = \frac{(\ ^0 \quad ^0)^2\ \ ^2 + ^2}{^2 + ^2\ ^3}$$

$$\Rightarrow \quad _1 = \frac{(\ ^0 \quad ^0)}{(c \quad s)\ ^2 + ^2} = \frac{(\ ^0 \quad ^0)}{^2 + ^2\ (c \quad s)}$$

$$\text{using} \quad = \frac{1}{}; \quad = \frac{1}{}; \quad ^0 = \frac{^0}{^2}; \quad ^0 = \frac{^0}{^2}$$

De nition 2.4. A circular helix is a space curve which lies on the surface of the circular cylinder, the axis of the helix being that of the cylinder and cutting the generators at constant angle

Example 2.6. Prove that the involute of a circular helix are plane curves.

Solution:

$$\text{For circular helix} \quad \frac{}{} = a \text{ (constant)}$$

$$i.e.; \quad ^0 = a\ ^0$$

Torsion of an involute of a given curve $\tilde{r} = \tilde{r}(s)$ is given by

$$\tau_1 = \frac{\rho\sigma' - \sigma\rho'}{(c-s)(\rho^2+\sigma^2)}$$

Put $\sigma = a\rho$ and $\sigma' = a\rho'$ in the above equation, then the equation reduces to $\tau_1 = 0$.

i.e., Torsion for the involute is zero and hence the involute is a plane curve.

Example 2.7. Find the involute of a circular helix $\tilde{r} = (a\cos u, a\sin u, bu)$

Solution:

$$\text{Given that } \tilde{r} = (a\cos u, a\sin u, bu)$$
$$\dot{\tilde{r}} = (-a\sin u, a\cos u, b)$$
$$\dot{s} = \sqrt{a^2+b^2}$$
$$\Rightarrow s = \int_0^u \sqrt{a^2+b^2}\,du = u\sqrt{a^2+b^2}$$
$$\text{Also, } \tilde{t} = \tilde{r}' = \frac{\dot{\tilde{r}}}{\dot{s}} = \frac{(-a\sin u, a\cos u, b)}{\sqrt{a^2+b^2}}$$

The equation of involute is

$$\tilde{r}_1 = \tilde{r} + (c-s)\tilde{t}$$
$$= (a\cos u, a\sin u, bu) + \frac{(c-s)}{\sqrt{a^2+b^2}}(-a\sin u, a\cos u, b)$$
$$= \left(a\cos u - \frac{(c-s)}{\sqrt{a^2+b^2}}\sin u,\; a\sin u + \frac{(c-s)}{\sqrt{a^2+b^2}}\cos u,\; bu + \frac{b(c-s)}{\sqrt{a^2+b^2}}\right)$$

where $s = \sqrt{a^2+b^2}\,u$.

Example 2.8. Find the involutes and evolutes of the circular helix $x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\tan\alpha$.

Solution:

$$\text{Given that } \tilde{r} = (a\cos\theta, a\sin\theta, a\theta\tan\alpha)$$
$$\dot{\tilde{r}} = a(-\sin\theta, \cos\theta, \tan\alpha)$$
$$\dot{s} = |\dot{\tilde{r}}| = a\sqrt{1+\tan^2\alpha} = a\sec\alpha$$
$$\Rightarrow \tilde{t} = \tilde{r}' = \frac{\dot{\tilde{r}}}{\dot{s}}$$
$$s = \int_0^\theta \dot{s}\,d\theta = \int_0^\theta a\sec\alpha\,d\theta = a\theta\sec\alpha$$

Equation of involutes are given by $\tilde{r}_1 = \tilde{r} + (c \quad s)\tilde{t}$:

$$\tilde{r}_1 = \tilde{r} + (c \quad s)\tilde{t}$$

$$i.e.; \quad \tilde{r}_1 = a(\cos \ ; \sin \ ; \ \tan \ ) + (c \quad a \ \sec \ )(\ \sin \ ; \cos \ ; \tan \ ) \cos \ :$$

If $\tilde{r}_1 = x\tilde{i} + y\tilde{j} + z\tilde{k}$; then the Cartesian equation of the involutes are

$$x = a \cos \quad \cos \ \sin \ (c \quad a \ \sec \ )$$

$$y = a \sin \ + \cos \ \cos \ (c \quad a \ \sec \ )$$

$$z = a \ \tan \ + \sin \ (c \ a \ \sec \ )$$

The equation of evolutes are given by

$$\tilde{r}_1 = \tilde{r} + \ \tilde{n} + \ \cot(\ + c)\tilde{b} \quad \text{where} \quad = \int ds$$

$$\tilde{t} = \frac{\tilde{r}}{s} = \cos \ (\ \sin \ ; \ \cos \ ; \ \tan \ )$$

$$\tilde{t}' = \tilde{n} = \frac{\cos^2}{a}(\ \cos \ ; \ \sin \ ; \ 0)$$

$$\Rightarrow \quad = \frac{\cos^2}{a}$$

$$i.e.; \quad = \frac{1}{} = a \sec^2$$

$$\tilde{n} = (\ \cos \ ; \ \sin \ ; 0)$$

$$\tilde{b} = \tilde{t} \ \tilde{n} = \cos \ (\sin \ \tan \ ; \cos \ \tan \ ; 1)$$

$$\tilde{b}' = \ \tilde{n} = \frac{\cos^2}{a}(\cos \ \tan \ ; \ \sin \ \tan \ ; 0) \qquad \left(\frac{\tilde{b}}{s} \right) =$$

$$i.e.; \quad = \frac{1}{a} \sin \ \cos$$

$$= \int ds = \int \frac{1}{a} \sin \ \cos \ ds = \frac{s}{a} \sin \ \cos \ = \sin \qquad [* \ s = a \ \sec \ ]$$

Thus, the equation of evolutes are given by

$$\tilde{r}_1 = a(\cos \ ; \sin \ ; \tan \ ) + a \sec^2 \ (\ \cos \ ; \ \sin \ ; 0)$$
$$+ a \sec^2 \ \cot(\ \sin \ + c) \cos \ (\sin \ \tan \ ; \cos \ \tan \ ; 1):$$

Let Us Sum Up:

In this unit, the students acquired knowledge to

nd the equation of osculating sphere and osculating circle.

nd the involute and evolute of a given curve .

Check Your Progress:

1. Find the equation of the osculating sphere and osculating circle at $(1, 2, 3)$ on the curve $\tilde{r} = (2u+1, 3u^2+2, 4u^3+3)$.

2. Show that the involutes of a circular helix are plane curves.

3. Find the involutes and evolutes of the twisted cubic given by $\tilde{r} = (u, u^2, u^3)$.

Answer:

1. $9(x^2+y^2+z^2) - 18x - 48y - 54z + 150 = 0$ and
   $9(x^2+y^2+z^2) - 18x - 48y - 54z + 150 = 0; \quad z - 3 = 0$

Glossaries:

Involute: Any curve of which a given curve is the evolute.

Suggested Readings:

1. T.J. Willmore, – *An Introduction to Differential Geometry*, Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, – *Differential Geometry of Three Dimensions*, University Press, Cambridge, 1930.

Block-I

UNIT-3

SPHERICAL INDICATRIX

Structure:

Objective

Overview

3. 1 The Spherical Indicatrices or Spherical Images

3. 1. 1 The Spherical Indicatrix (or spherical image) of the tangent

3. 1. 2 The Spherical Indicatrix (or spherical image) of the principal normal

3. 1. 3 The Spherical Indicatrix (or spherical image) of the binormal

3. 1. 4 Bertrand Curves

3. 2 Intrinisic equations, fundamental existence theorem for space curves

3.2.1 Fundamental theorem for space curves

3.2.2 Intrinsic Equations

3. 3 Helices

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be able to

- F nd spherical indicatrix of the tangent, principal normal and binormal.
- understand the concept of Bertrand curves and its properties.
- derive the fundamental theorem for space curves.

Overview

In this unit, we will explain how to nd the curvature and torsion of the spherical image of the principal normal and binormal.

3.1. The Spherical Indicatrices or Spherical Images:

When we move all unit tangent vectors $\tilde{t}$ of a curve C to a point, their extremities describes a curve C_1 on the unit sphere, this curve C_1 is called the spherical image of C (or) Spherical indicatrix of C. There is a one-one correspondence between C and C_1. Similarly, we can de ne the spherical indicatrix of the principal normal and the binormal.

3.1.1. The Spherical Indicatrix (or spherical image) of the tangent:

De nition 3.1. It is the locus of a point whose position vector is equal to the unit tangent $\tilde{t}$ at any point of a given curve is called the spherical indicatrix of the tangent. Since such locus lies on the surface of a unit sphere.

Bookwork 3.1. Find the curvature and torsion of the spherical indicatrix of the tangent.

Solution:

By de nition of indicatrix of tangent, we have $\tilde{r}_1 = \tilde{t};$ where $\tilde{r}_1$ is the position vector.

$$\tilde{r}_1 = \tilde{t}$$

Di erentiate both sides with respect to $s;$ we have

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{t}}{ds}$$

$$\tilde{t}_1\frac{ds_1}{ds} = \tilde{n}$$

$$i.e.; \quad \tilde{t}_1 = \tilde{n}\frac{ds}{ds_1}$$

From the above equation, we see that $\tilde{t}_1$ is parallel to $\tilde{n};$ we may measure s_1 such that

$$\tilde{t}_1 = \tilde{n} \tag{3.1}$$

$$\text{Then} \quad = \frac{ds_1}{ds} \tag{3.2}$$

Di erentiate with respect to $s;$ we get

$$\frac{d\tilde{t}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{n}}{ds} = \tilde{t} + \tilde{b}$$

$$_1\tilde{n}_1 = \tilde{t} + \tilde{b}$$

$$i.e.; \quad _1\,\tilde{n}_1 = \tilde{t} + \tilde{b} \qquad \text{(using (3.2))} \tag{3.3}$$

Squaring on both sides, we get

$${}_1^2\;{}^2 = {}^2 + {}^2$$

$$i.e.; \quad {}_1 = \frac{\sqrt{{}^2 + {}^2}}{} \tag{3.4}$$

$$\text{Now, } \tilde{b}_1 = \tilde{t}_1 \quad \tilde{n}_1 = \tilde{n} \quad \frac{\tilde{t} + \tilde{b}}{1} \quad \tilde{t}_1 = \tilde{n} \;\&\; \tilde{n}_1 = \frac{\tilde{t} + \tilde{b}}{1}$$

$$\text{i:e:;} \quad {}_1\tilde{b}_1 = \tilde{b} + \tilde{t} \tag{3.5}$$

Di erentiate equation (3.5) with respect to s; we get

$$1\frac{d\tilde{b}_1}{ds}\frac{ds_1}{ds} + \tilde{b}_1\frac{d}{ds}(\ {}_1\ ) = {}^0\tilde{t} + \tilde{n} \quad \tilde{n} + {}^0\tilde{b}$$

$$1\ {}_1\tilde{n}_1 + \tilde{b}_1\frac{d}{ds}(\ {}_1\ ) = {}^0\tilde{t} + {}^0\tilde{b} \tag{3.6}$$

Taking the dot product of (3.3) and (3.6), we get

$$^{2\ 3}{}_1 \quad {}_1 = {}^0 \quad {}^0$$

$$\text{i:e:;} \quad {}_1 = \frac{{}^0 \quad {}^0}{{}^{2\ 3}{}_1}$$

$$\text{But} \quad {}^2_1 = \frac{{}^2 + {}^2}{{}^2}$$

$$) \quad {}_1 = \frac{{}^0 \quad {}^0}{{}^2 + {}^2} \tag{3.7}$$

3.1.2. The Spherical Indicatrix (or spherical image) of the principal normal:

De nition 3.2. The locus of a point whose position vector is equal to the unit principal normal $\tilde{n}$ at any point of a given curve is called the spherical indicatrix of the principal normal.

Bookwork 3.2. Find the curvature and torsion of the spherical indicatrix of the principal normal.

Solution: By de nition of the spherical indicatrix of the principal normal, we have $\tilde{r}_1 = \tilde{n}$:

Di erentiate both sides with respect to s; we have

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{n}}{ds} = \tilde{t} + \tilde{b}$$

$$\text{i:e:;} \quad \tilde{t}_1\frac{ds_1}{ds} = \tilde{t} + \tilde{b} \tag{3.8}$$

Squaring both sides of (3.8), we get

$$\left(\frac{ds_1}{ds}\right)^2 = \tilde{t} + \tilde{b}^{\,2} = {}^2 + {}^2 \qquad (3.9)$$

$$\text{i.e.,} \quad \frac{ds_1}{ds} = \sqrt{{}^2 + {}^2}$$

Di erentiate (3.8), we have

$$\frac{d\tilde{t}_1}{ds_1}\left(\frac{ds_1}{ds}\right)^2 + \tilde{t}_1 \frac{d^2 s_1}{ds^2} = {}^0\tilde{b} \quad {}^2\tilde{n} \quad {}^0\tilde{t} \quad {}^2\tilde{n}$$

$${}_1\tilde{n}_1 \left(\frac{ds_1}{ds}\right)^2 + \tilde{t}_1 \frac{d^2 s_1}{ds^2} = {}^0\tilde{t} \quad {}^2 + {}^2\ \tilde{n} + {}^0\tilde{b} \qquad (3.10)$$

Taking cross product of (3.8) and (3.10), we get

$${}_1\left(\frac{ds_1}{ds}\right)^3 \tilde{b}_1 = {}^0\tilde{n} + {}^2 + {}^2\ \tilde{t} \quad {}^2 + {}^2\ \tilde{b} + {}^0\tilde{n}$$

$${}_1\left(\frac{ds_1}{ds}\right)^3 \tilde{b}_1 = {}^2 + {}^2\ \tilde{t} + {}^0 \quad {}^0\tilde{t}\ \tilde{n} + {}^2 + {}^2\ \tilde{b} \qquad (3.11)$$

Squaring (3.12), we get

$${}^2_1\left(\frac{ds_1}{ds}\right)^6 = {}^2\ {}^2 + {}^2\ {}^2 + {}^0\ {}^0\ {}^2 + {}^2\ {}^2 + {}^2\ {}^2$$

$${}^2_1 = \frac{{}^2 + {}^2\ {}^3 (\ {}^0\ \ {}^0\ )^2}{{}^2\ \ {}^2\ {}^3} \qquad \left(\frac{ds_1}{ds}\right)^2 = {}^2 + {}^2$$

$$\text{i.e.,} \quad {}^2_1 = 1 + \frac{(\ {}^0\ \ {}^0\ )^2}{{}^2 + {}^2\ {}^3} \qquad (3.12)$$

Since the indicatrix lies on the surface of a unit sphere, the torsion ${}_1 = \frac{1}{{}_1}$ and curvature ${}_1 = \frac{1}{{}_1}$ are given by the relation

$$= {}^2 + {}^2\ {}^0\ {}^2 = \frac{1}{{}^2_1} + \frac{1}{{}^2_1}\left(\frac{{}^0}{{}_1}\right)^2 \frac{1}{{}^2_1}$$

$${}_1 = \frac{{}^0_1}{\sqrt{{}^2_1 \quad 1}} \qquad (3.13)$$

Now, eliminating ${}_1$ between (3.12) and (3.13), we get

$${}^2_1 = {}^2 + {}^2\ {}^3 + {}^0\ \ {}^0\ {}^2 = {}^2 + {}^2\ 6\ {}^0_1\ {}^2$$

From (3.12), we have

$${}_1{}^2 \quad 1 \quad {}^2 + {}^2\ {}^3 = \ {}^2 \qquad (3.14)$$

Di erentiate equation (3.14), we get

$$3 \;{}^{2}_{1}\; 1 \;{}^{2} + {}^{2}\;{}^{2}\;{}^{0} + {}^{0} + {}^{0}_{1}\;{}_{1}\;{}^{2} + {}^{2}\;{}^{3} = {}^{0}\;{}^{0}\;{}^{00}\;{}^{00} \qquad (3.15)$$

From (3.15), we get the required value of ${}^{0}_{1};$ From (3.12) and (3.13), we get value of ${}_{1}:$

3.1.3. The Spherical Indicatrix (or spherical image) of the binormal:

De nition 3.3. The locus of a point whose position vector is equal to the unit binormal $\tilde{b}$ at any point of a given curve is called the spherical indicatrix of the binormal.

Bookwork 3.3. Find the curvature and torsion of the spherical indicatrix of the binormal.

Solution: By de nition of the spherical indicatrix of the binormal, we have $\tilde{r}_1 = \tilde{b}:$ Di erentiate both sides with respect to $s;$ we get

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{b}}{ds}$$

$$\tilde{t}_1\frac{ds_1}{ds} = \tilde{n}$$

$$i:e:; \quad \tilde{t}_1 = \tilde{n}\frac{ds}{ds_1} \qquad (3.16)$$

We may measure s_1 such that $\quad \tilde{t}_1 = \tilde{n} \qquad (3.17)$

From $(3.16);$ we have $\quad \frac{ds_1}{ds} = \qquad (3.18)$

Di erentiate (3.17), we get

$$\frac{d\tilde{t}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\tilde{n}}{ds} = \tilde{t} + \tilde{b}$$

$$i:e:; \quad {}_1\tilde{n}_1 = \tilde{t} \quad \tilde{b} \qquad (3.19)$$

Squaring, we get

$${}^{2}_{1}\;{}^{2} = {}^{2} + {}^{2} \Rightarrow {}_{1} = \frac{\sqrt{{}^{2} + {}^{2}}}{} \qquad (3.20)$$

$i:e:;$ ${}_1$ is the ratio of the screw curvature and the torsion of the given curve.

To nd the torsion of the indicatrix, take the cross of (3.17) and (3.19), we get

$$_1 \tilde{b}_1 = \tilde{t} + \tilde{b} \quad (3.21)$$

Di erentiate with respect to $s;$ we get

$$_1 \frac{d\tilde{b}_1}{ds_1}\frac{ds_1}{ds} + \tilde{b}_1 \frac{d}{ds}(\ _1\ ) = {}^{0}\tilde{t} + \tilde{n} + {}^{0}\tilde{b} \quad \tilde{n}$$

$$_1 \quad _1\tilde{n}_1 + \tilde{b}_1 \frac{d}{ds}(\ _1\ ) = {}^{0}\tilde{t} + {}^{0}\tilde{b} \quad (3.22)$$

Take the dot product of (3.19) and (3.22), we get

$$^{2}_{1}\ ^{3}\ _1 = \ ^{0}\ \ ^{0}\ ) \ _1 = \frac{^{0}\quad ^{0}}{^{2} + ^{2}} \quad (3.23)$$

3.1.4. Bertrand Curves:

De nition 3.4. A pair of curves C and C_1 which have the same principal normals are called Bertrand curves.

Properties of Bertrand Curves:

Property 1: The distance between corresponding points of two Bertrand curves is constant.

Proof.

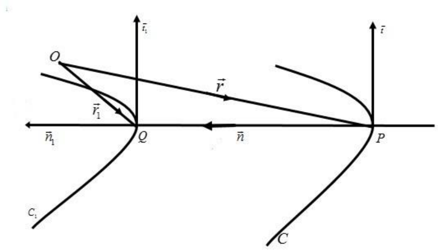


Figure 3.1: Evolute

Consider the principal normals to the curve C and C_1 in the same sense, by de nition

$$\tilde{n}_1 = \tilde{n} \quad (3.24)$$

Let $\tilde{r}$ be the position vector of the point P on C and $\tilde{r}_1$ be the position vector of the corresponding point Q on C_1 with respect to the origin $O:$

$$\tilde{r}_1 = \tilde{r} + \tilde{n} \quad \text{where} \quad \text{is a scalar function of } s \qquad (3.25)$$

Di erentiate both sides with respect to s; we get

$$\frac{d\tilde{r}_1}{ds_1}\frac{ds_1}{ds} = \tilde{r}^{\,0} + \tilde{n}^{\,0} + {}^{0}\tilde{n} = \tilde{t}_1 \frac{ds_1}{ds} = \tilde{t} + \quad \tilde{t} + \tilde{b} + {}^{0}\tilde{n}$$

$$\text{i:e:;} \quad \tilde{t}_1 \frac{ds_1}{ds} = (1 \quad)\tilde{t} + {}^{0}\tilde{n} + \quad \tilde{b} \qquad (3.26)$$

Taking the dot product of (3.24) and (3.26), we get

$$\tilde{n}_1 \quad \tilde{t}_1 \frac{ds_1}{ds} = \tilde{n} \quad (1 \quad)\tilde{t} + \quad {}^{0}\tilde{n} + \quad b$$

$$\text{i:e:;} \quad 0 = {}^{0} \quad) \quad = \text{constant}$$

Thus, the distance between P and Q is constant.

Property 2: The tangents at the corresponding points of two curves are inclined at a constant angle.

Proof.

$$\frac{d}{ds}\tilde{t} \quad \tilde{t}_1 = \frac{d\tilde{t}}{ds}\tilde{t} + \tilde{t}\frac{d\tilde{t}_1}{ds_1}\frac{ds_1}{ds} = \tilde{n} \quad \tilde{t}_1 + \tilde{t} \quad {}_1\tilde{n}_1 \frac{ds_1}{ds} \qquad * \ \tilde{n}_1 = \tilde{n}$$

$$= \tilde{n}_1 \quad \tilde{t}_1 + {}_1\frac{ds_1}{ds}\tilde{t} \quad \tilde{n} = 0$$

$$) \quad \tilde{t} \quad \tilde{t}_1 = \text{constant}$$

$$\text{i:e:;} \quad \cos \quad = \text{constant, where} \quad \text{is the angle between } \tilde{t} \text{ and } \tilde{t}_1$$

$$\text{i:e:;} \qquad = \text{constant}$$

Property 3: Curvature and torsion of either curves are connected by a linear relation.

Proof. From property (1), we have $^{0} = 0$:

Equation (3.26) in property (1), reduces to

$$\tilde{t}_1 \frac{ds_1}{ds} = (1 \quad)\tilde{t} + \quad \tilde{b} \qquad (3.27)$$

Taking dot product of both sides of (3.27) with $\tilde{b}_1$; we have

$$\tilde{b}_1 \quad \tilde{t}_1 \frac{ds_1}{ds} = (1 \quad)\tilde{t} \quad \tilde{b}_1 + \quad \tilde{b} \quad \tilde{b}_1$$

$$\text{i:e:;} \quad 0 = (1 \quad)\tilde{t} \quad \tilde{b}_1 + \quad \tilde{b} \quad \tilde{b}_1 \qquad (3.28)$$

Since the principal normals $\tilde{n}_1$ and $\tilde{n}$ coincide, the four vectors $\tilde{t}_1; \tilde{t}; \tilde{b}_1$ and $\tilde{b}$

are coplanar when they are localized at O:

$$\tilde{t} \cdot \tilde{b}_1 = \cos 90^0 \quad = \sin$$
$$\tilde{b} \cdot \tilde{b}_1 = \cos$$

Using the above equations, the equation (3.28) reduces to

$$0 = (1 \quad) \sin + \cos$$

The above relation shows that there exists a linear relation with constant coe cients between the curvature and torsion of the curve C:

Hence, the above relation can be written as

$$= \frac{1}{} \tan \quad (3.29)$$

Again the relation between the curves C and C_1 is reciprocal one, thus the point $P\ \tilde{r}$ is at a distance along the normal at $Q(\tilde{r}_1)$ and $\tilde{t}$ is inclined at an angle with $\tilde{t}_1$:

Thus for the curve C_1; we have relation corresponding to (3.29) as

$$_1 = \quad _1 + \frac{1}{} \tan$$

3.2. Intrinisic equations, fundamental existence theorem for space curves:

In this section, we express any point of a space curve by the equations $= (s)$ and $= (s)$ which are the intrinsic equations. Fundamental theorem of space curves is provided in two parts namely existence theorem and uniqueness theorem.

3.2.1. Fundamental theorem for space curves:

Theorem 3.1 (Existence theorem for space curves). If $= (s)$ and $= (s)$ are continuous functions of a real variable $s(s \quad 0)$; then there exists a space curve for which is the curvature and is the torsion, and s is the arc length measured from same suitable base point.

Proof. We have to show that there are four vector functions $\tilde{r} = \tilde{r}(s)$;

$\tilde{t} = \tilde{t}(s);\ \tilde{n} = \tilde{n}(s)$ and $\tilde{b} = \tilde{b}(s)$ such that $\tilde{t}; \tilde{n}; \tilde{b}$ are mutually perpendicular vectors satisfying Serret-Frenet formulae.

Then $\tilde{r} = \tilde{r}(s)$ will be the required curve

$$\frac{d}{ds} = \qquad \frac{d}{ds} = \quad + \qquad \frac{d}{ds} = \qquad (3.30)$$

where ; ; are unknown functions of s:

From the theory of di erential equations, we have that the above system has unique solution (s); (s); (s) which takes prescribed values at $s = 0$ (initial values).

In particular, this is a unique solution $_1(s);\ _1(s);\ _1(s)$ for which $_1(0) = 1;\ _1(0) = 0;\ _1(0) = 0$:

Similarly, we have another set of solutions $_2(s);\ _2(s);\ _2(s)$ for which $_2(0) = 0;\ _2(0) = 1;\ _2(0) = 0$ and another set of solutions are $_3(s);\ _3(s);\ _3(s)$ for which $_3(0) = 0;\ _3(0) = 0;\ _3(0) = 1$:

Next we shall show that $^2_1 + {}^2_1 + {}^2_1 = 1$:

$$\frac{d}{ds}\left(^2_1 + {}^2_1 + {}^2_1 \right) = 2\ _1\frac{d\ _1}{ds} + 2\ _1\frac{d\ _1}{ds} + 2\ _1\frac{d\ _1}{ds}$$

$$= 2\ _1(\quad _1) + 2\ _1(\quad _1 + \quad _1) + 2\ _1(\quad _1) = 0$$

$$\Rightarrow\ ^2_1 + {}^2_1 + {}^2_1 = \text{constant} = C_1 \text{ (say)}$$

Similarly, we can prove that $^2_2 + {}^2_2 + {}^2_2 = 1$ and $^2_3 + {}^2_3 + {}^2_3 = 1$

Now, we shall prove that $_1\ _2 + {}_1\ _2 + {}_1\ _2 = 0$:

$$\frac{d}{ds}(\ _1\ _2 + {}_1\ _2 + {}_1\ _2) = \ _1\frac{d\ _2}{ds} + \frac{d\ _1}{ds}\ _2 + {}_1\frac{d\ _2}{ds} + \frac{d\ _1}{ds}\ _2 + {}_1\frac{d\ _2}{ds} + \frac{d\ _1}{ds}\ _2$$

$$= 0 \quad \text{(using (3.30))}$$

$$\Rightarrow\ (\ _1\ _2 + {}_1\ _2 + {}_1\ _2) = \text{constant} = C_2 \text{ (say)}$$

Using the initial values at $s = 0$; we get

$$1(0) + 0(1) + 0(0) = C_2 \Rightarrow C_2 = 0$$

$$\Rightarrow \quad {}_1 {}_2 + {}_1 {}_2 + {}_1 {}_2 = 0$$

Similarly, we have ${}_2 {}_3 + {}_2 {}_3 + {}_2 {}_3 = 0$ and ${}_3 {}_1 + {}_3 {}_1 + {}_3 {}_1 = 0$.

Consider the matrix $A = \begin{pmatrix} {}_1 & {}_1 & {}_1 \\ {}_2 & {}_2 & {}_2 \\ {}_3 & {}_3 & {}_3 \end{pmatrix}$

$$AA^T = \begin{pmatrix} {}_1 & {}_1 & {}_1 \\ {}_2 & {}_2 & {}_2 \\ {}_3 & {}_3 & {}_3 \end{pmatrix}\begin{pmatrix} {}_1 & {}_2 & {}_1 \\ {}_1 & {}_2 & {}_3 \\ {}_1 & {}_2 & {}_3 \end{pmatrix}$$

$$\Rightarrow AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow AA^T = I \Rightarrow A^T = A^{-1}$$

$$\Rightarrow A^T A = A^{-1} A = I \Rightarrow A^T A = I$$

Thus, A is an orthogonal matrix.

$$A^T A = I$$

$$\Rightarrow \begin{pmatrix} {}_1 & {}_2 & {}_1 \\ {}_1 & {}_2 & {}_3 \\ {}_1 & {}_2 & {}_3 \end{pmatrix}\begin{pmatrix} {}_1 & {}_1 & {}_1 \\ {}_2 & {}_2 & {}_2 \\ {}_3 & {}_3 & {}_3 \end{pmatrix} = I$$

$$\begin{pmatrix} {}_1^2 + {}_2^2 + {}_3^2 & {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 & {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 \\ {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 & {}_1^2 + {}_2^2 + {}_3^2 & {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 \\ {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 & {}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 & {}_1^2 + {}_2^2 + {}_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow {}_1^2 + {}_2^2 + {}_3^2 = 1$$

$${}_1^2 + {}_2^2 + {}_3^2 = 1$$

$${}_1^2 + {}_2^2 + {}_3^2 = 1$$

and ${}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 = 0$

$${}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 = 0$$

$${}_1 {}_1 + {}_2 {}_2 + {}_3 {}_3 = 0$$

Let $\tilde{t} = {}_1\tilde{i} + {}_2\tilde{j} + {}_3\tilde{k}$

$$\tilde{n} = {}_1\tilde{i} + {}_2\tilde{j} + {}_3\tilde{k}$$

$$\tilde{b} = {}_1\tilde{i} + {}_2\tilde{j} + {}_3\tilde{k}$$

Then $\tilde{t} = 1;\ \tilde{n} = 1;\ \tilde{b} = 1$ and $\tilde{t}\ \tilde{n} = 0;\ \tilde{n}\ \tilde{b} = 0; \tilde{b}\ \tilde{t} = 0.$

Therefore, $\tilde{t}; \tilde{n}$ and $\tilde{b}$ are mutually perpendicular unit vectors.

For each value of $s;$ we let $\tilde{r} = \int_0^s \tilde{t} ds;$ then $\frac{d\tilde{r}}{ds} = \tilde{t} \Rightarrow \tilde{r} = \tilde{t}.$

This $\tilde{r} = \tilde{r}(s)$ is the required curve with s as its arc length. Clearly for this $\tilde{r};$ the unit vectors $\tilde{t}; \tilde{n}$ and $\tilde{b}$ satisfy Serret-Frenet formula (which are the given di erential equations) with given functions as curvature and torsion.

Hence the existence of the curve is proved.

Theorem 3.2 (Uniqueness theorem for space curves). If two curves have the same intrinsic equation then they are congruent.

Proof. If possible, let there be two curves C and C_1 having equal curvature and equal torsion for the same values of $s.$ For any arc length $s;$ let the corresponding points be P and P_1 on C and C_1 respectively. Denoting the corresponding triads for the two curves C and C_1 by $\tilde{t}; \tilde{n}; \tilde{b}$ and $\tilde{t}_1; \tilde{n}_1; \tilde{b}_1.$

Now, consider

$$\frac{d}{ds}\left(\tilde{t}\ \tilde{t}_1 + \tilde{n}\ \tilde{n}_1 + \tilde{n}\ \tilde{b}_1\right) = \tilde{t}\ \tilde{t}_1{}' + \tilde{t}'\ \tilde{t}_1 + \tilde{n}\ \tilde{n}_1{}' + \tilde{n}'\ \tilde{n}_1 + \tilde{b}'\ \tilde{b}_1 + \tilde{b}'\ \tilde{b}_1$$

$$= \tilde{t}\ \tilde{n}_1 + \tilde{n}\ \tilde{t}_1 + \tilde{t}_1 + \tilde{b}_1\ \tilde{n} + \tilde{t} + \tilde{b}\ \tilde{n}_1$$

$$+ \tilde{b}\ \tilde{n}_1 + \tilde{n}\ \tilde{b}_1 = 0$$

$$\Rightarrow \frac{d}{ds}\left(\tilde{t}\ \tilde{t}_1 + \tilde{n}\ \tilde{n}_1 + \tilde{n}\ \tilde{b}_1\right) = 0$$

$$\Rightarrow \tilde{t}\ \tilde{t}_1 + \tilde{n}\ \tilde{n}_1 + \tilde{n}\ \tilde{b}_1 = \text{constant} = c \text{ (say)}$$

If C_1 is moved in such a manner that at $s = 0$ the two triads $\tilde{t}; \tilde{n}; \tilde{b}$ and $\tilde{t}_1; \tilde{n}_1; \tilde{b}_1$ coincide then at that point $\tilde{t} = \tilde{t}_1;\ \tilde{n} = \tilde{n}_1;\ \tilde{b} = \tilde{b}_1.$

Thus, we have

$$\Rightarrow \tilde{t}\ \tilde{t} + \tilde{n}\ \tilde{n} + \tilde{b}\ \tilde{b} = c$$

$$\Rightarrow \tilde{t}^2 + \tilde{n}^2 + \tilde{b}^2 = c \Rightarrow c = 3$$

$$\Rightarrow \tilde{t}\ \tilde{t}_1 + \tilde{n}\ \tilde{n}_1 + \tilde{b}\ \tilde{b}_1 = 3$$

$$\Rightarrow \tilde{t}\ \tilde{t}_1 \cos + \tilde{n}\ \tilde{n}_1 \cos + \tilde{b}\ \tilde{b}_1 \cos = 3$$

$$\Rightarrow \cos + \cos + \cos = 3$$

$$\Rightarrow \cos = 1;\ \cos = 1;\ \cos = 1$$

$$\Rightarrow = 0;\ = 0;\ = 0$$

i.e., angle between $\tilde{t}$ and $\tilde{t}_1$; $\tilde{n}$ and $\tilde{n}_1$; $\tilde{b}$ and $\tilde{b}_1$ are zero.

Hence $\tilde{t} = \tilde{t}_1$; $\tilde{n} = \tilde{n}_1$; $\tilde{b} = \tilde{b}_1$

Also, $\tilde{t} = \tilde{t}_1$ gives $\frac{d\tilde{r}_1}{ds} = \frac{d\tilde{r}}{ds} \Rightarrow d\tilde{r}_1 = d\tilde{r}.$

Integrating, we get $\tilde{r}_1 = \tilde{r} + \tilde{a}$; where $\tilde{a}$ is a constant vector. $\Rightarrow \tilde{r}_1 - \tilde{r} = \tilde{a}.$

At $s = 0$, we have $\tilde{r}_1 = \tilde{r}.$ $\Rightarrow \tilde{a} = 0.$

Thus, we have $\tilde{r}_1 = \tilde{r}$ for all $s.$

Hence the two curves C and C_1 coincides (or) the two curves are congruent. This proves the uniqueness.

3.2.2. Intrinsic Equations:

We have de ned the curve with respect to a set of three mutually perpendicular axes but in case the same curve be referred to a di erent set of Cartesian coordinate axes, then its equations are altogether di erent and it is not at all clear that they refer to the same curve. This can be expressed by the curvature and torsion at any point as functions of arc length s say $= (s)$ and $= (s).$ These are called the intrinsic equations of the curve.

Example 3.1. Show that the intrinsic equation of the curve given by $x = ae^u \cos u$; $y = ae^u \sin u$ and $z = be^u$ are $= \frac{a\sqrt{2}}{s\sqrt{2a^2+b^2}}$; $= \frac{b}{s\sqrt{2a^2+b^2}}.$

Solution:

$$\text{Given that } \tilde{r} = (ae^u \cos u, ae^u \sin u, be^u)$$

$$\dot{\tilde{r}} = (ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u)$$

$$\dot{s} = e^u\sqrt{2a^2+b^2}$$

$$s = \int^u \dot{s}\,du = \int^u e^u\sqrt{2a^2+b^2}\,du = e^u\sqrt{2a^2+b^2} = s$$

$$\tilde{r}' = \frac{\dot{\tilde{r}}}{\dot{s}} = \frac{[a(\cos u - \sin u), a(\sin u + \cos u), b]e^u}{e^u\sqrt{2a^2+b^2}}$$

$$\tilde{r}'' = \tilde{n} = \frac{[-a(\sin u + \cos u), a(\cos u - \sin u - \sin u), 0]}{\sqrt{2a^2+b^2}}\frac{1}{s}$$

Taking modulus on both sides

$$\tilde{r}'' = -\frac{a\sqrt{2}}{\sqrt{2a^2+b^2}}\frac{1}{s} = \frac{a\sqrt{2}}{\sqrt{2a^2+b^2}}\frac{1}{s}$$

$$s\tilde{r}'' = \frac{[-a(\sin u+\cos u), a(\cos u - \sin u), 0]}{\sqrt{2a^2+b^2}}\frac{1}{s}$$

Differentiate both sides with respect to s, we get

$$s\tilde{r}''' + \tilde{r}'' = \frac{[-a(\cos u - \sin u), -a(\sin u+\cos u), 0]}{\sqrt{2a^2+b^2}}\frac{1}{s}$$

$$\text{i.e.,}\quad s^2\tilde{r}''' + s\tilde{r}'' = \frac{[-a(\cos u - \sin u), -a(\sin u+\cos u), 0]}{\sqrt{2a^2+b^2}}$$

$$\Rightarrow\quad s^2\tilde{r}''' = \frac{[2a\sin u, -2a\cos u, 0]}{\sqrt{2a^2+b^2}}$$

$$\text{Now,}\quad \left[\tilde{r}', s\tilde{r}'', s^2\tilde{r}'''\right] = \begin{vmatrix} -a(\cos u - \sin u) & -a(\sin u+\cos u) & b \\ -a(\sin u+\cos u) & a(\cos u - \sin u) & 0 \\ 2a\sin u & -2a\cos u & 0 \end{vmatrix}\frac{1}{(2a^2+b^2)^{3/2}}$$

$$\text{i.e.,}\quad s^3\left[\tilde{r}', \tilde{r}'', \tilde{r}'''\right] = \frac{1}{(2a^2+b^2)^{3/2}}\,2a^2b$$

$$\text{i.e.,}\quad s^3\kappa^2\tau = \frac{1}{(2a^2+b^2)^{3/2}}\,2a^2b$$

$$\Rightarrow\quad \tau = \frac{b}{\sqrt{2a^2+b^2}}\frac{1}{s}, \qquad \because\ \kappa^2 = \frac{2a^2}{(2a^2+b^2)s^2}.$$

3.3. Helices:

Definition 3.5 (Cylindrical Helices). A helix is a space curve which is traced on the surface of a cylinder and cuts the generator at constant angle.

Note 3.1. The tangent to a helix makes a constant angle (say) with fixed direction, this fixed line (direction) is known as axis (or) generator of the cylinder.

Definition 3.6 (Circular helix). A helix which lies on the surface of a circular cylinder is called a circular helix (or) right circular helix.

Theorem 3.3 (Theorem of Lancret (Characteristic property of helices)). A necessary and sufficient condition for a curve to be helix is that at all points curvature bears a constant ratio with Torsion.

Proof. Necessary part: Let $\tilde{a}$ be a constant vector and $\tilde{t}$ be the unit tangent vector to the helix.

$$\Rightarrow \tilde{t}\ \tilde{a} = \tilde{t}\ \tilde{a}\cos$$
$$\Rightarrow \tilde{t}\ \tilde{a} = a\cos$$

Di erentiate with respect to s; we get

$$\tilde{t}\ (0) + \tilde{t}'\ \tilde{a} = 0$$
$$i.e.,\quad \tilde{n}\ \tilde{a} = 0$$
$$\Rightarrow \tilde{n}\ \tilde{a} = 0 \Rightarrow \tilde{n} \text{ is perpendicular to } \tilde{a}$$

$i.e.,$ the principal normal is everywhere perpendicular to generators.

But the principal normal is everywhere perpendicular to the rectifying plane, hence the generators must be parallel to the rectifying plane (containing $\tilde{t}$ and $\tilde{b}$).

Since $\tilde{a}$ makes constant angles with $\tilde{t}$, it follows that it makes constant angle with $\tilde{b}$ also. $i.e.,$ 90 :

$$\text{we have } \tilde{n}\ \tilde{a} = 0$$

Di erentiate both sides with respect to s; we get

$$\tilde{n}\ (0) + \tilde{n}'\ \tilde{a} = 0$$
$$\Rightarrow \tilde{t} + \tilde{b}\ \ \tilde{a} = 0$$
$$\Rightarrow \tilde{t}\ \tilde{a} + \tilde{b}\ \tilde{a} = 0$$
$$\Rightarrow \tilde{t}\ \tilde{a}\cos + \tilde{b}\ \tilde{a} = 0$$
$$\Rightarrow a\cos + \tilde{b}\ \tilde{a}\cos(90\ \ ) = 0$$
$$\Rightarrow a\cos + a\sin = 0$$
$$i.e.,\quad a\cos = a\sin$$
$$\Rightarrow - = \tan = \text{constant.}$$

Su cient Part:

$$\text{Assume that } - = \text{constant.}$$
$$\text{Let } - = C \Rightarrow = C$$
$$\text{We know that } \tilde{t}' = \tilde{n} = C\ \tilde{n}$$
$$\text{and } \tilde{b}' = \tilde{n} \Rightarrow \tilde{n} = \tilde{b}'$$
$$\Rightarrow \tilde{t}' = C\tilde{b}' \quad \text{(using (3.31))}$$
$$\text{Integrating, } \tilde{t} + C\tilde{b} = \tilde{a} \quad \text{(a constant vector)}$$

Taking dot product with $\tilde{t}$

$$\Rightarrow \tilde{t} \quad \tilde{t} + C\tilde{b} \quad = \quad \tilde{t} \quad \tilde{a}$$
$$\Rightarrow 1 + 0 \quad = \quad \tilde{t} \quad \tilde{a}$$
$$\Rightarrow 1 \quad = \quad \tilde{t} \quad \tilde{a} \cos$$
$$i.e., \quad a \cos \quad = \quad 1$$
$$i.e., \quad \cos \quad = \quad \frac{1}{a}$$
$$i.e., \quad = \quad \text{constant}$$

Thus the curve is a helix.

Example 3.2. Show that a necessary and su cient condition that a curve be an helix is that $\left[\tilde{r}\,''; \tilde{r}\,'''; \tilde{r}^{(iv)}\right] = \quad ^5 \frac{d}{ds}\left(\overline{\quad}\right) = 0.$

Solution:

$$\tilde{r}\,' = \frac{d\tilde{r}}{ds} = \tilde{t}$$
$$\tilde{r}\,'' = \tilde{t}\,'$$
$$\frac{d^2\tilde{r}}{ds^2} = \quad \tilde{n}$$
$$\tilde{r}\,''' = \frac{d}{ds} \quad \tilde{n} = \quad '\tilde{n} + \quad \tilde{t} + \quad \tilde{b}$$
$$= \quad '\tilde{n} \quad ^2\tilde{t} + \quad \tilde{b}$$

Similarly $\quad \tilde{r}^{(iv)} = \quad '' \quad ^3 \quad ^2 \quad \tilde{n} \quad 3 \quad '\tilde{t} + 2 \quad ' \quad + \quad ' \quad \tilde{b}$

$$\left[\tilde{r}\,''; \tilde{r}\,'''; \tilde{r}^{(iv)}\right] = \quad 0 \quad 2 \quad ' \quad 3 \quad ' \quad '' \quad 2 \quad 3 \quad 2 \quad ' + \quad ' \quad 0$$
$$= \quad ^3 \quad ' \quad ' = \quad ^5 \frac{\quad ' \quad '}{2}$$
$$= \quad ^5 \frac{d}{ds}\left(\overline{\quad}\right)$$

For an helix $\overline{\quad} = \text{constant}.$

$$\Rightarrow \frac{d}{ds}\left(\overline{\quad}\right) = 0$$

$\Rightarrow$ the curve is an helix.

Let Us Sum Up:

In this unit, the students acquired knowledge

- to nd the spherical image of the principal normal.
- to nd the spherical image of the principal tangent.
- to nd the spherical image of the principal binormal.

Check Your Progress:

1. Show that the spherical indicatrix of a curve is a circle if and only if the curve is an helix.
2. Prove that the curve given by $x = a\sin^2 u; y = a\sin u\cos u; \ z = a\cos u$ lies on a sphere.
3. De ne Intrinsic equations of the curve.
4. State and Prove fundamental theorem for space curves.

Choose the correct or more suitable answer:

1. A pair of curves C and C_1 which have the same $: : : : : :$ are called Bertrand Curves
   (a) principal tangent
   (b) principal normal
   (c) principal binormal
   (d) none of these.
2. Curvature and torsion of either curves are connected by a $: : : : : :$
   (a) linear relation
   (b) quadratic relation
   (c) cubic relation
   (d) none of these.

Answer:

(1) b (2) a

Glossaries:

Bertrand Curve: One of pair of curves having the same principal normals.

Suggested Readings:

1. T.J. Willmore, An Introduction to Di erential Geometry , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, Di erential Geometry of Three Dimensions , University Press, Cambridge, 1930.

Block-II

Unit-4: Theory of Surfaces.

Unit-5: Metric.

Unit-6: Families of Curves.

Block-II

UNIT-4

THEORY OF SURFACES

Structure:

Objective

Overview

4. 1 De nition of a surface

4. 1. 1 Regular (or Ordinary) point and Singularities on a surface

4. 2. 1 Parametric Curves

4. 2. 2 Tangent Plane and Normal

4. 3 Surface of Revolution

4. 3. 1 The Spheres

4. 3. 2 The general surface of revolution

4. 3. 3 The anchor ring

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Suggested Readings

Objectives

After completion of this unit, students will be to

- understand the concept of proper transformation.
- find the parametric curves, condition for the parametric curves to be orthogonal.
- find the equation of tangent plane and normal.

Overview

In this unit, we will explain the concept of regular point and singularities on a surface and also discussed different types of singularities.

4.1. Definition of a surface

In the previous chapter, we have defined a curve as the locus of a point whose Cartesian coordinates (x, y, z) are functions of a single parameter.

Definition 4.1 (Surface). A surface is defined as the locus of a point whose Cartesian coordinates (x, y, z) or whose position vector $\tilde{r}$ are functions of two parameters u and v. i.e., $x = f(u, v)$; $y = g(u, v)$; $z = h(u, v)$ or $\tilde{r} = \tilde{r}(u, v)$ are the parametric equations of surface.

Definition 4.2. The two parameters u, v are called the curvilinear coordinates of a current point on the surface.

Any point (x, y, z) on the surface, the values of u and v are determined uniquely and that point is referred as (u, v)

Definition 4.3. If the parameters u, v are eliminated from the parametric equation of a surface then the obtained relations $F(x, y, z) = 0$ is called the constraint equation of the surface.

Examples of a surface:

$$x = u; \quad y = v; \quad z = u^2 - v^2 \tag{4.1}$$

After eliminating the parameters u and v; we get $x^2 \; y^2 = z$ which represents a hyperbolic paraboloid surface.

Note 4.1. Now consider

$$x = u + v; \quad y = u \quad v; \quad z = 4uv \tag{4.2}$$

On eliminating the parameters $u; v$ we get $x^2 \quad y^2 = z$, the same paraboloid.

Thus, the parametric equation (4.1) and (4.2) represent the same surface $x^2 \; y^2 = z$:

Sometimes, after eliminating the parameters and then obtained constraint equation represents more than the given surface, so that parametric equations and constraint equations are not equivalent.

Consider the surface given by the parametric equations

$$x = u \cosh v; \quad y = u \sinh v; \quad z = u^2 \tag{4.3}$$

where the parameters u and v are takes real values. Upon eliminating the parameters, obtained constraint equation is $x^2 \; y^2 = z$ which represents the whole of the paraboloid. The parametric equations (4.3) represents only that part of the surface for which $z \quad 0$; since u takes only real values.

Hence the parametric equation of a given surface are not unique.

De nition 4.4 (Monge form of the surface). The equation $F(x; y; z) = 0$ will represent a surface. Here $x = f(u; v)$; $y = g(u; v)$ and $z = h(u; v)$ when we eliminate the parameters u and v; we get the surface. Instead of three variables $x; y; z$; it can be expressed in terms of two variables x and y i:e:; $z = f(x; y)$: Then $F(x; y; z) = 0 = F(x; y; f(x; y))$: This is called the Monge's form of a given surface.

De nition 4.5 (Class of surface). If $x = f(u; v)$; $y = g(u; v)$; $z = h(u; v)$ be the parametric equations of a given surface, then the surface is said to be of class r, if the functions $f; g; h$ are single valued continuous functions and possess derivatives of the r^{th} order.

Note 4.2. If partial di erentiation with respect to the parameters u and v are denoted by the su xes are 1 and 2 respectively.

Thus $\tilde{r}_1 = \dfrac{\partial \tilde{r}}{\partial u}$; $\tilde{r}_2 = \dfrac{\partial \tilde{r}}{\partial v}$; $\tilde{r}_{11} = \dfrac{\partial^2 \tilde{r}}{\partial u^2}$; $\tilde{r}_{12} = \dfrac{\partial^2 \tilde{r}}{\partial u \partial v} = \tilde{r}_{21}$; $\tilde{r}_{22} = \dfrac{\partial^2 \tilde{r}}{\partial v^2}$

4.1.1. Regular (or Ordinary) point and Singularities on a surface:

Consider a point P on the surface whose position vector $\tilde{r} = \tilde{r}(u, v)$, where $x = x(u, v)$;

$y = y(u, v)$ and $z = z = (u, v)$.

Then $\tilde{r}_1 = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$; $\tilde{r}_2 = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$

The point P is called regular point or ordinary point if $\tilde{r}_1 \times \tilde{r}_2 \neq 0$

$i.e.,$ if the rank of the matrix $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$ is two.

But, if $\tilde{r}_1 \times \tilde{r}_2 = 0$ at a point P, we say that the point P is called the singular point or we can say that the point P is a singularity of the surface.

Types of Singularities:

There are two types of singularities, namely Essential singularity and Artificial Singularity.

Essential Singularity: These are inherent singularities, $i.e.,$ these singularities are due to the nature (or geometric features) of the surface and these are independent of the choice of parametric representation.

For example, the vertex of the cone is an essential singularity.

Artificial Singularity:

These singularities arises from the choice of particular parametric representation of the surface.

For example, the pole (or origin) in the plane, referred to polar coordinates is an artificial singularity.

Consider $\tilde{r} = (r\cos\theta, r\sin\theta, 0)$, here r and θ are the parameters.

$$\tilde{r}_1 = (\cos\theta, \sin\theta, 0)$$
$$\tilde{r}_2 = (-r\sin\theta, r\cos\theta, 0)$$
$$\tilde{r}_1 \times \tilde{r}_2 = r\tilde{k} = 0 \quad (\text{if } r = 0 \text{ at the pole})$$

Thus, at the pole $r = 0$ is an artificial singularity as it is not due to inherent property of the surface, but it has arisen due to the choice of parametric representation.

Definition 4.6 (Proper Transformation:).

Consider the surface given by the parametric equations

$x = u + v;\ y = u - v;\ z = 4uv$ and $x = u;$ $y = v;\ z = u^2 - v^2.$ These two representations represent the same surface such as $x^2 - y^2 = z$ and are related by the parameter transformation of the form $u' = \phi(u, v);\quad v' = \psi(u, v).$

This transformation is said to be proper transformation, if ϕ and ψ are single values and having non-vanishing Jacobian, i.e.,

$$\frac{\partial(\phi, \psi)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial \phi}{\partial u} & \frac{\partial \psi}{\partial u} \\ \frac{\partial \phi}{\partial v} & \frac{\partial \psi}{\partial v} \end{vmatrix} \neq 0$$

Property of point transformation:

A regular point is transformed to a regular point by a proper parametric transformation.

Let $\tilde{r} = \tilde{r}(u, v)$ be the equation of the surface.

The parameters be transformed by the relations $u' = \phi(u, v)$; $v' = \psi(u, v).$ Moreover, this transformation is a point transformation and hence by definition $\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0.$

$$\tilde{r}_1 = \frac{\partial \tilde{r}}{\partial u'}\frac{\partial u'}{\partial u} + \frac{\partial \tilde{r}}{\partial v'}\frac{\partial v'}{\partial u}$$

$$\text{i.e.,}\quad \tilde{r}_1 = \frac{\partial \tilde{r}}{\partial u'}\frac{\partial \phi}{\partial u} + \frac{\partial \tilde{r}}{\partial v'}\frac{\partial \psi}{\partial u}$$

$$\text{Similarly}\quad \tilde{r}_2 = \frac{\partial \tilde{r}}{\partial u'}\frac{\partial \phi}{\partial v} + \frac{\partial \tilde{r}}{\partial v'}\frac{\partial \psi}{\partial v}$$

$$\tilde{r}_1 \times \tilde{r}_2 = \frac{\partial \tilde{r}}{\partial u'} \times \frac{\partial \tilde{r}}{\partial v'}\frac{\partial(\phi, \psi)}{\partial(u, v)}$$

Now if the given parametric representation of the surface is proper i.e., $\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0$; then if $\tilde{r}_1 \times \tilde{r}_2 \neq 0$ (for an ordinary point) then $\frac{\partial \tilde{r}}{\partial u'} \times \frac{\partial \tilde{r}}{\partial v'}$ is also not zero.

Hence a proper parametric transformation transfers regular (ordinary point) into a regular (ordinary) point.

Definition 4.7. A representation R of a surface S of class r in E_3 is a set of points in E_3 covered by a system of overlapping point $\bigcup V_j$ each part V_j being given by parametric equations of class $r.$ Each point lying in the overlap of two parts V_iV_j is such that the change of parameters from those of one part to those of the other part is proper and class r.

Definition 4.8. Two representations R, R' are said to be r-equivalent if the composite family of parts $\{V_iV_j'\}$ satisfies the condition that at each point P lying in the overlap of any two parts, the change of parameters from those of one part to those of another is proper and class r.

Definition 4.9. A surface S of class r in E_3 is an r-equivalence class of representations.

4.2. Curves on a surface:

We know that a curve is the locus of a point whose position vector $\tilde{r}$ can be expressed as a single parameter.

Let us consider a surface $\tilde{r} = \tilde{r}(u, v)$ defined on a domain D and if u and v are functions of a single parameter t, then the position vector $\tilde{r}$ becomes a function of a single parameter t and hence its locus is a curve lying on the surface $\tilde{r} = \tilde{r}(u, v)$. Let $u = u(t), v = v(t)$, then $\tilde{r} = \tilde{r}(u(t), v(t))$ is a curve lying on the surface $\tilde{r} = \tilde{r}(u, v)$ in D.

The equations $u = u(t), v = v(t)$ are called curvilinear equations of the curve lying on the surface $\tilde{r} = \tilde{r}(u, v)$.

4.2.1. Parametric Curves:

Let $\tilde{r} = \tilde{r}(u, v)$ be the equation of a surface. Now by keeping $u =$ constant or $v =$ constant we get curves of special importance and are called parametric curves.

If $v =$ constant, say c then u varies, the point $\tilde{r} = \tilde{r}(u, c)$ describes a parametric curve called the u curve or the parametric curve $v = c$.

Similarly, if $u =$ constant say c then v varies, the point $\tilde{r}(c, v)$ traces a parametric curve called the v curve or the parametric curve $u = c$.

For u-curve, u is the parameter and determines a sense along the curve. The tangent to the curve in the sense of u-increasing is along the vector

$\tilde{r}_1$. Similarly the tangent to v-curve in the sense of v increasing is along the vector $\tilde{r}_2$.

Thus, we have two systems of parametric curves, viz., u-curve and v-curve and since we know that $\tilde{r}_1 \times \tilde{r}_2 \neq 0$, therefore the parametric curves of different system can not touch each other.

If $\tilde{r}_1 \cdot \tilde{r}_2 = 0$ at a point P, the two parametric curves through the point P are orthogonal. If this condition is satisfied at every point $i.e.,$ for all values of u and v in the domain D, the two systems of parametric curves are orthogonal.

4.2.2. Tangent plane and Normal:

Let $\tilde{r}(u, v)$ be the equation of the surface in terms of the parameters u and v.

$$\frac{d\tilde{r}}{dt} = \frac{\partial \tilde{r}}{\partial u}\frac{du}{\partial t} + \frac{\partial \tilde{r}}{\partial v}\frac{dv}{\partial t}$$

$$\frac{d\tilde{r}}{dt} = \tilde{r}_1\frac{du}{dt} + \tilde{r}_2\frac{dv}{dt}$$

$$\text{or} \quad d\tilde{r} = \tilde{r}_1 du + \tilde{r}_2 dv$$

The tangent to any curve drawn on a surface is called the tangent line to the surface. Now $\tilde{r}_1, \tilde{r}_2$ are non-zero and independent so that tangents to the curve through a point P lie in the plane which contains $\tilde{r}_1$ and $\tilde{r}_2$. This plane is the required tangent plane at P. Since it contains $\tilde{r}_1$ and $\tilde{r}_2$ therefore $\tilde{r}_1 \times \tilde{r}_2$ gives the normal to the plane. If $\tilde{R}$ be the position vector of a current point on the plane then its equation is

$$\left(\tilde{R} - \tilde{r}\right) \cdot \left(\tilde{r}_1 \times \tilde{r}_2\right) = 0$$

$$\text{or} \quad \left[\tilde{R} - \tilde{r}, \tilde{r}_1, \tilde{r}_2\right] = 0$$

From the above , we can say that $\tilde{R} - \tilde{r}, \tilde{r}_1, \tilde{r}_2$ are coplanar and as such one of them can be expressed as a linear combination of the other two.

$$\Rightarrow \tilde{R} - \tilde{r} = a\tilde{r}_1 + b\tilde{r}_2$$

$$i.e., \quad \tilde{R} = \tilde{r} + a\tilde{r}_1 + b\tilde{r}_2$$

which is the equation of the tangent plane at P; where a and b are parameters.

Normal line:

Normal to the tangent plane at P is the line passing through $P\,(\tilde{r})$ and is parallel to the vectors $\tilde{r}_1 \times \tilde{r}_2$; hence the equation of the normal line at P to the surface is given by $R = \tilde{r} + \lambda(\tilde{r}_1 \times \tilde{r}_2)$.

The normal to the surface at P is the same as the normal to the tangent plane at P and therefore the unit normal

$$\tilde{N} = \frac{\tilde{r}_1 \times \tilde{r}_2}{|\tilde{r}_1 \times \tilde{r}_2|} = \frac{\tilde{r}_1 \times \tilde{r}_2}{H} \quad \text{where } \tilde{r}_1 \times \tilde{r}_2 \neq 0 \text{ or } H\tilde{N} = \tilde{r}_1 \times \tilde{r}_2$$

Also, $\tilde{r}_1, \tilde{r}_2, \tilde{N}$ form a right handed system and this gives the direction of the normal.

Example 4.1. Find the equation of the tangent plane and normal to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

Solution:

$$\begin{aligned}
\text{Let } F(x, y, z) &= z - x^2 - y^2 = 0\\
\frac{\partial F}{\partial x} &= -2x = -2 \quad \text{at } (1, -1, 2)\\
\frac{\partial F}{\partial y} &= -2y = 2 \quad \text{at } (1, -1, 2)\\
\frac{\partial F}{\partial z} &= 1 \quad \text{at } (1, -1, 2)
\end{aligned}$$

Thus, the equation of the tangent plane at the point $(1, -1, 2)$ is

$$\begin{aligned}
(x-1)(-2) + (y+1)(2) + (z-2)(1) &= 0\\
\text{i.e.,}\quad -2x + 2 + 2y + 2 + z - 2 &= 0\\
\text{i.e.,}\quad 2x - 2y - z &= 2
\end{aligned}$$

Equation of the normal is

$$\frac{X - x}{\frac{\partial F}{\partial x}} = \frac{Y - y}{\frac{\partial F}{\partial y}} = \frac{Z - z}{\frac{\partial F}{\partial z}}$$

$$\text{i.e.,}\quad \frac{x - 1}{-2} = \frac{y + 1}{2} = \frac{z - 2}{1}$$

Example 4.2. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Solution:

$$\text{Let } F(x, y, z) = x^2y + 2xz - 4 = 0$$

$$\frac{\partial F}{\partial x} = 2xy + 2z = -2 \text{ at } (2, -2, 3)$$

$$\frac{\partial F}{\partial y} = x^2 = 4 \text{ at } (2, -2, 3)$$

$$\frac{\partial F}{\partial z} = 2x = 4 \text{ at } (2, -2, 3)$$

The vector $\tilde{N}$ normal to the surface is given by

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (-2, 4, 4)$$

$$|\tilde{N}| = \sqrt{4 + 16 + 16} = 6$$

$$\Rightarrow \text{ Unit normal vector } = \left(\frac{-2}{6}, \frac{4}{6}, \frac{4}{6}\right)$$

4.3. Surface of Revolution:

4.3.1. The Sphere:

When the polar angles (i.e.,) Co-latitude u and the langitude v are taken as parameters on a sphere of centre O and radius a, the position vector is

$$\tilde{r} = (\sin u \cos v, \sin u \sin v, \cos u)$$

The poles $u = 0$ and $u = \pi$ are artificial singularities and domain of u, v is $0 < u < \pi, \ 0 \le v < 2\pi$.

The parametric curves $v =$ constant are the meridians and $u =$ constant are the parallels.

$$\tilde{r}_1 = a(\cos u \cos v, \cos u \sin v, -\sin u)$$

$$\tilde{r}_2 = a(-\sin u \sin v, \sin u \cos v, 0)$$

Now $\tilde{r}_1 \cdot \tilde{r}_2 = 0$ at all points.

Thus, the two system of the parametric curves are orthogonal.

$$\text{Now}\quad \tilde{r}_1 \times \tilde{r}_2 = a^2\left(\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos v\right)$$
$$H = |\tilde{r}_1 \times \tilde{r}_2| = a^2 \sin u$$
$$\tilde{N} = = \frac{\tilde{r}_1 \times \tilde{r}_2}{H} = (\sin u\cos v, \sin u\sin v, \cos u) = \frac{1}{a}\tilde{r}$$

which is directed outwards from the sphere.

4.3.2. The general surface of revolution:

Taking z-axis for the axis of revolution, let the generating curve in the xz plane be given by the parametric equations

$$x = g(u), \quad y = 0, \quad z = f(u)$$

Then, if v is the angle of rotation about the z axis, the position vector of the point (u, v) is

$$\tilde{r} = \left(g(u)\cos v, g(u)\sin v, f(u)\right)$$

and the domain of u, v is $0 \le v < w$ together with the range of u.

As in the case of sphere v = constant are the meridians given by the various position of the generating curve and u = constant are parallels, circles in planes, parallel to the xy plane.

The vectors $\tilde{r}_1$ and $\tilde{r}_2$ are given by

$$\tilde{r}_1 = \left(g'(u)\cos v, g'(u)\sin v, f'(u)\right)$$
$$\tilde{r}_2 = (-g(u)\sin v, g(u)\cos v, 0)$$

Thus $\tilde{r}_1 \cdot \tilde{r}_2 = -g(u)g'(u)\sin v\cos v + g(u)g'(u)\cos v\sin v = 0$ for all u, v i.e., the parameters are orthogonal.

The unit normal vector $\tilde{N}$ is given by

$$\tilde{N} = \frac{\tilde{r}_1 \times \tilde{r}_2}{H} = \frac{(-f'(u)\cos v, -f'(u)\sin v, g'(u))}{\left(f'^2(u) + g'(u)^2\right)^{1/2}}$$

using the fact that $g \neq 0$ at an ordinary point.

If $g(u) = u$, the right circular cone of semi-vertical angle α, for example $g(u) = u, \quad f(u) = u\cot\alpha.$

$\Rightarrow \tilde{r} = (u\cos v, u\sin v, u\cot\ )$.

4.3.3. The anchor ring:

The anchor ring is obtained by rotating a circle of radius a about a line in its plane and at a distance $b(> a)$ from its centre.

Therefore, $g(u) = b + a\cos u, \quad f(u) = a\sin u.$

Thus, $\tilde{r} = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u)$ and the domain of u, v is $0 < u < 2\ , \quad 0 < v < 2\ .$

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of singularities on a surface.
- the concept of proper transformation.
- nd the equation of tangent plane and normal.

Check Your Progress:

1. De ne Parametric curves.
2. Prove that a regular point is transformed to a regular point by a parametric transformation.
3. Find a unit normal vector to the surface $2xz^2\ \ 3xy\ \ 4x = 7$ at the point $(1,\ \ 1, 2)$.

Answer:

3. $\left(\frac{7}{\sqrt{122}},\ \frac{5}{\sqrt{122}}, \frac{8}{\sqrt{122}}\right)$

Choose the correct or more suitable answer:

1. If : : : : : : at a Point P, the two parametric curves through the point P are orthogonal.

   (a) $\tilde{r}_1 \quad \tilde{r}_2 = 0$ (b) $\tilde{r}_1 \quad \tilde{r}_2 \neq 0$ (c) $\tilde{r}_1 \quad \tilde{r}_2 = 0$ (d) $\tilde{r}_1 \quad \tilde{r}_2 \neq 0$

2. The pole in the plane, referred to polar coordinates is : : : : : :
   (a) an essential singularity
   (b) removal singularity
   (c) arti cial singularity
   (d) none of these.

3. vertex of cone is an : : : : : :
   (a) an essential singularity
   (b) removal singularity
   (c) arti cial singularity
   (d) none of these.

4. The transformation is said to be point transformation, if
   (a) and are multiple variables and having vanishing Jacobian.
   (b) and are multiple variables and having non-vanishing Jacobian.
   (c) and are single variables and having vanishing Jacobian.
   (d) and are single variables and having non-vanishing Jacobian.

Answer:

(1) c (2) c (3) a (4) d

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-II

UNIT-5

METRIC

Structure

Objective

Overview

5. 1 Helicoids

5. 1. 1 Right helicoid

5. 1. 2 The general helicoid

5. 2 Metric

5. 2. 1 Geometrical Interpretation of metric

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be to

- F nd the relationship between the fundamental coe cients.
- F derive the equation of the metric and understanding its geometrical interpretation.

Overview

In this unit, we will illustrate to nd the relationship between the fundamental coe cients and geometrical interpretation of metric also explained.

5.1. Helicoids:

A helicoid is a surface generated by the screw motion of a curve about a xed line, the axis. The various position of the generating curve are obtained by rst translating it through a distance parallel to the axis and then rotating it through an angle about the axis, where $-$ = has a constant value :

The constant 2 is called the pitch of the helicoid.

5.1.1. Right helicoid:

This is the helicoid generated by a straight line which meets the axis at right angles. Taking the axis to be the z-axis, the position vector is

$$\tilde{r} = (u \cos v; u \sin v; av)$$

where u and v are respectively the distance from the axis and the distance from the angle of rotation. The generator being the x-axis when $v = 0$: Here u and v take real values.

$$\tilde{r}_1 = (\cos v;\ \sin v;\ 0)$$
$$\tilde{r}_2 = (\ u \sin v;\ u \cos v;\ a)$$
$$\Rightarrow \tilde{r}_1\ \tilde{r}_2 = 0$$

Thus, the curves $v =$ constant are the generators and $u =$ constant are circular helices, $\tilde{r}_1\ \tilde{r}_2 = 0;$ the helices are orthogonal to the generators.

5.1.2. The general helicoid:

The general helicoid is given by the equation

$$x = g(u);\quad y = 0;\quad z = f(u)$$

The position vector of a point on the surface is

$$\tilde{r} = (g(u) \cos v; g(u) \sin v;\ f(u) + av)$$

The curves $v =$ constant are the generators and $u =$ constant are circular helices.

When parametric curves are orthogonal, we get a helicoid (or) $a = 0$ which gives a surface of revolution.

Example 5.1. A helicoid is generated by a screw motion of a straight line skew to the axis. Find the curve coplanar with the axis which generates the same helicoid.

Solution: If c is the shortest distance and is the angle between the axis and the given skew line, then this line can be taken as $x = c;$ $y = u \sin\ ;\ z = u \cos$ where u is the parameter. Rotating through an angle v about the z axis and translating a distance av parallel to this axis, the position vector of a point on the helicoid is found to be

$$\tilde{r} = (c \cos v \quad u \sin \quad \sin v; c \sin v + u \sin \quad \cos v; u \cos \quad + av) \tag{5.1}$$

The required plane curve is the section of this surface by the plane $y = 0$ and is given by $u \sin \quad \cos v = \quad c \sin v:$ $i:e:;$ $u \sin \quad = \quad c \tan v:$

Substituting this in equation (5.1), we get

$x = c \cos v;\quad y = 0;\ z = av \quad c \cot \quad \tan v$ where v is a parameter for the curve:

In the notation used above for the general helicoid, $g(u) = c \sec u;$ and $f(u) = au \quad c \cot \quad \tan u:$

5.2. Metric:

Let $\underset{\sim}{r} = \underset{\sim}{r}(u, v)$ be the equation of the surface. Consider the curve defined by $u = u(t)$; $v = v(t)$ on the surface, then $\underset{\sim}{r}$ is a function of t along the curve and the arc length s is related to the parameter t by

$$\begin{aligned}
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{d\underset{\sim}{r}}{dt}\right)^2 = \left(\underset{\sim}{r}_1\frac{du}{dt} + \underset{\sim}{r}_2\frac{du}{dt}\right)^2 \\
&= \underset{\sim}{r}_1^{\,2}\left(\frac{du}{dt}\right)^2 + 2\underset{\sim}{r}_1\cdot\underset{\sim}{r}_2\frac{du}{dt}\frac{dv}{dt} + \underset{\sim}{r}_2^{\,2}\left(\frac{dv}{dt}\right)^2 \\
&= E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}\frac{dv}{dt}
\end{aligned}$$

where $E = \underset{\sim}{r}_1^{\,2}$; $F = \underset{\sim}{r}_1\cdot\underset{\sim}{r}_2$; $G = \underset{\sim}{r}_2^{\,2}$

The above equation can be expressed conveniently in the following quadratic differential form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{5.2}$$

The right hand side of equation (5.2) does not involve the parameter t except in so far as u and v depends on t.

Definition 5.1 (Metric).

The quadratic differential form $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ in du and dv is called metric or first fundamental form of the surface and the quantities E, F, G are called the first fundamental coefficients or fundamental magnitudes of first order.

5.2.1. Geometrical Interpretation of metric:

Let $\underset{\sim}{r} = \underset{\sim}{r}(u)$ be a given surface. Let P and Q be two neighbouring points on the curve with position vectors $\underset{\sim}{r}$ and $\underset{\sim}{r} + \delta\underset{\sim}{r}$ respectively.

$$\begin{aligned}
d\underset{\sim}{r} &= \frac{\partial\underset{\sim}{r}}{\partial u}du + \frac{\partial\underset{\sim}{r}}{\partial v}dv = \underset{\sim}{r}_1 du + \underset{\sim}{r}_2 dv \\
\text{Let } \overline{PQ} &= ds, \text{ then } ds = |d\underset{\sim}{r}| \\
\therefore ds^2 &= (d\underset{\sim}{r})^2 = (\underset{\sim}{r}_1 du + \underset{\sim}{r}_2 dv)^2 \\
&= Edu^2 + 2Fdudv + Gdv^2
\end{aligned}$$

If ds can be interpreted as the infinitesimal distance from the point P to the point Q on the surface. Thus, the first fundamental form is used to calculate the arc lengths on the surface.

Relation between the fundamental coefficients:

$$\text{Now,}\quad (\tilde{r}_1 \times \tilde{r}_2)^2 = (\tilde{r}_1 \times \tilde{r}_2)\cdot(\tilde{r}_1 \times \tilde{r}_2)$$
$$= \tilde{r}_1^{\,2}\tilde{r}_2^{\,2} - (\tilde{r}_1 \cdot \tilde{r}_2)^2$$
$$H^2 = EG - F^2 \quad \text{where } H = |\tilde{r}_1 \times \tilde{r}_2|$$

The coefficients E, G and H^2 satisfy $E > 0; G > 0; H^2 = EG - F^2 > 0.$

Since $E > 0;$ we may take

$$\begin{aligned} ds^2 &= Edu^2 + 2Fdudv + Gdv^2 \\ &= \frac{1}{E}\left[E^2du^2 + 2FEdudv + EGdv^2\right] \\ &= \frac{1}{E}\left[(Edu + Fdv)^2 + (EG - F^2)dv^2\right] \\ &= \frac{1}{E}\left[(Edu + Fdv)^2 + H^2dv^2\right] \geq 0 \end{aligned}$$

$$\begin{aligned} \left[Edu^2 + 2Fdudv + Gdv^2\right] &= 0 \\ \Rightarrow \frac{1}{E}\left[(Edu + Fdv)^2 + H^2dv^2\right] &= 0 \\ \Rightarrow (Edu + Fdv)^2 + H^2dv^2 &= 0 \\ \Rightarrow Edu + Fdv &= 0; \quad \text{and } H^2dv^2 = 0 \\ \Rightarrow Edu + Fdv &= 0; \quad \text{and } dv = 0 \\ \Rightarrow du &= 0 \text{ and } dv = 0 \end{aligned}$$

But both du and dv cannot vanish together.

Hence, the metric $Edu^2 + 2Fdudv + Gdv^2 = 0$ is a positive definite quadratic form in du and $dv.$

Example 5.2. Compute the first fundamental magnitudes for the surface $\tilde{r} = (u\cos v; u\sin v; f(u))$.

Solution:

$$\begin{aligned} \tilde{r}_1 &= (\cos v; \sin v; f'(u)) \\ \tilde{r}_2 &= (-u\sin v; u\cos v; 0) \\ E &= \tilde{r}_1 \cdot \tilde{r}_1 = \cos^2 v + \sin^2 v + f'^2(u) = 1 + f'^2(u) \end{aligned}$$

$$F = \tilde{r}_1 \cdot \tilde{r}_2 = -u \sin v \cos v + u \cos v \sin v = 0$$
$$G = \tilde{r}_2 \cdot \tilde{r}_2 = u^2 \left(\cos^2 v + \sin^2 v\right) = u^2$$
$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = \left(1 + f'^2(u)\right) du^2 + u^2 dv^2$$

Example 5.3. Calculate the fundamental coefficients E, F, G and H for the paraboloid $\tilde{r} = \left(u, v, u^2 - v^2\right)$.

Solution:

$$\text{Given that } \tilde{r} = \left(u, v, u^2 - v^2\right)$$
$$\tilde{r}_1 = (1, 0, 2u)$$
$$\tilde{r}_2 = (0, 1, -2v)$$
$$E = \tilde{r}_1 \cdot \tilde{r}_1 = 1 + 4u^2$$
$$F = \tilde{r}_1 \cdot \tilde{r}_2 = -4uv$$
$$G = \tilde{r}_2 \cdot \tilde{r}_2 = 1 + 4v^2$$
$$H = \sqrt{EG - F^2} = \left[\left(1 + 4u^2\right)\left(1 + 4v^2\right) - 16u^2v^2\right]^{1/2}$$
$$= \left(1 + 4u^2 + 4v^2\right)^{1/2}$$

Angle between parametric curves:

Let P be the point of intersection of the parametric curves $u =$ constant and $v =$ constant. Let $\tilde{r}$ be the position vector of the point P; $\tilde{r}_1$ and $\tilde{r}_2$ are the tangent vectors to the two curves at P respectively.

The angle ω $(0 < \omega < \pi)$ between them are given by

$$\cos \omega = \frac{\tilde{r}_1 \cdot \tilde{r}_2}{|\tilde{r}_1|\,|\tilde{r}_2|} = \frac{F}{\sqrt{EG}}$$
$$\sin \omega = \frac{|\tilde{r}_1 \times \tilde{r}_2|}{|\tilde{r}_1|\,|\tilde{r}_2|} = \frac{H}{\sqrt{EG}}$$
$$\tan \omega = \frac{H}{F}$$

The parametric curves are cut orthogonal when $F = 0$ i.e., $\tilde{r}_1 \cdot \tilde{r}_2 = 0$.

Element of Area:

Consider the following figures with four vertices (u, v), $(u + \delta u, v)$, $(u + \delta u, v + \delta v)$ and $(u, v + \delta v)$ joined by the parametric curves.

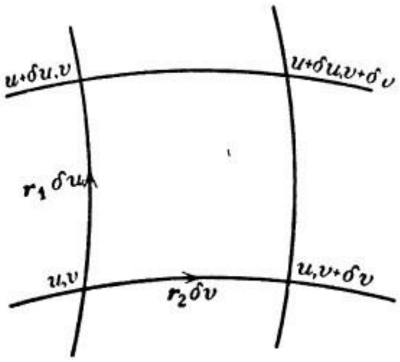


Figure 5.1

If δu and δv are small and positive, then this figure is approximately equal to parallelogram with adjacent sides given by $\tilde{r}_1\,\delta u$ and $\tilde{r}_2\,\delta v$.

Now, if δs be the area of the parallelogram, then

$$\delta s = \left|\tilde{r}_1\,\delta u \times \tilde{r}_2\,\delta v\right| = \left|\tilde{r}_1 \times \tilde{r}_2\right| \delta u\,\delta v = H\,\delta u\,\delta v$$

Example 5.4. For the anchor ring,

$\tilde{r} = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u)$. Calculate the area corresponding to the domain $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

Solution:

$$\begin{aligned}
\text{Given that } \tilde{r} &= ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u) \\
\tilde{r}_1 &= (-a\sin u\cos v, -a\sin u\sin v, a\cos v) \\
\tilde{r}_2 &= (-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0) \\
E &= \tilde{r}_1 \cdot \tilde{r}_1 \\
&= a^2\left(\sin^2 u\cos^2 v + \sin^2 u\sin^2 v\right) + a^2\cos^2 u \\
&= a^2\left(\sin^2 u + \cos^2 u\right) = a^2 \\
F &= \tilde{r}_1 \cdot \tilde{r}_2 = 0 \\
G &= \tilde{r}_2 \cdot \tilde{r}_2 = (b + a\cos u)^2\left(\sin^2 v + \cos^2 v\right) = (b + a\cos u)^2 \\
H &= \sqrt{EG - F^2} = \sqrt{a^2(b + a\cos u)^2} = a(b + a\cos u) \\
\text{Thus, element of area} &= H\,du\,dv = a(b + a\cos u)\,du\,dv \\
\therefore \text{ The total area} &= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u)\,du\,dv = 4\pi^2 ab
\end{aligned}$$

Example 5.5. Show that the metric is invariant under a parameter transformation.

Solution: Let $\tilde{r} = \tilde{r}(u, v)$ be the equation of the surface. The parameters

u, v are transformed into the parameters u' and v' by the relations

$$u' = (u, v)\,; \quad v' = (u, v) \qquad (5.3)$$

$$\tilde{r}_1{}' = \frac{\partial \tilde{r}}{\partial u'} = \frac{\partial \tilde{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \tilde{r}}{\partial v}\frac{\partial v}{\partial u'}$$

$$\tilde{r}_1{}' = \tilde{r}_1 \frac{\partial u}{\partial u'} + \tilde{r}_2 \frac{\partial v}{\partial u'} \qquad (5.4)$$

In a similar way, we can write
$$\tilde{r}_2{}' = \tilde{r}_1 \frac{\partial u}{\partial v'} + \tilde{r}_2 \frac{\partial v}{\partial v'} \qquad (5.5)$$

Now
$$\begin{aligned}
E'du'^2 + 2F'du'dv' + G'dv'^2 &= \tilde{r}_1{}'^2 du'^2 + 2\tilde{r}_1{}'\tilde{r}_2{}' du'dv' + \tilde{r}_2{}'^2 dv'^2 \\
&= \left(\tilde{r}_1{}' du' + \tilde{r}_2{}' dv'\right)^2 \\
&= \left[\left(\tilde{r}_1 \frac{\partial u}{\partial u'} + \tilde{r}_2 \frac{\partial v}{\partial u'}\right) du' + \left(\tilde{r}_1 \frac{\partial u}{\partial v'} + \tilde{r}_2 \frac{\partial v}{\partial v'}\right) dv'\right]^2 \\
&= \left[\tilde{r}_1\left(\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'\right) + \tilde{r}_2\left(\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'\right)\right]^2 \\
&= \left(\tilde{r}_1 du + \tilde{r}_2 dv\right)^2 \\
&= \tilde{r}_1^2 du^2 + 2\tilde{r}_1\tilde{r}_2 dudv + \tilde{r}_2^2 dv^2 \\
&= Edu^2 + 2Fdudv + Gdv^2
\end{aligned}$$

Thus the metric is invariant under parametric transformation.

Let Us Sum Up:

In this unit, the students acquired knowledge

- to know the concept of helicoid and right helicoid.
- to know the relationship between the fundamental coe cients.

Check Your Progress:

1. Explain geometrical interpretation of metric.
2. Prove that the metric is invariant under a transformation of parameters.

Choose the correct or more suitable answer:

1. For the paraboloid $x = u;\ y = v;\ z = u^2 \quad v^2$, the value of E is

   (a) $1 + 4u$ (b) $1 \quad 4u$ (c) $1 + 4u^2$ (d) $1 \quad 4u^2$

2. Relation between the coe cients $E; F; G$ and H is

   (a) $H^2 = EG + F^2$

   (b) $H = EG + F^2$

   (c) $H^2 = EG \quad F^2$

   (d) $H^2 = EG + F$

Answer:

(1) c (2) c

Glossaries

Helicoid: A helicoid with generating line perpendicular to its axis.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-II

UNIT-6

FAMILIES OF CURVES

Structure:

Objective

Overview

6. 1 Direction coe cients

 6. 1. 1 Angle between the directions

6. 2 Relation between direction coe cients and direction ratios

6. 3 Families of Curves

 6. 3. 1 Orthogonal Trajectories

 6. 3. 2 Double family of curves

6. 4 Isometric correspondence

6. 5 Intrinsic properties

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be to

- F nd the direction coe cients and angle between the directions.
- F nd the condition for orthogonal direction.
- F understand the concept of families of curves and derive the equation for families of curves.
- F de ne the isometric-correspondence between two points on two surfaces.

Overview

In this unit, we will illustrate the relationship between direction coe cients and direction ratios.

6.1. Direction coe cients:

At each point P of a surface $\tilde{r} = \tilde{r}(u; v)$ there are three independent vectors $\tilde{N}; \tilde{r}_1$ and $\tilde{r}_2:$ Every vector $\tilde{a}$ at P can be expressed in the form

$$\tilde{a} = a_n\tilde{N} + \tilde{r}_1 + \tilde{r}_2$$

where scalars $a_n; ;$ are de ned uniquely by this relation.

This gives $\tilde{a}$ as the sum of two vectors $a_n\tilde{N}$ normal to the surface and $\tilde{r}_1 + \tilde{r}_2$ is the tangent plane at $P:$ The scalar a_n is called the normal component of $\tilde{a}$ and is given by $a_n = \tilde{a} \quad \tilde{N}:$ The vector $\tilde{r}_1 + \tilde{r}_2$ is called the tangential part of $\tilde{a}$ and $;$ are the tangential components of $\tilde{a}:$

A direction in the tangent plane at P is conveniently described by the components of unit vector in this direction. These components are called direction coe cients and written as $(l; m):$ The direction coe cients satisfy the identity $El^2 + 2Flm + Gm^2 = 1:$

6.1.1. Angle between the directions:

If $(l;m)$ and $(l^o; m^o)$ are coe cients of two directions at the same point, then the corresponding unit vectors are

$$\tilde{a} = l\tilde{r}_1 + m\tilde{r}_2; \quad \tilde{a}^o = l^o\tilde{r}_1 + m^o\tilde{r}_2$$

The angle between these directions, measured in the sense described above is given by

$$\cos = \tilde{a} \; \tilde{a}^o$$
$$\Rightarrow \cos = Ell^o + F\,lm^o + l^o m + Gmm^o$$
$$\text{and } \sin \tilde{N} = \tilde{a} \; \tilde{a}^o \Rightarrow \sin = H \; lm^o \quad l^o m$$

Note 6.1. The direction coe cients opposite to $(l; m)$ is $(\; l; \; m):$

6.2. Relation between direction coe cients and direction ratios:

Direction rations are proportional to direction coe cients, therefore

$$\frac{l}{} = \frac{m}{} = k$$
$$\Rightarrow l = k; \quad m = k$$

Since $(l; m)$ are direction coe cients, so we have

$$El^2 + 2Flm + Gm^2 = 1$$
$$i:e:; \quad E \;{}^2k^2 + 2F(\;k)(\;k) + Gk^2 \;{}^2 = 1$$
$$k^2 \; E \;{}^2 + 2F \quad + G \;{}^2 = 1$$

$$k = \frac{1}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

$$\Rightarrow \quad l = \lambda k = \frac{\lambda}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

$$\text{Similarly,} \quad m = \mu k = \frac{\mu}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

Thus, the direction ratios, the numbers $(\lambda; \mu)$ proportional to $(l; m)$ have the relations

$$(l; m) = \frac{(\lambda; \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

Note 6.2. **The condition for orthogonal direction:**

If $\theta = 90^\circ$; then the directions with direction coefficients $(l; m)$ and $(l'; m')$ are orthogonal for which the condition will be

$$Ell' + F(lm' + l'm) + Gmm' = 0$$

$$\text{or } E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0$$

Note 6.3. The vectors $\tilde{r}_1$ and $\tilde{r}_2$ have components $(1; 0)$ and $(0; 1)$: Then the direction coefficients are

$$\frac{(1; 0)}{\sqrt{E + 0 + 0}} = \left(\frac{1}{\sqrt{E}}; 0\right) \text{ and } \frac{(0; 1)}{\sqrt{0 + 0 + G}} = \left(0; \frac{1}{\sqrt{G}}\right)$$

6.3. Families of Curves:

$$\text{Let } \tilde{r} = \tilde{r}(u; v) \text{ represent a surface} \qquad (6.1)$$

Two parameters $u; v$ are connected by the relation $\phi(u; v) = c$ (6.2)

where $\phi(u; v)$ is a single valued function and have continuous derivatives ϕ_1 and ϕ_2 which do not vanish together and c is a real parameter.

The equation (6.2) shows that a family of curves lying on the surface (6.1).The different curves belonging to the family (6.2) and it lying on the surface (6.1) for different values of c. Also (6.2) represent one member of the family, when c is a constant.

Note 6.4. One curve of the family of curves (6.2) passing through every point (u, v) of the surface (6.1).

Differential Equation of family of curves:

Let $\phi(u, v) = c$ represents a family of curves.

$$\text{Differentiating , we get } \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0$$
$$\Rightarrow \phi_1 du + \phi_2 dv = 0$$
$$i.e., \frac{du}{dv} = -\frac{\phi_1}{\phi_2}$$

Thus, $(-\phi_2, \phi_1)$ are direction ratios of the tangent at the point (u, v) to the member of family (6.2) which passes through that point.

Suppose, if ϕ_1, ϕ_2 both vanish together at any point, the directions are indeterminate which means that we shall not have a definite tangent at that point. Thus the above restriction is necessary.

Conversely, every first order differential equation of the form

$$P(u, v)du + Q(u, v)dv = 0 \tag{6.3}$$

where P and Q are class 1 functions which do not vanish together, always define a family of curves. With this, the equation (6.3) is always integrable so that every function $\lambda(u, v) \neq 0$ and $\phi(u, v)$ such that $\lambda P = \phi_1; \lambda Q = \phi_2.$

Thus the equation (6.3) becomes

$$\frac{1}{\lambda}(\phi_1 du + \phi_2 dv) = 0$$
$$i.e., \phi_1 du + \phi_2 dv = 0$$

The solution of the above equation is therefore $\phi(u, v) =$ constant.

Also the tangent at the point (u, v) for the family of curves are given by (6.3) has direction ratios $(-Q, P)$ since these are directly proportional to (du, dv).

6.3.1. Orthogonal Trajectories:

Definition 6.1. Let $\phi(u,v) = c$ be a given family of curves lying on a surface $\tilde{r} = \tilde{r}(u,v)$ then if there exists another family of curves $\psi(u,v) = k$ lying on the same surface such that every point of the surface the two curves one from each family are orthogonal, then the family of curves $\psi(u,v) = k$ is called orthogonal trajectory of the family of curves $\phi(u,v) = c.$

Bookwork 6.1. Derive the differential equation of the orthogonal trajectories.

Let $\tilde{r} = \tilde{r}(u,v)$ be the equation of the surface and let $\phi(u,v)$ be the equation of given family of curves on $\tilde{r}(u,v).$

$$\begin{aligned} \text{Differentiating } \phi(u,v) &= c && (6.4)\\ d\phi &= 0 \\ \Rightarrow \frac{\partial\phi}{\partial u}du + \frac{\partial\phi}{\partial v}dv &= 0 \\ \Rightarrow Pdu + Qdv &= 0 \text{ (say)} \\ \Rightarrow Pdu &= -Qdv \\ \Rightarrow \frac{du}{-Q} &= \frac{dv}{P} && (6.5) \end{aligned}$$

Therefore $(-Q, P)$ are direction ratios of tangent at any point (u,v) of member of family $\phi(u,v) = c.$

Let the direction ratios of orthogonal trajectories of (6.4) be denoted by $(du, dv).$

Thus, by condition of orthogonality, we have

$$\begin{aligned} E\lambda\lambda_1 + F(\lambda\mu_1 + \lambda_1\mu) + G\mu\mu_1 &= 0 \\ \Rightarrow E(-Q)du + F(-Qdv + Pdu) + GPdv &= 0 \\ \Rightarrow (FP - EQ)\,du + (GP - FQ)\,dv &= 0 \end{aligned}$$

The coefficients du and dv are continuous and do not vanish together since $EG \neq F^2$ and $P, -Q$ do not vanish together.

This is the required differential equation of the orthogonal trajectories of the family of curves $\phi(u,v) = c.$

6.3.2. Double family of curves:

The quadratic di erential equation of the form

$$Pdu^2 + 2Qdudv + Rdv^2 = 0 \qquad (6.6)$$

where P, Q, R are continuous functions of u and v and do not vanish together represent two family of curves on the surface provided $Q^2 \quad PR > 0.$

Thus, the equation (6.6) can be written in the form

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0 \qquad (6.7)$$

which is a quadratic in $\frac{du}{dv}.$

Bookwork 6.2. Derive the condition that the quadratic di erential equation $Pdu^2 + 2Qdudv + Rdv^2 = 0$ represents orthogonal families of curves.

Let the direction ratios of the curves of the two families given by (6.6) through a point (u, v) on the surface be $(\; ; \;)$ and $(\;^0 ; \;^0)$. Then $\frac{}{}$ and $\frac{^0}{^0}$ are the roots of the quadratic equation (6.7).

$$\text{Sum of the roots} = \frac{}{} + \frac{^0}{^0} = \frac{2Q}{P}$$

$$\text{Product of the roots} = \frac{}{}\frac{^0}{^0} = \frac{R}{P}$$

The directions $(\; ; \;)$ and $(\;^0 ; \;^0)$ are orthogonal if

$$E \;^0 + F (\;^0 + \;^0) + G \;^0 = 0$$

$$i.e., \quad E\frac{}{}\frac{^0}{^0} + F\left(\frac{}{} + \frac{^0}{^0}\right) + G = 0$$

$$\Rightarrow \quad E\frac{R}{P} \quad \frac{2Q}{P}F + G = 0$$

$$i.e., \quad ER \quad 2QF + GP = 0$$

If $P = R = 0$ in (6.6), then the equation reduces to $dudv = 0$ giving the two families of parametric curves. Thus, the condition for parametric curves to be orthogonal is $F = 0$:

Example 6.1. On the paraboloid $x^2 \; y^2 = z$; nd the orthogonal trajectories of sections by the planes $z = \text{constant}$.

Solution: Given a surface $x^2 \;\; {}^2 = z$: Let $x = u; y = v$ so that $z = u^2 \quad v^2$:

Given curve $z = c \Rightarrow u^2 \quad v^2 = c$:

Therefore, equation of paraboloid can be written in vector form as

$$\begin{aligned}
\tilde{r} &= u\tilde{i} + v\tilde{j} + \; u^2 \quad v^2 \; \tilde{k} \\
\tilde{r}_1 &= \tilde{i} + 0\tilde{j} + 2u\tilde{k} \\
\tilde{r}_2 &= 0\tilde{i} + \tilde{j} \quad 2v\tilde{k} \\
\text{Now,} \quad E &= \tilde{r}_1 \;\; \tilde{r}_1 = 1 + 4u^2 \\
F &= \tilde{r}_1 \;\; \tilde{r}_2 = \quad 4uv \\
G &= \tilde{r}_2 \;\; \tilde{r}_2 = 1 + 4v^2 \\
\text{Given curve,} \; u^2 \quad v^2 &= c \\
\Rightarrow 2udu \quad 2vdv &= 0 \Rightarrow \frac{du}{v} = \frac{dv}{u}
\end{aligned}$$

Therefore, the tangents at $(u; v)$ has direction ratios $(v; u)$:

Let $(du; dv)$ be direction ratios of orthogonal to the direction $(u; v)$:

$\Rightarrow \quad = v; \quad = u; \quad {}_1 = du; \quad {}_1 = dv$

So, by orthogonality condition, we have

$$\begin{aligned}
E \quad {}_1 + F(\quad {}_1 + \quad {}_1) + G \quad {}_1 &= 0 \\
\Rightarrow \quad 1 + 4u^2 \; vdu + (\quad 4uv)[vdv + udu] + \; 1 + 4v^2 \; udv &= 0 \\
\Rightarrow vdu + udv &= 0 \\
\Rightarrow d(uv) &= 0 \\
\Rightarrow uv &= \text{constant} \\
\Rightarrow xy &= \text{constant}
\end{aligned}$$

These are orthogonal trajectories of given curves.

Example 6.2. Show that on a right helicoid, the family of curves orthogonal to the curves $u \cos v = \text{constant}$ is the family $u^2 + v^2 \; \sin^2 v = \text{constant}$.

Solution:

We know that the equation of right helicoid is $\tilde{r} = (u \cos v; u \sin v; av)$:

$$\tilde{r} = u\cos v\tilde{i} + u\sin v\tilde{j} + av\tilde{k}$$
$$\tilde{r}_1 = \cos v\tilde{i} + \sin v\tilde{j} + 0\tilde{k}$$
$$\tilde{r}_2 = \quad u\sin v\tilde{i} + u\cos v\tilde{j} + a\tilde{k}$$

Now,
$$E = \tilde{r}_1 \ \tilde{r}_1 = \cos^2 v + \sin^2 v = 1$$
$$F = \tilde{r}_1 \ \tilde{r}_2 = \quad u\sin v\cos v + u\sin v\cos v + 0 = 0$$
$$G = \tilde{r}_2 \ \tilde{r}_2 = u^2 + a^2$$

Family of given curves: $u\cos v =$ constant.

Di erentiating both sides, we get $u(\quad \sin v dv) + \cos v du = 0$

$$\Rightarrow \cos v du = u\sin v dv$$
$$\Rightarrow \frac{du}{u\sin v} = \frac{dv}{\cos v}$$

The direction ratios of tangent at (u, v) is $(u\sin v, \cos v)$.

Let (du, dv) be orthogonal to the direction ratios of orthogonal to the given curve.

$\Rightarrow \quad = u\sin v; \quad = \cos v; \quad _1 = du; \quad _1 = dv.$

By orthogonality condition, we have

$$E \quad _1 + F(\quad _1 + \quad _1) + G \quad _1 = 0$$
$$\Rightarrow 1(u\sin v)du + 0 + \ u^2 + a^2 \ \cos v dv = 0$$
$$\Rightarrow u\sin v du = \quad u^2 + a^2 \ \cos v dv$$
$$\Rightarrow \frac{udu}{u^2 + a^2} = \frac{\cos v}{\sin v} dv$$

Integrating, we get
$$\log \ u^2 + a^2 = 2\log(\sin v) + \log c$$
$$\Rightarrow \ u^2 + a^2 \ \sin^2 v = c$$

which is the required family of curves.

Example 6.3. A helicoid is generated by the screw motion of a straight line which meets the axis at an angle : Find the orthogonal trajectories of the generators. Find also the metric of the surface referred to the generators and their orthogonal trajectories as parametric curves.

Solution:The equation of given helicoid is

$$\tilde{r} = u \sin \quad \cos v\tilde{i} + u \sin \quad \sin v\tilde{j} + (u \cos \quad + av)\tilde{k}$$

$$\tilde{r}_1 = \sin \quad \cos v\tilde{i} + \sin \quad \sin v\tilde{j} + \cos \quad \tilde{k}$$

$$\tilde{r}_2 = \quad u \sin \quad \sin v\tilde{i} + u \sin \quad \cos v\tilde{j} + a\tilde{k}$$

$$E = \tilde{r}_1 \quad \tilde{r}_1 = 1$$

$$F = \tilde{r}_1 \quad \tilde{r}_2 = a \cos$$

$$G = \tilde{r}_2 \quad \tilde{r}_2 = u^2 \sin^2 \quad + a^2$$

generators are given by $v =$ constant.

$$v = c \Rightarrow dv = 0 \text{ (or) } dv = 0(du)$$

$$\Rightarrow \frac{dv}{0} = \frac{du}{1}$$

$$\Rightarrow \frac{du}{1} = \frac{dv}{0}$$

Therefore, the direction ratio of the given family of curves is $(1; 0)$: Let $(du; dv)$ be the direction ratios orthogonal to $(1; 0)$:

We get $\quad = 1; \quad = 0; \quad _1 = du; \quad _1 = dv$:

By orthogonality condition, we have

$$E \quad _1 + F(\quad _1 + \quad _1) + G \quad _1 = 0$$

$$1 \quad 1 \quad du + a \cos \quad (1 \quad dv + du \quad 0) + \quad u^2 \sin^2 \quad + a^2 \quad 0 \quad dv = 0$$

$$\Rightarrow du + a \cos \; dv = 0$$

Integrating, we get $u + av \cos \quad =$ constant

This is the required orthogonal trajectories of given family of curves.

To examine these trajectories note that $u = 0$ for some value of v on every curve, so that every trajectory meets the axis of the helicoid.

For a particular curve there is no loss of generality in taking its intersection with the axis to be the origin.

Then $u = \quad av \cos \quad$ and the curve is given by

$$\tilde{r} = a \sin \quad (\quad v \cos \quad \cos v; \quad v \cos \quad \sin v; \; v \sin \quad)$$

with v as parameter. It is the intersection of the cone $x^2 + y^2 = z^2 \cot^2$ and the cylinder whose cross section by the xy plane is the spiral $r = \frac{1}{2}a \sin 2 \;$:

A transformation which takes the generators and their orthogonal trajectories into parametric curves is

$$u^o = u + av\cos\ ; \quad v^o = v$$

$$) \ du = du^o \quad a\cos \ dv^o; \quad dv = dv^o$$

The metric is $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$= 1{:}du^2 + 2a\cos \ dudv + \ a^2 + u^2\sin \ \ dv^2$$

will become

$$ds^2 = du^{o\,2} + \sin^2 \ a^2 + \ u^o \quad av^o\cos \ ^2 dv^{o\,2}$$

and the new coe cients are

$$E^o = 1; \ F^o = 0; \ G^o = \sin^2 \ a^2 + \ u^o \quad av^o\cos \ ^2$$

Example 6.4. Show that the curves $du^2 \ u^2 + a^2 \ dv^2 = 0$ form an orthogonal system on the right helicoid.

Solution: Given di erential form represent a double family of curves which form an orthogonal system if $ER \quad 2PQ + GP = 0$.

We have $Pdu^2 + 2Qdudv + Rdv^2 = 0$

Comparing with $du^2 \quad u^2 + a^2 \ dv^2 = 0$ we get

$$P = 1; Q = 0; \quad R = \ u^2 + a^2$$

The equation to the right helicoid is

$$\tilde{r} = (u\cos v; u\sin v; av)$$

$$\tilde{r}_1 = (\cos v; \sin v; 0)$$

$$\tilde{r}_2 = (\ u\sin v; u\cos v; a)$$

$$) \ E = \tilde{r}_1 \ \tilde{r}_1 = 1; \quad F = \tilde{r}_1 \ \tilde{r}_2 = 0; \quad G = \tilde{r}_2 \ \tilde{r}_2 = u^2 + a^2$$

$$ER \quad 2FQ + GP = \ u^2 + a^2 + u^2 \ a^2 = 0$$

Therefore, the given curves form an orthogonal net.

Example 6.5. The metric of a surface is $v^2du^2 + u^2dv^2$. Find the equation of the family of curves orthogonal to the curves $uv =$ constant.

Solution:

$$\text{Given metric of the surface is } ds^2 = v^2du^2 + u^2dv^2 \quad (6.8)$$

$$\text{We know that } ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad (6.9)$$

$$\text{Comparing (6.8) and (6.9), we get } E = v^2; \quad F = 0; \quad G = u^2$$

$$\text{Equation of the given family of curve is } uv = \text{constant}$$

$$\begin{aligned} \text{Differentiating, we get } udv + vdu &= 0 \\ \Rightarrow vdu &= -udv \\ \Rightarrow \frac{du}{u} &= -\frac{dv}{v} \end{aligned}$$

Therefore, the direction ratios of given family is $(-u, v)$.

Let (du, dv) be the direction ratios of required family orthogonal to the given family.

Let $\lambda = -u; \quad \mu = v; \quad \lambda_1 = du; \quad \mu_1 = dv.$

By orthogonality condition, we have

$$\begin{aligned} E\lambda\lambda_1 + F(\lambda\mu_1 + \mu\lambda_1) + G\mu\mu_1 &= 0 \\ \Rightarrow v^2(-u)du + 0 + u^2 v dv &= 0 \\ \Rightarrow \frac{dv}{v} &= \frac{du}{u} \\ \text{Integrating, we get } \log v &= \log u + \log c \\ \Rightarrow \frac{v}{u} &= \text{constant} \end{aligned}$$

This gives the orthogonal trajectories.

Example 6.6. If θ is the angle at the point (u, v) between the two directions given by $Pdu^2 + 2Qdudv + Rdv^2 = 0$ then prove that $\tan\theta = \dfrac{2H\left(Q^2 - PR\right)^{1/2}}{ER - 2FQ + GP}$.

Solution: Let (λ, μ) and (λ', μ') be ratios of two directions given by

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0$$

Then $\frac{\lambda}{\mu}$ and $\frac{\lambda'}{\mu'}$ are the roots of the above equation.

$$\frac{\lambda}{\mu}+\frac{\lambda'}{\mu'} = \frac{2Q}{R}$$

$$\frac{\lambda}{\mu}\,\frac{\lambda'}{\mu'} = \frac{R}{P}$$

$$\text{Now, } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{H(\lambda\mu' - \lambda'\mu)}{E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu'}$$

$$= \frac{H\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right)}{E\frac{\lambda}{\mu}\frac{\lambda'}{\mu'} + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G}$$

$$= \frac{H\left[\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right)^2 - 4\frac{\lambda}{\mu}\frac{\lambda'}{\mu'}\right]^{1/2}}{E\frac{\lambda}{\mu}\frac{\lambda'}{\mu'} + F\left(\frac{\lambda}{\mu} + \frac{\lambda'}{\mu'}\right) + G}$$

$$\Rightarrow \tan\theta = \frac{H\left[\left(\frac{2Q}{P}\right)^2 - \frac{4R}{P}\right]^{1/2}}{E\frac{R}{P} + F\left(\frac{2Q}{P}\right) + G}$$

$$\tan\theta = \frac{\frac{H\left(4Q^2 - 4PR\right)^{1/2}}{P}}{\frac{ER - 2FQ + GP}{P}}$$

$$\text{i.e., } \tan\theta = \frac{2H\left(Q^2 - PR\right)^{1/2}}{ER - 2FQ + GP}$$

6.4. Isometric correspondence:

We shall consider examples of classes of surfaces with the property that surface in the same class are specially related to each other. The fundamental ideal behind this is that of correspondence of points between two surfaces and the two surfaces are regarded as equivalent, if this correspondence (or) mapping preserves geometrical rules on that surfaces.

An isometric correspondence between points P on a surface S and the points P' on S' such that as P traces out an arc on S then P' traces out an arc of equal length on S'.

An isometric mapping preserves both distance and angles, whereas conformal mapping preserves angles only.

We are concerned only with local properties of a surface, and in discussing correspondence between surfaces S and S'. Now, we shall consider only correspondence between parts of the surfaces. If the point (u', v') on S' corresponds to the point (u, v) on S, then u', v' are single valued functions of u and v, say

$$u' = \phi(u, v); \quad v' = \psi(u, v) \tag{6.10}$$

If surfaces S and S' are of class r and r' respectively, we may assume that ϕ and ψ are functions of class $\min(r, r')$ with non-vanishing Jacobian in the domain of u, v. Also we assume that the mapping is one to one throughout this domain.

We have restricted the maps between the part of S and part of S' to be differentiable homeomorphisms of sufficiently high class regular at each point of the domain of u, v.

Consider a curve C of class 1 passing through P and lying on S, given parametrically by equations $u = u(t)$, $v = v(t)$. If the surface S is related to surface S' by the equation (6.10), then C will map into a curve C' on S' passing through P', with parametric equations.

$$u' = \phi(u(t), v(t))$$
$$v' = \psi(u(t), v(t))$$

The direction of the tangent to the curve C at P will map into definite direction at P' namely that of the tangent to C', given by the direction ratios $(\dot{u}', \dot{v}')$, where

$$\left.\begin{aligned} \dot{u}' &= \frac{\partial \phi}{\partial u}\dot{u} + \frac{\partial \phi}{\partial v}\dot{v} \\ \dot{v}' &= \frac{\partial \psi}{\partial u}\dot{u} + \frac{\partial \psi}{\partial v}\dot{v} \end{aligned}\right\} \tag{6.11}$$

Solving the equations (6.11) for $(\dot{u}, \dot{v})$, we get

$$\dot{u} = \frac{\dot{u}'\frac{\partial \psi}{\partial v} - \dot{v}'\frac{\partial \phi}{\partial v}}{J}$$

$$\dot{v} = \frac{\dot{v}'\frac{\partial \phi}{\partial u} - \dot{u}'\frac{\partial \psi}{\partial u}}{J}$$

Since J is a non vanishing Jacobian, it follows that to a given direction at P^0 will corresponds to a definite direction at P.

Now we shall show that a proper parameter transformation in S^0 (or) S; so that corresponding point P, P^0 carry identical parameter values.Since, the functions ; of equation (6.10) satisfy the condition for a proper parameter transformation and after transforming the parameters of S^0 in this way the correspondence $S \to S^0$ gives $(u, v) \to (u, v)$ as required.

Definition 6.2 (Isometric or Applicable surfaces). Two surfaces S; S^0 are said to be isometric (or) applicable if there is a correspondence between the points of S and S^0 such that corresponding arcs of curves have the same length. The correspondence is called an *isometry*.

For example, consider a region S (not too big) of a plane and a region S^0 of a cylinder. The plane can be considered as being fitted onto the cylinder so that S coincides with S^0; and since no part of S is cut or stretched in this process the length of an arc in S remains unaltered.

Geometrically, S is continuously deformed in space until it coincides with S^0 so that continuity and arc length is preserved in S preserved. Points of S and S^0 which ultimately coincide are corresponding points of the isometry. This gives a clear idea of the relation between two isometric surfaces and explain the fact that S and S^0 need not be congruent in order to be isometric.

Locally Isometric:

The application of a plane to a circular cylinder gives the idea of local isometry. If the whole plane S is wrapped round the cylinder S^0; infinitely many points of S corresponds to the same point of S^0 so that the correspondence $S \to S^0$ is not one-one but many one. The plane and cylinder are not isometric in the large, they are however locally isometric because every point of the plane has a neighbourhood which is isometric with a region of the cylinder.

Note 6.5. For an isometry, the length of any arc in S must be equal to the length of corresponding arc in S^0: This means that $ds = ds^0$ where ds and ds^0 are corresponding linear elements of arc and this must be true for all u, v, du, dv

and the corresponding $u'; v'; du'; dv'$: The metric S therefore transforms into the metric of S' under the transformation (6.10).

If surfaces S and S' are isometric, there exists correspondence (6.10) between their parameters where and are single valued and non-vanishing Jacobians such that the metric of S transforms into the metric of S':

Example 6.7. Find a surface of revolution which is isometric with a region of right helicoid.

Solution: We know that the surface of revolution is given by

$$\tilde{r} = (g(u)\cos v; g(u)\sin v; f(u))$$

$$\tilde{r}_1 = (g_1(u)\cos v; g_1(u)\sin v; f_1(u))$$

$$\tilde{r}_2 = (\; g(u)\sin v; g(u)\cos v; 0)$$

Now $E = \tilde{r}_1 \; \tilde{r}_1 = g_1^2(u) + f_1^2(u); \quad F = \tilde{r}_1 \; \tilde{r}_2 = 0; \quad G = \tilde{r}_2 \; \tilde{r}_2 = g^2(u)$

For some functions $f(u)$ and $g(u)$ and its metric is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = g_1^2(u) + f_1^2(u) \; du^2 + 0 + g^2(u)dv^2$$

i:e:; $ds^2 = g_1^2(u) + f_1^2(u) \; du^2 + g^2(u)dv^2$

The right helicoid of pitch $2 \; a$ is given by

$$\tilde{r} = u' \cos v' + u' \sin v' ; av'$$

$$\tilde{r}_1 = \cos v' + \sin v' ; 0 \qquad (6.12)$$

$$\tilde{r}_2 = u' \sin v' \;\; u' \cos v' ; a$$

$$E' = \tilde{r}_1^{\,2} = 1; \quad F' = \tilde{r}_1 \; \tilde{r}_2 = 0; \quad G' = u'^2 + a^2$$

Therefore, its metric is given by $ds'^2 = du'^2 + u'^2 + a^2 \; dv'^2$ (6.13)

We have to find a transformation $(u; v) = (u'; v')$ so that $ds = ds'$:

Taking $v' = v; \; u' = (u);$ we have

$$dv' = dv; \quad du' = {}_1(u)du$$

$$\Rightarrow \quad ds'^2 = {}_1^2 du'^2 + {}^2 + a^2 \; dv'^2$$

So the metrices ds and ds' are identical if

$$g_1^2(u) + f_1^2(u) = {}_1^2 \qquad (6.14)$$

$$g^2(u) = {}^2 + a^2 \qquad (6.15)$$

These are two equations in three functions namely f, g and ϕ.

If we eliminate ϕ, there remains a differential equation for f as a function of g.

By putting $g(u) = a\cosh u$; $\phi(u) = a\sinh u$ to satisfy equation (6.15), we have from equation (6.14)

$$f_1^2(u) = \phi_1^2(u) - g_1^2(u)$$
$$\Rightarrow f_1^2(u) = a^2\cosh^2 u - a^2\sin^2 u = a^2$$
$$\Rightarrow f_1(u) = a$$

Integrating, we get $f(u) = au$

Hence the right helicoid is isometric with the surface obtained by revolving the curve $x = g(u)$;
$y = 0; z = f(u)$ i.e., $x = a\cosh u$; $y = 0$; $z = au$ about z-axis.

Note 6.6. The generating curve is the catenary $x = a\cosh\frac{z}{a}$ with parameter a and the directrix the z-axis and the surface of revolution is a catenoid.

The correspondence $u' = a\sinh u$; $v' = v$ shows that the generators $v' =$ constant on the helicoid correspond to the meridians $v =$ constant on the catenoid, and the helices $u' =$ constant correspond to the parallels $u =$ constant.

On the helicoid u' and v' can take all values but on the catenoid $0 \leq v < 2\pi$. The correspondence is therefore an isometry only for the region of the helicoid $0 \leq v' < 2\pi$. Hence, one period of a right helicoid of pitch $2\pi a$ corresponds isometrically to the whole catenoid of parameter a.

Example 6.8.

A surface of revolution defined by the equations $x = \cos u\cos v$; $y = \cos u\sin v$; $z = \sin u + \log\tan\left(\frac{\pi}{4} + \frac{u}{2}\right)$ where $0 < u < \frac{\pi}{2}$; $0 < v < 2\pi$. Show that the metric is $\tan^2 u\,du^2 + \cos^2 u\,dv^2$ and prove that the region $0 < u < \frac{\pi}{2}$; $0 < v < \pi$ is mapped isometrically on the region $\frac{\pi}{3} < u' < \frac{\pi}{2}$; $0 < v' < 2\pi$ by the correspondence $u' = \cos^{-1}\left(\frac{\cos u}{2}\right)$; $v' = 2v$.

Solution:

$$\text{we have } \tilde{r} = \cos u \cos v;\ \sin v \cos u;\ \ \sin u + \log\tan\frac{}{4} + \frac{u}{2}$$

$$\tilde{r}_1 = (\ \sin u \cos v;\ \ \sin u \sin v;\ \ \cos u + \sec u)$$

$$\tilde{r}_2 = (\ \cos u \sin v;\ \cos u \cos v;\ 0) \qquad (6.16)$$

$$E = \tilde{r}_1{}^2 = \tan^2 u$$

$$F = \tilde{r}_1\ \tilde{r}_2 = \cos^2 u$$

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = \tan^2 udu^2 + \cos^2 udv^2 \quad (6.17)$$

$$\text{put } v = \frac{v^o}{2};\ \ \cos u = 2\cos u^o \text{ in (6.17); we obtain}$$

$$dv = \frac{1}{2}dv^o;\ \ \ \sin u du = \ \ 2\sin u^o du^o$$

$$\Rightarrow\ ds^{o\,2} = \tan^2 u^o du^{o\,2} + \cos^2 u^{o\,2} dv^{o\,2} \qquad (6.18)$$

From (6.17) and (6.18), we find that the two metrics are identical and hence the transformed surface are given by

$$x = 2\cos u^o \cos\frac{v^o}{2};\ \ y = 2\cos u^o \sin\frac{v^o}{2}$$

$$z = \ \ \sin\ \cos^{\ 1}\ 2\cos u^o \ \ + \log\tan\ \frac{}{4} + \frac{1}{2}\cos^{\ 1}\ 2\cos u^o$$

is isometric to the given surface.

Also $(u; v) \to (u^o; v^o)$ with $v = \frac{v^o}{2};\ \ \cos u = 2\cos u^o$:

The given region is $0 < u < \frac{}{2};\ \ 0 < v <$

$$u = 0 \Rightarrow 2\cos u^o = 1 \Rightarrow \cos u^o = \frac{1}{2} \Rightarrow u^o = \frac{}{3}$$

$$u = \frac{}{2} \Rightarrow \cos u^o = 0 \Rightarrow u^o = \frac{}{2}$$

$$\Rightarrow\ 0 < u < \frac{}{2} \text{ corresponds to } \frac{}{3} < u^o < \frac{}{2}$$

Similarly for $0 < v <$ corresponds to $0 < v^o < 2$:

Definition 6.3 (Isometric lines, Isometric system). The parametric curves u = constant, v = constant on the surface S given by $\tilde{r} = \tilde{r}(u; v)$ are called isometric lines if the metric on S can be put in the form $ds^2 = (Udu^2 + Vdv^2)$; where is a function of u and v; U is a function of u alone and V is a function of v alone. The parameters u and v are called isometric parameters.

Example 6.9. Show that the meridians and parallels on a sphere form an isometric system and also determine the isometric parameters.

Solution: The position vector of any point on a sphere is $\tilde{r} = a(\sin u \cos v, \sin u \sin v, \cos u)$.

Here the parametric curves

$$\begin{aligned} v &= \text{constant are the meridian and} \\ u &= \text{constant are the parallels} \\ \text{Now, } \tilde{r}_1 &= a(\cos u \cos v, \cos u \sin v, -\sin u) \\ \tilde{r}_2 &= a(-\sin u \sin v, \sin u \cos v, 0) \\ E &= \tilde{r}_1^{\,2} = a^2; \quad F = \tilde{r}_1 \cdot \tilde{r}_2 = 0; \quad G = \tilde{r}_2^{\,2} = a^2 \sin^2 u \\ ds^2 &= Edu^2 + 2Fdudv + Gdv^2 = a^2du^2 + 0 + a^2 \sin^2 udv^2 \\ \Rightarrow ds^2 &= a^2 \sin^2 u \left[\operatorname{cosec}^2 udu^2 + dv^2\right] \end{aligned}$$

This is of the form $ds^2 = \lambda\left(Udu^2 + Vdv^2\right)$, where $\lambda = a^2 \sin^2 u$; $U = \operatorname{cosec}^2 u$; $V = 1$.

Thus, the system is an isometric system.

To find the parametric curves, we use the transformation $(u, v) \to (u', v')$ given by $du' = \sqrt{U}du$ and $dv' = \sqrt{V}dv$.

$$\begin{aligned} \Rightarrow du' &= \sqrt{\operatorname{cosec}^2 u}\,du \\ \Rightarrow u' &= \int \operatorname{cosec} u\,du = \log \tan \frac{u}{2} \\ \text{and } dv' &= dv \Rightarrow v' = v. \end{aligned}$$

Therefore, the parametric curves are

$u' = \text{constant} \Rightarrow \log \tan \frac{u}{2} = \text{constant}$ and $v' = \text{constant} \Rightarrow v = \text{constant}$.

6.5. Intrinsic properties:

Let E, F, G be any real single valued continuous functions of u and v satisfying $E > 0$ and $EG - F^2 > 0$ in some domain D of u, v. Then it will be seen that every point of D has a neighbourhood D' (in D) in which $Edu^2 + 2Fdudv + Gdv^2$ is the metric of the surface referred to u and v as parameters. This is the first fundamental existence theorem and shows that there is no hidden identity relating E, F and G. It asserts to existence of a vector function $\tilde{r}(u, v)$ satisfying the partial differential equations $\tilde{r}_1^{\,2} = E$; $\tilde{r}_1 \cdot \tilde{r}_2 = F$; $G = \tilde{r}_2 \cdot \tilde{r}_2$ in some domain D'.

The surface having a given metric is certainly not unique, however, even apart from rigid displacements in any space. Any two isometric surfaces, for example, have the same metric when the corresponding points are assigned the same parameters, although they are not congruent. The class of surfaces having a given metric is the class of isometric with any one member.

It follows that any formula (or) property of a surface which is deducible from the metric alone, without recourse to the vector function $\tilde{r}(u; v)$; automatically applies to the whole class of isometric surface. Properties of this kind will be described as intrinsic other wise is non-intrinsic.

If a formula equation (or) theorem is intrinsic, it should be possible to derive it by an intrinsic arguments without introducing normal properties. It paves the way for Riemannian geometry which is mainly intrinsic. The quadratic differential form of metric is itself deducted from $\tilde{r}(u; v)$: The square root of a quadratic differential form (or) any other homogeneous form of degree 2.

A vector in the tangent plane may be defined by its components (;) and is intrinsic, all such vectors at a point form a vector space with a norm (magnitude) defined so that norm of $(du; dv)$ is the linear elements ds given by the metric. The vector (;) = (;) where is very small can be regarded as the small displacement from the point $(u; v)$ to the point $(u + ; v +)$:

The angle between two vectors (;) and ($'$; $'$) at a point $(u; v)$ can be defined by the Euclidean cosine formula applied to the small triangle with vertices $(u; v)$; $(u + ; v +)$ and $(u + ' '; v + ' ')$; where and $'$ are small. It can be verified that this definition of angle is consistent.

Now we can study the intrinsic property of a surface at any point namely linear and area elements, vector components, vector magnitudes, direction coefficients and angle formulas.

Let Us Sum Up:

In this unit, the students acquired knowledge

- to know the relation between direction coefficients and direction ratios.
- to know the concept of orthogonal trajectories.

to know the concept of isometric correspondence.

Check Your Progress:

1. Show that parametric curves are orthogonal on the surface $x = u\cos v;\ \ y = u\sin v;\ z = a\log\left(u + \left(u^2 - a^2\right)^{1/2}\right):$
2. Show that the parametric curves on the sphere $\tilde{r} = a(\sin u\cos v; \sin u \sin v; \cos u)\ \ 0 < u < \frac{\pi}{2};\ \ 0 < v < 2\pi$ form an orthogonal set.

Choose the correct or more suitable answer:

1. The direction coefficients satisfy the identity
   - (a) $El^2 + 2Flm + Gm^2 = 1$
   - (b) $El^2 - 2Flm + Gm^2 = 1$
   - (c) $El^2 + 2Flm - Gm^2 = 1$
   - (d) $El^2 - 2Flm - Gm^2 = 1$
2. An isometric mapping preserves
   - (a) distance only.
   - (b) angles only.
   - (c) both distance and angles only.
   - (d) neither distance nor angles.

Answer:

(1) a (2) c

Glossaries:

Orthogonal Trajectory: The locus of a point whose path cuts each curve of a family of curves at right angles.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-III

Unit-7: Geodesics-I.

Unit-8: Geodesics-II.

Unit-9: Geodesics-III.

Block-III

UNIT-7

GEODESICS-I

Structure

Objective

Overview

7. 1 Geodesics

7. 2 Canonical geodesics equations

7. 3 Normal property of Geodesics

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the concept of Geodesics.
- derive the equations of the Geodesics.
- normal properties of Geodesics.

Overview

In this unit, we illustrated the properties of special intrinsic curves, called geodesics which is related to straight lines in Euclidean space because they are curves of shortest distance.

7.1. Geodesics:

The problem is given any two points A and B on the surface, we can find the least arc length by joining all the possible arcs between A and B. As we already familiar that the equation of the curve is given by $u = u(t); \quad v = v(t)$. Every curve given by these equations is called geodesic, whether the curve is of shortest distance (or) not, and geodesic may be regarded as curves of stationary, rather than strictly shortest distance on the surface.

Definition 7.1 (Geodesics). If two points A and B on a surface S be joined by curves lying on S; then the curve which possesses a stationary length for small variations is called geodesics.

Bookwork 7.1. Derive the differential equation of geodesics.

Let A and B be two points on the surface $\tilde{r} = \tilde{r}(u; v)$.

Consider all the possible arcs which join A and B are given by the equations $u = u(t); \quad v = v(t)$ where $u(t)$ and $v(t)$ are functions of class 2. Without loss of generality it can be assumed that every arc γ; $t = 0$ at $A = 0$ (A is called the initial point) and $t = 1$ at B (B is called the end point). we assume that for every arc γ is given by $0 \leq t \leq 1$.

Let be one such arc and let $s(\ )$ be the arc joining A and B measured along :

$$\text{We have } ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

$$\Rightarrow s^2 = Eu^2 + 2Fuv + Gv^2$$

$$\text{Now arc length } s(\ ) = \int_0^1 s\,dt$$

$$\Rightarrow s(\ ) = \int_0^1 \sqrt{Eu^2 + 2Fuv + Gv^2}\,dt \qquad (7.1)$$

Let be slightly deformed to obtain $'$ keeping the end points A and B xed.

Then $'$ has the equations

$$u_1(t) = u(t) + g(t)$$

$$v_1(t) = v(t) + h(t)$$

where is small and g, h are arbitrary functions such that $g(0) = h(0) = 0$ and $g(1) = h(1) = 0$.

$$\Rightarrow \text{Arc length of } ' = s(') = \int_0^1 \sqrt{Eu_1^2 + 2Fu_1v_1 + Gv_1^2}\,dt$$

(replace u, v by u', v' in (7.1))

The variation in $s(\ )$ is in $s(\ )\ \ s(')$ and in general it is of order :

If is such that this variation is atmost of order ^{2}; for all variations in ; then $s(\ )$ is said to be stationary and the curve is geodesic.

$$\text{Let } T(u, v, u, v) = \frac{Eu^2 + 2Fuv + Gv^2}{2}$$

$$\text{then } T = \frac{1}{2}s^2$$

$$\Rightarrow s^2 = 2T \Rightarrow s = \sqrt{2T} = f \text{ (say)}$$

$$\Rightarrow s(\ ) = \int_0^1 s\,dt = \int_0^1 f\,dt; \quad \text{where } f = f(u, v, u, v)$$

Now,
$$s(\gamma') - s(\gamma) = \int_0^1 f(u_1, v_1, \dot u_1, \dot v_1)\,dt - \int_0^1 f(u, v, \dot u, \dot v)\,dt$$
$$= \int_0^1 \left[f(u_1, v_1, \dot u_1, \dot v_1) - f(u, v, \dot u, \dot v) \right] dt$$
$$s(\gamma') - s(\gamma) = \int_0^1 \left[f\left(u + \epsilon g, v + \epsilon h, \dot u + \epsilon \dot g, \dot v + \epsilon \dot h\right) - f(u, v, \dot u, \dot v) \right] dt$$
$$= \int_0^1 \left\{ f(u, v, \dot u, \dot v) + \epsilon\left(g\frac{\partial f}{\partial u} + h\frac{\partial f}{\partial v} + \dot g\frac{\partial f}{\partial \dot u} + \dot h\frac{\partial f}{\partial \dot v}\right) + O\left(\epsilon^2\right) - f(u, v, \dot u, \dot v) \right\} dt$$

(expanding by Taylor's theorem for several variables)

$$\Rightarrow s(\gamma) - s(\gamma') = \epsilon\int_0^1 \left(g\frac{\partial f}{\partial u} + h\frac{\partial f}{\partial v} + \dot g\frac{\partial f}{\partial \dot u} + \dot h\frac{\partial f}{\partial \dot v}\right) dt + O(\epsilon^2) \quad (7.2)$$

Consider
$$\int_0^1 \dot g \frac{\partial f}{\partial \dot u}\,dt = \int_0^1 \frac{\partial f}{\partial \dot u}\,\dot g\,dt = \int_0^1 U\,dV$$
$$\text{where } U = \frac{\partial f}{\partial \dot u}; \quad dV = \dot g\,dt$$
$$\Rightarrow dU = d\left(\frac{\partial f}{\partial \dot u}\right); \quad V = g$$
$$= [UV]_0^1 - \int_0^1 V\,dU$$
$$= \left[\frac{\partial f}{\partial \dot u}\, g\right]_0^1 - \int_0^1 g\, d\left(\frac{\partial f}{\partial \dot u}\right)$$
$$= 0 - 0 - \int_0^1 g \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) dt$$
$$\because g(1) = 0; \quad g(0) = 0$$
$$\Rightarrow \int_0^1 \dot g \frac{\partial f}{\partial \dot u}\,dt = -\int_0^1 g \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) dt$$

In a similarly way, we can get
$$\int_0^1 \dot h \frac{\partial f}{\partial \dot v}\,dt = -\int_0^1 h \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right) dt$$

Thus, equation (7.2), becomes
$$s(\gamma) - s(\gamma') = \epsilon\int_0^1 \left[g\frac{\partial f}{\partial u} + h\frac{\partial f}{\partial v} - g\frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) - h\frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right) \right] dt$$
$$= \epsilon\int_0^1 \left\{ g\left(\frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial \dot u}\right) + h\left(\frac{\partial f}{\partial v} - \frac{d}{dt}\frac{\partial f}{\partial \dot v}\right) \right\} dt + O(\epsilon^2)$$
$$= \epsilon\int_0^1 \left(gL + hM \right) dt + O(\epsilon^2)$$

where $L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right)$; $M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right)$

By definition $s(\ )$ is stationary, $s(\ ) - s(\ ^{o})$ is almost of order ^{2}.

Therefore is a geodesic if $\int_0^1 (gL + hM)\, dt = 0$

$$\Rightarrow L = 0 \text{ and } M = 0 \quad (\because g, h \text{ are arbitrary functions})$$

$$\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0 \tag{7.3}$$

$$\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) = 0 \tag{7.4}$$

These are the differential equations for geodesic. But, $f = \sqrt{2T}$; so we write the differential equations involving T rather than f. Thus, equation (7.3) becomes

$$\frac{\partial}{\partial u}(\sqrt{2T}) - \frac{d}{dt}\left(\frac{\partial \sqrt{2T}}{\partial u}\right) = 0$$

$$\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{T}}\frac{\partial T}{\partial u} - \frac{1}{\sqrt{T}}\frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) + \frac{\partial T}{\partial u}\left(2T^{-3/2}\frac{dT}{dt}\right) = 0$$

Multiplying both sides by $\sqrt{T}$, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{\partial T}{\partial u}\frac{dT}{dt} \tag{7.5}$$

Similarly from equation (7.4) we get (7.6)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{\partial T}{\partial v}\frac{dT}{dt} \tag{7.7}$$

For convenience, we denote left hand side members of equations (7.5) and (7.7) by U and V:

$$\Rightarrow U = \frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{\partial T}{\partial u}\frac{dT}{dt} \tag{7.8}$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{\partial T}{\partial v}\frac{dT}{dt} \tag{7.9}$$

$$\text{Equation (7.8)} \Rightarrow U = \frac{1}{2T}\frac{\partial T}{\partial u}\frac{dT}{dt} \tag{7.10}$$

$$\text{Equation (7.9)} \Rightarrow V = \frac{1}{2T}\frac{\partial T}{\partial v}\frac{dT}{dt} \tag{7.11}$$

Eliminate $\frac{dT}{dt}$ from equations (7.10) and (7.11), we get

$$U\frac{\partial T}{\partial v} \quad V\frac{\partial T}{\partial u} = 0 \tag{7.12}$$

This is the necessary for a curve on a surface to be geodesic.

Note 7.1. The expressions U and V so defined are important in relation to any curve, whether it is geodesics or not. They satisfy the identity

$$uU + vV = \frac{dT}{dt} \tag{7.13}$$

Example 7.1. Prove that the curves of the family $\frac{v^3}{u^2}$ = constant are geodesics on the surface with metric $v^2du^2 \quad 2uvdudv + 2u^2dv^2$ $(u > 0; v > 0)$:

Solution: Given curve $\frac{v^3}{u^2} = c \ (c > 0)$; the parametric equation of the given curve can be written as

$$\left.\begin{aligned} u &= ct^3; \quad v = ct^2 \\ u &= 3ct^2; \quad v = 2ct \end{aligned}\right\} \tag{7.14}$$

$$\begin{aligned}
ds^2 &= v^2du^2 \quad 2uvdudv + 2u^2dv^2 \\
\Rightarrow s^2 &= v^2u^2 \quad 2uvuv + 2u^2v^2 \\
\text{Let } T &= \frac{1}{2}s^2 \\
\Rightarrow T &= \frac{1}{2}\ v^2u^2 \quad 2uvuv + 2u^2v^2 \\
\frac{\partial T}{\partial u} &= \frac{1}{2}\ 0 \quad 2vuv + 4uv^2 = \ vuv + 2uv^2 \\
\frac{\partial T}{\partial u} &= \ ct^2 \quad 3ct^2 \ (2ct) + 2 \ ct^3 \ 4 \ c^2t^2 = 2c^3t^5 \\
\text{Similarly, } \frac{\partial T}{\partial v} &= 3c^3t^6 \\
\frac{\partial T}{\partial u} &= c^3t^6 \\
\frac{\partial T}{\partial v} &= c^3t^7 \\
\text{we have } U &= \frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) \quad \frac{\partial T}{\partial u} = \frac{d}{dt}\ c^3t^6 \quad 2c^3t^5 = 4c^3t^5 \\
\text{and } V &= \frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) \quad \frac{\partial T}{\partial v} = \frac{d}{dt}\ c^3t^7 \quad 3c^3t^6 = 4c^3t^6 \\
U\frac{\partial T}{\partial v} \quad V\frac{\partial T}{\partial u} &= \ 4c^3t^5 \quad c^3t^7 \quad 4c^3t^6 \quad c^3t^6 = 0
\end{aligned}$$

Hence the curve is a geodesic for all values of c:

Example 7.2. Prove that the curves of the family $u+v=\text{constant}$ are geodesics on the surface with metric $\left(1+u^2\right)du^2-2uv\,du\,dv+\left(1+v^2\right)dv^2.$

Solution: Given curve $u+v=c$; the parametric equations of the given curve can be written as

$$\left.\begin{aligned} u &= t; \quad v = c-t \\ \dot{u} &= 1; \quad \dot{v} = -1 \end{aligned}\right\} \tag{7.15}$$

$$\begin{aligned}
ds^2 &= \left(1+u^2\right)du^2-2uv\,du\,dv+\left(1+v^2\right)dv^2\\
\Rightarrow \dot{s}^2 &= \left(1+u^2\right)\dot{u}^2-2uv\dot{u}\dot{v}+\left(1+v^2\right)\dot{v}^2\\
\text{Let } T &= \frac{1}{2}\dot{s}^2\\
\Rightarrow T &= \frac{1}{2}\left[\left(1+u^2\right)\dot{u}^2-2uv\dot{u}\dot{v}+\left(1+v^2\right)\dot{v}^2\right]\\
\frac{\partial T}{\partial u} &= \frac{1}{2}\left(2u\dot{u}^2-2v\dot{u}\dot{v}+0\right)=u\dot{u}^2-v\dot{u}\dot{v}\\
\frac{\partial T}{\partial u} &= t(1)-(c-t)(1)(-1)=t+c-t=c\\
\text{Similarly, } \frac{\partial T}{\partial v} &= c\\
\frac{\partial T}{\partial \dot{u}} &= 1+ct\\
\frac{\partial T}{\partial \dot{v}} &= ct-1-c^2\\
\text{we have } U &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)-\frac{\partial T}{\partial u}=\frac{d}{dt}(1+ct)-c=0\\
\text{and } V &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)-\frac{\partial T}{\partial v}=\frac{d}{dt}\left(ct-1-c^2\right)-c=0\\
U\frac{\partial T}{\partial \dot{v}}-V\frac{\partial T}{\partial \dot{u}} &= 0-0=0
\end{aligned}$$

Hence the curve is a geodesic for all values of $c.$

Example 7.3. Prove that on a general surface, a necessary and sufficient condition for the parametric curve $v=\text{constant}$ to the geodesic is $EE_2+FE_1-2EF_1=0.$

Solution: On the curve $v=c$; we may take u as parameter. Therefore $u=t.$

$$\begin{aligned}
u &= t; \quad v=c\\
\Rightarrow \dot{u} &= 1; \quad \dot{v}=0\\
\text{we have } T &= \frac{1}{2}\left(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2\right)\\
\frac{\partial T}{\partial u} &= \frac{1}{2}\left(E_1\dot{u}^2+2F_1\dot{u}\dot{v}+G_1\dot{v}^2\right)
\end{aligned}$$

$$\text{where } E_1=\frac{\partial E}{\partial u}; \quad F_1=\frac{\partial F}{\partial u}; \quad G=\frac{\partial G}{\partial u}$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}E_1$$

$$\text{Similarly,}\quad \frac{\partial T}{\partial v} = \frac{1}{2}E_2$$

$$\frac{\partial T}{\partial \dot{u}} = E$$

$$\Rightarrow \frac{\partial T}{\partial \dot{v}} = F$$

$$\text{Now,}\quad U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = E_1 - \frac{1}{2}E_1 = \frac{1}{2}E_1$$

$$\text{and}\quad V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = F_1 - \frac{1}{2}E_2$$

We know that the necessary and sufficient condition for a curve to be a geodesic is

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$$

$$\Rightarrow \frac{1}{2}E_1 F - \left(F_1 - \frac{1}{2}E_2\right)E = 0$$

$$\Rightarrow EE_2 + FE_1 - 2EF_1 = 0$$

This is the required condition.

7.2. Canonical geodesic equations:

The geodesic equations are given by

$$\left.\begin{aligned} U &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}} \\ V &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}} \\ &\text{where } T(u, v, \dot{u}, \dot{v}) = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \end{aligned}\right\} \quad (7.16)$$

Here t is a parameter without loss of generality we can take s as parameter, so $\dot{u}, \dot{v}$ are replaced by u', v' and

$$T\left(u, v, u', v'\right) = Eu'^2 + 2Fu'v' + Gv'^2 \quad (7.17)$$

Along the curve u' and v' satisfy the identity of direction coefficients. Hence $T = \frac{1}{2}$; $\frac{dT}{ds} = 0$ and equations (7.16) becomes the canonical equations for geodesics

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \qquad V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \qquad (7.18)$$

In these equations, the partial derivatives of T are calculated from equation (7.16) before values of u' and v' are substituted. T is not equal to $\frac{1}{2}$ identically for all values of u, v, u', v' but only along the curve. We get the identity namely $u'U + v'V = 0.$

The equation (7.18) are not independent. For a curve other than a parametric curve $u' \neq 0, v' \neq 0$ and the conditions $U = 0, V = 0$ are equivalent other being sufficient for a geodesic. For a parametric curve $u =$ constant, $u' = 0, v' \neq 0$ and $V = 0$ for all $s.$ The condition for a geodesic is $U = 0.$ Similarly, $V = 0$ is the sufficient condition for the curve $v =$ constant to be geodesic.

Example 7.4. Find the geodesics on a surface of revolution.

Solution: Let the surface be given by

$$\begin{aligned}
\tilde{r} &= (g(u)\cos v, g(u)\sin v, f(u)) \\
\tilde{r}_1 &= (g'(u)\cos v, g'(u)\sin v, f'(u)) \\
\tilde{r}_2 &= (-g(u)\sin v, g(u)\cos v, 0) \\
E &= \tilde{r}_1^{\,2} = g'^2(u) + f'^2(u); \\
G &= \tilde{r}_2 \cdot \tilde{r}_2 = g^2(u); \\
F &= \tilde{r}_1 \cdot \tilde{r}_2 = 0 \\
\text{Hence} \quad ds^2 &= Edu^2 + 2Fdudv + Gdv^2 \\
&= \left[g'^2(u) + f'^2(u)\right] du^2 + g^2(u)dv^2 \\
T &= \left(f_1^2 + g_1^2\right) u'^2 + g^2 V'^2 \qquad \text{where} \quad f_1 = f' = \frac{df}{du}
\end{aligned}$$

From above, we see that $\frac{\partial T}{\partial v} = 0$ then the canonical equation $V = 0$ reduces to

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) = 0$$

Upon integrating, we get $g^2 v' = \; > 0,$ where is an arbitrary constant. If $= 0,$ then V is constant and every meridian is a geodesic. Now we assume that is positive. Then the first order differential equation can be written as

$$g^4 dv^2 = {}^2 ds^2 = {}^2 \left[(f_1^2 + g_1^2) du^2 + g^2 dv^2 \right]$$

$$\Rightarrow \sqrt{f_1^2 + g_1^2}\, du \quad g \sqrt{g^2 \quad {}^2}\, dv = 0 \qquad (7.19)$$

Even though is an arbitrary constant, being included, because $\frac{dv}{du}$ may change sign along the same geodesic. If $g^2 \neq {}^2;$ then equation (7.19) becomes

$$\sqrt{f_1^2 + g_1^2}\, du = g \sqrt{g^2 \quad {}^2}\, dv = 0$$

$$\Rightarrow \frac{dv}{du} = \frac{\sqrt{f_1^2 + g_1^2}}{g \sqrt{g^2 \quad {}^2}}$$

Upon integration we get $v = \quad + \int \frac{\sqrt{f_1^2 + g_1^2}}{g \sqrt{g^2 \quad {}^2}} du = \quad + \quad (u; \quad)$ (say)

where and are arbitrary constants.

If $g^2 = {}^2;$ then from equation (7.19), we get $u =$ constant. For curves $u =$ constant, the equation $V = 0$ is satis ed. To check whether the curve $u = c$ is geodesic , it is necessary to apply the condition that $U = 0$: Since $u' = 0$ and $v' = g^{\ 1}$ from the identity for direction coe cients.

$$\frac{\partial T}{\partial u'} = 0; \quad \frac{\partial T}{\partial u} = \frac{g_1}{g}; \quad U = \frac{g_1}{g}$$

The curve $u = c$ is therefore a geodesic if and only if $g_1(c) = 0$: Since g is the radius of the parallel $u = c$ on the surface of revolution, a parallel is a geodesic if its radius is stationary.

Example 7.5. Discuss the nature of geodesics on the right helicoid $x = u\cos v;\ y = u\sin v;\ z = av$:

Solution:

$$\tilde{r}(u) = (u\cos v; u\sin v; av)$$

$$\tilde{r}_1(u) = (\cos v; \sin v; 0)$$

$$\tilde{r}_2(u) = (\ \ u\sin v; u\cos v; a)$$

$$E = \tilde{r}_1^{\ 2} = 1; \quad F = 0; \quad G = u^2 + a^2$$

Now the canonical equation $V = 0$ except these for which $u =$ constant, are the geodesics on the surface. Also $u = c$ is a geodesic if and only if $V = 0$.

The metric is

$$ds^2 = du^2 + \left(u^2 + a^2\right) dv^2$$

$$\text{and } T = \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2\right)$$

$$= \frac{1}{2}\left(u'^2 + \left(u^2 + a^2\right) v'^2\right)$$

$$\frac{\partial T}{\partial v} = 0; \quad \frac{\partial T}{\partial v'} = \left(u^2 + a^2\right) v'$$

$$V = \frac{\partial}{\partial s}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{\partial}{\partial s}\left(\left(u^2 + a^2\right) v'\right) - 0$$

$$= \frac{\partial}{\partial s}\left(\left(u^2 + a^2\right) v'\right)$$

$$V = 0 \Rightarrow \frac{\partial}{\partial s}\left(\left(u^2 + a^2\right) v'\right) = 0$$

Integrating, we get $\left(u^2 + a^2\right) v' = k$ where k is an arbitrary constant

If $k = 0$, then we get $v' = 0$ (or) $v =$ constant. Thus every meridian $v = c$ is a geodesic on the right helicoid.

Squaring, we get

$$\left(u^2 + a^2\right)^2 \left(\frac{dv}{ds}\right)^2 = k^2$$

$$\left(u^2 + a^2\right)^2 dv^2 = k^2 ds^2$$

$$\left(u^2 + a^2\right)^2 dv^2 = k^2\left(Edu^2 + 2Fdudv + Gdv^2\right)$$

$$\left(u^2 + a^2\right)\left(u^2 + a^2 - k^2\right) dv^2 = k^2 du^2$$

$$dv = \frac{k}{\sqrt{\left(u^2 + a^2\right)\left(u^2 + a^2 - k^2\right)}}\, du \quad (7.20)$$

Case 1: Let $u^2 + a^2 - k^2 \neq 0$. Integrating (7.20), we get the equation of geodesic.

$$v = k_1 - k \int \frac{k}{\sqrt{\left(u^2 + a^2\right)\left(u^2 + a^2 - k^2\right)}}\, du$$

where k_1 is an arbitrary constant

Case 2: Let $u^2 + a^2 - k^2 = 0$. Then from equation (7.20), we see that $du = 0$ (or) $u =$ constant, the equation $V = 0$ is automatically satisfied. Further, the necessary and sufficient condition for the curve $u = c$ to be geodesic is that $U = 0$: Since $F = 0$ for this surface, the curve $u = c$ will be a geodesic if and only if $G_1 = 0$ then $u =$ constant for all values of v:

$$\begin{aligned} G &= u^2 + a^2 \\ G_1 &= 2u + a^2; \quad u = c \end{aligned}$$

Thus, $G_1 = 0$ implies that $u =$ constant will be a geodesic if and only if $2c + a^2 = 0 \Rightarrow c = \frac{-a^2}{2}$:

The parametric curve is $u = \frac{-a^2}{2}$ is also a geodesic.

7.3. Normal property of Geodesics:

In this section, we are going to study the properties of Geodesics and the application of Tensors in the study of Geodesics.

Bookwork 7.2. A characteristic property of a geodesic is that at every point its principal normal is normal to the surface.

Proof. The geodesic equations can be expressed in terms of $\tilde{r}(u; v)$ in terms of the following identities which hold for any functions $u(t); v(t)$ of a general parameter t:

$$\left.\begin{aligned} \frac{\partial T}{\partial u} &= \tilde{r} \cdot \tilde{r}_1; & \frac{\partial T}{\partial v} &= \tilde{r} \cdot \tilde{r}_2 \\ U(t) &= \tilde{r} \cdot \tilde{r}_1; & V(t) &= \tilde{r} \cdot \tilde{r}_2 \end{aligned}\right\} \tag{7.21}$$

where $T = \frac{1}{2}\left(Eu^2 + 2Fuv + Gv^2\right)$:

To prove these, consider the relations

$$\begin{aligned} T &= \frac{1}{2}\left(\tilde{r}_1^2 u^2 + 2\tilde{r}_1 \cdot \tilde{r}_2 uv + \tilde{r}_2^2 v^2\right) = \frac{1}{2}\left(\tilde{r}_1 u + \tilde{r}_2 v\right)^2 \\ T &= \frac{1}{2}\tilde{r}^2 \qquad (\because \tilde{r} = \tilde{r}_1 u + \tilde{r}_2 v) \\ \frac{\partial T}{\partial u} &= \tilde{r} \cdot \frac{\partial \tilde{r}}{\partial u} = \tilde{r} \cdot \tilde{r}_1 \\ \frac{\partial T}{\partial u} &= \tilde{r} \cdot \frac{\partial \tilde{r}}{\partial u} = \tilde{r} \cdot \frac{\partial}{\partial u}\left(\tilde{r}_1 u + \tilde{r}_2 v\right) = \tilde{r} \cdot \frac{d}{dt}\tilde{r}_1 \\ \Rightarrow U(t) &= \frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) - \frac{\partial T}{\partial u} \\ &= \frac{d}{dt}\left(\tilde{r} \cdot \tilde{r}_1\right) - \tilde{r} \cdot \frac{d}{dt}\tilde{r}_1 = \tilde{r} \cdot \frac{d}{dt}\tilde{r}_1 \end{aligned}$$

Similarly, we can show that $V(t) = \tilde{r} \frac{d}{dt} \tilde{r}_2$:

If s as parameter, then the geodesic equations are $U(s) = 0$; $V(s) = 0$: These can be written as

$$\tilde{r}'' \tilde{r}_1 = 0; \quad \tilde{r}'' \tilde{r}_2 = 0$$

This shows that $\tilde{r}''$ is perpendicular to both $\tilde{r}_1$ and $\tilde{r}_2$ and therefore along the normal to the surface. Since $\tilde{r}_1$ and $\tilde{r}_2$ lie in the tangent plane to the surface. But $\tilde{r}''$ is along the principal normal to the curve. Hence we see that at every point P of a geodesic, the principal normal is normal to the surface.

Note 7.2. Every great circle of a sphere have the normal property of geodesics, therefore every great circle on a sphere is a geodesic.

Example 7.6. A particle is constrained to move on a smooth surface under no force except the normal reaction. Prove that its path is a geodesic.

Solution: Let $\tilde{r}$ be the position vector of a moving point and the parameter t is the time.

i:e:; $\tilde{r} = \tilde{r}(t)$:

$$\text{Then the velocity vector} = \frac{d\tilde{r}}{dt} = \tilde{r}$$

$$\text{and acceleration vector} = \frac{d\tilde{r}}{dt} = \tilde{r}$$

Given that the only force acting on the particle is the normal reaction.

We know that $F = m\hat{r}$ * Force = mass acceleration

Given that the force is along the normal to the surface, so $\tilde{r}$ must be along normal to the surface.

Since $\tilde{r}$ is tangential to the path of the particle, it must be along tangential to the surface.

$$\tilde{r} \; ? \; \tilde{r}$$

$$\tilde{r} \; \tilde{r} = 0$$

$$) \; 2\tilde{r} \; \tilde{r} = 0 \;) \; \frac{d}{dt} \tilde{r}^2 = 0$$

$$) \; \tilde{r}^2 = 0 \;) \; \tilde{r}^2 = \text{constant} \;) \; \tilde{r} = c$$

$$) \; \text{speed } s = c$$

$$\text{Now, } \tilde{r} = \frac{d\tilde{r}}{dt} = \frac{d\tilde{r}}{ds}\frac{ds}{dt} = \tilde{t}s \quad \text{where } \tilde{t} = \frac{d\tilde{r}}{ds}$$

is the unit tangent to the path of the particle and $\tilde{r} = c\tilde{t}$:

$$\tilde{r} = c\frac{d\tilde{t}}{dt}$$
$$= c\frac{d\tilde{t}}{ds}\frac{ds}{dt} = c\tilde{t}'s' = c\tilde{t}'c = c^2\tilde{t}'$$
$$\Rightarrow \tilde{r} = c^2 \tilde{n}$$

where $\tilde{n}$ is the unit principal normal to the path of the particle $\Rightarrow \tilde{r} \parallel \tilde{n}$.

i.e., Surface normal is parallel to unit principal normal.

Therefore, by the normal property path of the particle is geodesic.

Example 7.7. Show that every helix on a cylinder is a geodesic.

Solution: Let C be a helix on a cylinder whose generators are parallel to a constant vector $\tilde{a}$.

Let P be any point on C. Let $\tilde{t}$ and $\tilde{n}$ be the unit tangent and unit principal normal to C at P. Let $\tilde{N}$ be the unit normal surface at P (to the cylinder).

Since C is an helix, we have $\tilde{t} \cdot \tilde{a} =$ constant (by definition of helix).

$$\text{Differentiate with respect to } s,\ \tilde{t}(0) + \frac{d\tilde{t}}{ds} \cdot \tilde{a} = 0$$
$$\Rightarrow \tilde{t}' \cdot \tilde{a} = 0$$
$$\Rightarrow \kappa\tilde{n} \cdot \tilde{a} = 0$$
$$\Rightarrow \tilde{n} \cdot \tilde{a} = 0 \Rightarrow \tilde{n} \perp \tilde{a}$$
$$\text{Also, } \tilde{n} \perp \tilde{t}$$

Thus $\tilde{n}$ is perpendicular to both $\tilde{a}$ and $\tilde{t}$.

$\Rightarrow \tilde{n}$ is parallel to $\tilde{a} \times \tilde{t}$.

Since $\tilde{a}$ and $\tilde{t}$ are tangential to the surface of the cylinder at P, $\tilde{a} \times \tilde{t}$ is along the surface normal $\tilde{N}$ at P.

Thus $\tilde{n}$ and $\tilde{N}$ are parallel.

Hence by the normal property, it follows that C is geodesic on the cylinder.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- derive the canonical equations for the Geodesics.
- understand the normal properties of Geodesics.

Check Your Progress:

1. Define Geodesic.
2. Derive the canonical equation for the geodesic.
3. Prove that every helix on a cylinder is a geodesic.
4. Derive the normal property of a geodesic.

Choose the correct or more suitable answer:

1. The curve $u =$ constant is a geodesic if and only if

   (a) $GG_1 + FG_2 - 2GF_2 = 0$

   (b) $GG_1 - FG_2 - 2GF_2 = 0$

   (c) $GG_1 - FG_2 + 2GF_2 = 0$

   (d) $GG_1 + FG_2 + 2GF_2 = 0$.

2. A characteristic property of a geodesic is that at every point its principal normal is : : : : : : to the surface

   (a) tangent (b) binormal

   (c) normal (d) none of these.

3. Every helix on a : : : : : : is geodesic.

   (a) $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) + \frac{\partial T}{\partial u} = 0; \quad V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0$

   (b) $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0; \quad V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) + \frac{\partial T}{\partial v} = 0$

   (c) $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) + \frac{\partial T}{\partial u} = 0; \quad V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) + \frac{\partial T}{\partial v} = 0$

   (d) $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0; \quad V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0$

Answer:

(1) a (2) c (3) d

Glossaries:

Geodesics: The shortest path between two points on the surface.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-III

UNIT-8

GEODESICS-II

Structure

Objective

Overview

8. 1 Existing Theorems

8. 2 Geodesic Parallels

8. 3 Geodesic Curvature

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the concept of Geodesic parallels, Geodesic coordinates and Geodesic polars.
- derive the expression for Geodesic curvature.
- derive Liouville's formula for $_g$:

Overview

In this unit, we will illustrate the basic concepts of Geodesic parallels and Geodesic curvature.

8.1. Existing Theorems:

With s as parameter the geodesic equations can be written in the form

$$\overset{oo}{u} = f\left(u, v, \overset{o}{u}, \overset{o}{v}\right); \quad \overset{oo}{v} = g\left(u, v, \overset{o}{u}, \overset{o}{v}\right)$$

where f and g are quadratic forms in u^o, v^o with single-valued continuous functions of u and v as coefficients. These are simultaneous second order differential equations for u and v as function of $s;$ and from the theory of such equations if f and g are of class 1 a solution exists and is determined uniquely by arbitrary initial values of u^o and $v^o.$ Hence

A geodesic can be found to pass through any given point and have any given direction at that point. The geodesic is determined uniquely by these initial conditions.

From the above existence theorem, it is to be expected that if a point Q is sufficiently close to any point P then it is possible to find the direction at P such that the geodesic through P in this direction also passes through $Q.$ We have the following theorem where we assume that the surface is of class $3.$

Every point P of the surface has a neighbourhood N with the property that every point of N can be joined to P by a unique geodesic are which lies wholly in N .

Note 8.1. The above theorem asserts that we can say at present about the existence of geodesic joining two given points, it says that Q can be joined to P if it is sufficiently close to $P.$ Nothing more than that can be said as long as the region of the surface have been considered arbitrary. However, when a complete surface has been defined it will appear that any two points can be joined by atleast one geodesic.

Definition 8.1 (Convex Region). A region R is convex if any two points can be joined by a geodesic lying wholly in R and is simple if there is not more than one such geodesic arc.

Note 8.2. In the Euclidean plane a convex region is necessarily simple but this is not so for a surface in general. The surface of a sphere for example is convex but not simple.

An existence theorem due to J.H.C. Whitehead states that every point of P of a surface has a neighbourhood which is convex and simple .

8.2. Geodesic Parallels:

A family of geodesics is given, and that a parameter system is chosen so that the geodesics of the family are the curves $v =$ constant and their orthogonal trajectories are the curves $u =$ constant. Then $F = 0$ and condition for the $v = c$ curve be a geodesic is $EE_2 + FE_1 \quad 2EF_1 = 0.$ This implies $v =$ constant to be geodesic becomes $E_2 = 0.$ Thus, the metric is of the form

$$ds^2 = E(u)du^2 + G(u, v)dv^2$$

Consider the distance between any two of the orthogonal trajectories, say $u = u_1$ and $u = u_2$ measured along the geodesic $v = c.$

Along $v = c$ and $dv = 0$ and $ds = \sqrt{E(u)}du.$ This implies $s = \int_{u_1}^{u_2} \sqrt{E(u)}du.$ Which is independent of $c.$ Thus the distance is same along whichever geodesic, $v =$ constant is measured. For this reason, the orthogonal trajectories are called geodesic parallels.

When $dv = 0$ and $ds = du$ implies $E(u) = 1.$ Thus the metric is reduced to $ds^2 = du^2 + G(u, v)dv^2$ where u is the new parameter determines the distance from some xed parallel the parallel, determines by u measured along the geodesic $v =$ constant.

Geodesic Coordinates: If the parametric curves are orthogonal and one of the family of parametric curves are geodesics then the coordinate of any point on the surface are called a set of geodesic coordinates.

Geodesic Polars: A particular system of geodesics and parallels is found by taking the geodesics which pass through a given point $O.$ By the second existence theorem, there is a neighbourhood of O in which, when the point O is excluded, the geodesic constitute a family. Parameters u, v can be chose as above. In particular u can be taken as the distance measured from O along the geodesics and v can be taken as the angle measured at O between a xed geodesics $v = 0$ and the one determined by v:

In this way, the parameters u and v corresponds to polar coordinates r and in the plane.

Thus the metric is given by

$$ds^2 = du^2 + Gdv^2$$

where G is such that when u is small, the metric approximates to plane polar form with $u; v$ in place of $r;$: i:e:; to $du^2 + u^2dv^2$: Hence G u^2:

$$\lim_{u \to 0} \frac{\sqrt{G}}{u} = 1$$

In geodesic polar parameters the parallel $u =$ constant are geodesic circles.

8.3. Geodesic Curvature:

For any curve on a surface, curvature vector at P is $\tilde{r}'' = \tilde{n}$ where is the curvature and $\tilde{n}$ is the unit principal normal.

Since any vector at P is a linear combination of $\tilde{r}_1; \tilde{r}_2$ and $\tilde{N}$; we can write $\tilde{r}''$ as

$$\tilde{r}'' = \tilde{r}_1 + \tilde{r}_2 + {}_n\tilde{N} \qquad (8.1)$$

where $_n$ is the normal component of $\tilde{r}''$; called the normal curvature P: The vectors $\tilde{r}_1 + \tilde{r}_2$ with components (;) is intrinsic so that the magnitudes measures in some sense the deviation of the curve from geodesic.

$$\tilde{r}''\ \tilde{r}_1 = {}_n\tilde{N} + \tilde{r}_1 + \tilde{r}_2\ \ \tilde{r}_1 \qquad (* \tilde{N}\ \tilde{r}_1 = 0)$$

$$U = \tilde{r}''\ \tilde{r}_1 = \tilde{r}_1{}^2 + \tilde{r}_1\ \tilde{r}_2 = E + F$$

$$V = \tilde{r}''\ \tilde{r}_2 = \tilde{r}_1\ \tilde{r}_2 + \tilde{r}_2{}^2 = F + G \qquad (* \tilde{N}\ \tilde{r}_2 = 0)$$

Solving the above two equations, we get the values of and :

i:e:; $= \dfrac{GU \quad FV}{H^2}$; $= \dfrac{EV \quad FU}{H^2}$:

Geodesic curvature vector $\tilde{r}_1 + \tilde{r}_2$ is denoted by $\tilde{\ }_g$ and its magnitude by $_g$: The vector (;) is called the geodesic curvature of the vector.

Bookwork 8.1. Prove that the geodesic curvature vector of any curve is orthogonal to the curve.

Proof. Now we shall prove that the geodesic curvature vector $\tilde{\ }_g$ of any curve is orthogonal to the curve. We have

$$\tilde{r}'' = \tilde{r}_1 + \tilde{r}_2 + {}_n\tilde{N}$$

$$\tilde{r}'' = \tilde{\ }_g + {}_n\tilde{N} \tag{8.2}$$

Taking scalar product of equation (8.2) with $\tilde{r}'$;

$$\tilde{r}' \ \tilde{r}'' = \tilde{r}' \ \tilde{\ }_g + {}_n \tilde{r}' \ \tilde{N}$$

$$\Rightarrow 0 = \tilde{r}' \ \tilde{\ }_g + 0 \qquad \left[* \ \tilde{r}' \ \tilde{r}'' = 0; \ \tilde{r}' \ \tilde{N} = 0 \right]$$

$$\Rightarrow \tilde{t} \ \tilde{\ }_g = 0$$

This shows that $\tilde{\ }_g$ is orthogonal to the curve.

Bookwork 8.2. For a geodesic, the geodesic curvature is zero.

Proof. Now, $\tilde{r}' \ \tilde{N} = 0 \Rightarrow \tilde{r}' ? \tilde{N}$ and $\tilde{N} \ \tilde{r}'$ is perpendicular to both $\tilde{N}$ and $\tilde{r}'$:

Therefore, $\tilde{r}'; \tilde{N} \ \tilde{r}'; \tilde{N}$ form a right handed system of unit vectors.

Thus, the geodesic curvature vector $\tilde{\ }_g$ can be expressed as $\tilde{\ }_g = {}_g \tilde{N} \ \tilde{r}'$:

$$\text{Equation (8.2)} \Rightarrow \tilde{r}'' = {}_g \tilde{N} \ \tilde{r}' + {}_n\tilde{N}$$

Taking dot product with $\tilde{N} \ \tilde{r}'$; we get

$$\tilde{N} \ \tilde{r}' \ \tilde{r}'' = {}_g \left(\tilde{N} \ \tilde{r}' \right)^2 + {}_n \left(\tilde{N} \ \tilde{r}' \ \tilde{N} \right)$$

$$\Rightarrow \left[\tilde{N}; \tilde{r}'; \tilde{r}'' \right] = {}_g \tilde{N} \ \tilde{r}'^{\,2} + {}_n \left[\tilde{N}; \tilde{r}'; \tilde{N} \right]$$

$$\Rightarrow \left[\tilde{N}; \tilde{r}'; \tilde{r}'' \right] = {}_g (1) + {}_n (0)$$

$$\Rightarrow {}_g = \left[\tilde{N}; \tilde{r}'; \tilde{r}'' \right]$$

If the curvature is a geodesic, then $\tilde{r}'' = {}_n\tilde{N}$:

$$\kappa_g = [\tilde N, \tilde r', \kappa_n \tilde N]$$
$$= \kappa_n [\tilde N, \tilde r', \tilde N] = 0$$

$$\Rightarrow \kappa_g = 0$$

Bookwork 8.3. Derive an expression for Geodesic Curvature.

Proof. As we already proved that the geodesic curvature vector κ_g of a curve is orthogonal to the curve. κ_g lies on the tangent plane and therefore perpendicular to the surface $\tilde N$. Thus, κ_g is orthogonal to the unit vector $\tilde N \times \tilde r'$.

Therefore, the geodesic curvature vector is $\kappa_g (\tilde N \times \tilde r')$ and hence it can be written as

$$\tilde r'' = \kappa_n \tilde N + \kappa_g (\tilde N \times \tilde r') \qquad (8.3)$$

Taking dot products with the unit vectors $\tilde N \times \tilde r$, we have

$$(\tilde N \times \tilde r)\cdot \tilde r'' = (\tilde N \times \tilde r)\cdot \kappa_n \tilde N + (\tilde N \times \tilde r)\cdot \kappa_g (\tilde N \times \tilde r)$$
$$[\tilde N, \tilde r', \tilde r''] = 0 + \kappa_g \qquad \because (\tilde N \times \tilde r)\cdot(\tilde N \times \tilde r) = 1; \quad (\tilde N \times \tilde r)\cdot \tilde N = 0$$
$$\text{i.e.,}\quad \kappa_g = [\tilde N, \tilde r', \tilde r'']$$

If we replace the parameter s by t, we have

$$\Rightarrow \tilde r' = \frac{\dot{\tilde r}}{\dot s}; \quad \tilde r'' = \frac{\dot s \ddot{\tilde r} - \dot{\tilde r} \ddot s}{\dot s^2}; \quad \text{and } \tilde r' \times \tilde r'' = \frac{\dot{\tilde r} \times \ddot{\tilde r}}{\dot s^2}$$

Therefore, we have $\kappa_g = \dfrac{1}{\dot s^3}[\tilde N, \dot{\tilde r}, \ddot{\tilde r}]$

$$\text{But, } \tilde N = \frac{1}{H}(\tilde r_1 \times \tilde r_2)$$
$$\Rightarrow \kappa_g = \frac{1}{H\dot s^3}(\tilde r_1 \times \tilde r_2)\cdot(\dot{\tilde r} \times \ddot{\tilde r}) = \frac{1}{H\dot s^3}\begin{vmatrix} \tilde r_1 \cdot \dot{\tilde r} & \tilde r_1 \cdot \ddot{\tilde r} \\ \tilde r_2 \cdot \dot{\tilde r} & \tilde r_2 \cdot \ddot{\tilde r} \end{vmatrix}$$

Also, we know that $\dfrac{\partial T}{\partial u} = \dot{\tilde r}\cdot \tilde r_1$; $\dfrac{\partial T}{\partial v} = \dot{\tilde r}\cdot \tilde r_2$; $U(t) = \ddot{\tilde r}\cdot \tilde r_1$; $V(t) = \ddot{\tilde r}\cdot \tilde r_2$.

Thus, we have

$$\kappa_g = \frac{1}{H\dot s^3}\begin{vmatrix} \frac{\partial T}{\partial u} & U(t) \\ \frac{\partial T}{\partial v} & V(t) \end{vmatrix} = \frac{1}{H\dot s^3}\left[V(t)\frac{\partial T}{\partial u} - U(t)\frac{\partial T}{\partial v}\right]$$

Replacing the parameter t by s, we get

$$\kappa_g = \cdots = \frac{1}{H}\left[V(t)\frac{\partial T}{\partial u'} - U(t)\frac{\partial T}{\partial v'} \right]$$

This is the expression for κ_g.

Example 8.1. Find the geodesic curvature of the parametric curves v = constant.

Solution:

Taking u as the parameter.

$$\begin{aligned}
\text{i.e.,}\quad u &= t; \quad v = c \\
\Rightarrow \quad u' &= 1; \quad v' = 0 \\
T &= \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2 \right) \\
\frac{\partial T}{\partial u'} &= E; \quad \frac{\partial T}{\partial v'} = F \\
U &= \frac{d}{dt}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(E) - \frac{1}{2}E_1 = \frac{1}{2}E_1 \\
\text{and}\quad V &= \frac{d}{dt}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(F) - \frac{1}{2}E_2 = F_1 - \frac{1}{2}E_2 \\
\Rightarrow \quad \kappa_g &= \frac{1}{Hs'^3}\left[\frac{\partial T}{\partial u'}V(t) - \frac{\partial T}{\partial v'}U(t) \right] \\
&= \frac{1}{H E^{3/2}}\left[E\left(F_1 - \frac{1}{2}E_2\right) - F\left(\frac{1}{2}E_1\right) \right] \\
&= \frac{1}{2HE^{3/2}}\left[2EF_1 - EE_2 - FE_1 \right]
\end{aligned}$$

Example 8.2. Derive the formula for geodesic curvature when the arc length s is chosen as parameter.

Solution: We know that

$$\begin{aligned}
k_g &= \frac{1}{H}\left[\frac{\partial T}{\partial u'}V(s) - \frac{\partial T}{\partial v'}U(S) \right] \\
&= \frac{1}{H}V(s)\left[\frac{\partial T}{\partial u'} - \frac{\partial T}{\partial v'}\frac{U(S)}{V(S)} \right] \\
&= \frac{1}{H}\frac{V(s)}{u'}\left[u'\frac{\partial T}{\partial u'} + v'\frac{\partial T}{\partial v'} \right] \\
&= \frac{1}{H}\frac{V(s)}{u'}
\end{aligned}$$

[Since $T = \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2 \right) = \frac{1}{2}$ is a homogeneous function of second degree in u' and v'.

Hence $u'\frac{\partial T}{\partial u'} + v'\frac{\partial T}{\partial v'} = 2T = 2\left(\frac{1}{2}\right) = 1.$]

In a similar fashion, we can prove that $_g = \frac{1}{H}\frac{U(s)}{v'}$:

Thus $_g = \frac{1}{H}\frac{V(s)}{u'} = \frac{1}{H}\frac{U(s)}{v'}$:

Example 8.3. Show that the components ; of the geodesic curvature vector are given by the following formula

$$ = \frac{1}{H^2}\frac{U}{v'}\frac{@T}{@v'} = \frac{1}{H^2}\frac{V}{u'}\frac{@T}{@v'}$$

$$ = \frac{1}{H^2}\frac{V}{u'}\frac{@T}{@u'} = \frac{1}{H^2}\frac{U}{v'}\frac{@T}{@u'}$$

Where s is the parameter.

Solution: We know that

$$ = \frac{1}{H^2}[GU \quad FV]; \quad = \frac{1}{H^2}[EV \quad FU]$$

Now, $ = \frac{U}{H^2}\left(G \quad F\frac{V}{U}\right)$

If s is a parameter, then $u'U + v'V = 0$ i.e.; $\frac{V}{U} = \frac{u'}{v'}$:

Thus,
$$ = \frac{U}{H^2}\left(G + F\frac{u'}{v'}\right)$$
$$ = \frac{U}{H^2 v'}\left(Gv' + Fu'\right)$$
$$ = \frac{U}{H^2 v'}\frac{@T}{@v'}$$
$$\left[* \; T = \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2\right)\right]$$

Again
$$ = \frac{V}{H^2}\left(\frac{GU}{V} \quad F\right)$$
$$ = \frac{V}{H^2}\left(G\left(\frac{v'}{u'}\right) \quad F\right)$$
$$ = \frac{V}{u'H^2}\left(Gv' + Fu'\right)$$
$$ = \frac{V}{u'H^2}\frac{@T}{@v'}$$

In a similar way, we can prove the other results.

Example 8.4. Prove that if (;) is the geodesic curvature vector, then $_g = \frac{H}{Fu' + Gv'} = \frac{H}{Eu' + Fv'}$:

Solution: We know that

$$
\begin{aligned}
&= \frac{1}{H^2}[GU \quad FV] = \frac{U}{H^2}\left(G \quad F\frac{V}{U}\right)\\
&= \frac{U}{H^2}\left(G + F\frac{u'}{v'}\right) \qquad\qquad * \; u'U + v'V = 0\\
&= \frac{U}{H^2 v'}\left(Gv' + Fu'\right)\\
&= \frac{g}{H}\left(Fu' + Gv'\right)\\
\Rightarrow \quad g &= \frac{H}{Fu' + Gv'}
\end{aligned}
$$

Similarly, we can prove the other results.

Liouville's formula for $_g$:

Bookwork 8.4. If is the angle which the curve under consideration makes with parametric curves $v =$ constant, then according to Liouville's formula $_g$ is expressed by

$$
\begin{aligned}
_g &= {}' + Pu' + Qv'\\
\text{where} \quad P &= \frac{2EF_1 \quad FE_1 \quad EE_2}{2HE}\\
Q &= \frac{EG_1 \quad FE_2}{2HE}
\end{aligned}
$$

Proof. The direction coe cients of the parametric curve $v =$ constant are $\left(\frac{1}{\sqrt{E}}; 0\right)$ and the direction coe cients of given curve be $(u'; v')$: We have

$$
\begin{aligned}
\cos &= Ell_1 + F(lm_1 + l_1m) + Gmm_1\\
\Rightarrow \cos &= E\frac{1}{\sqrt{E}}u' + F\left(\frac{1}{\sqrt{E}}v' + 0\right) + G(0)\\
\Rightarrow \cos &= \frac{Eu' + Fv'}{\sqrt{E}} \qquad (8.4)\\
\text{we have } T &= \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2\right)\\
\Rightarrow \frac{\partial T}{\partial u} &= \frac{1}{2}\left(E_1u'^2 + 2F_1u'v' + G_1v'^2\right) \qquad (8.5)\\
\text{and } \frac{\partial T}{\partial u'} &= \frac{1}{2}\left(2Eu' + 2F(1)v' + 0\right)\\
\Rightarrow \frac{\partial T}{\partial u'} &= Eu' + Fv' \qquad (8.6)
\end{aligned}
$$

Using equation (8.6) in (8.4), we get

$$\cos\theta = \frac{1}{\sqrt{E}}\frac{\partial T}{\partial u'} \qquad (8.7)$$

$$\Rightarrow \cos\theta = E^{-1/2}\frac{\partial T}{\partial u'}$$

Differentiate both sides with respect to s; we get

$$-\sin\theta\frac{d\theta}{ds} = E^{-1/2}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) + \frac{\partial T}{\partial u'}\left(-\frac{1}{2}E^{-3/2}\frac{dE}{ds}\right)$$

$$-\sin\theta\,\theta' = \frac{1}{\sqrt{E}}\left[U(s) + \frac{\partial T}{\partial u}\right] - \frac{1}{2}\frac{1}{E^{3/2}}(Eu' + Fv')\frac{dE}{ds}$$

$$= \frac{1}{\sqrt{E}}\left[U(s) + \frac{\partial T}{\partial u}\right] - \frac{1}{2E^{3/2}}(Eu' + Fv')\left(\frac{\partial E}{\partial u}\frac{du}{ds} + \frac{\partial E}{\partial v}\frac{dv}{ds}\right)$$

$$-\sin\theta\,\theta' = \frac{1}{\sqrt{E}}\left(U + \frac{\partial T}{\partial u}\right) - \frac{1}{2E^{3/2}}(Eu' + Fv')(E_1u' + E_2v')$$

$$\Rightarrow -\sin\theta\,\theta' = \frac{1}{\sqrt{E}}\left[U + \frac{1}{2}\left(E_1u'^2 + 2F_1u'v' + G_1v'^2\right)\right] - \frac{1}{2E^{3/2}}\left[EE_1u'^2 + (EE_2 + FE_1)u'v' + FE_2v'^2\right]$$

We know that $\sin\theta = H(lm_1 - l_1m) = H\left(\frac{1}{\sqrt{E}}v' - 0\right) = \frac{Hv'}{\sqrt{E}}$

$$\Rightarrow -\frac{Hv'\theta'}{\sqrt{E}} = \frac{1}{\sqrt{E}}\left[U + \frac{1}{2}\left(E_1u'^2 + 2F_1u'v' + G_1v'^2\right)\right] - \frac{1}{2E^{3/2}}\left[EE_1u'^2 + (EE_2 + FE_1)u'v' + FE_2v'^2\right]$$

$$-Hv'\theta' = U + \frac{1}{2}\left[E_1u'^2 + 2F_1u'v' + G_1v'^2\right] - \frac{1}{2E}\left[EE_1u'^2 + (EE_2 + FE_1)u'v' + FE_2v'^2\right]$$

$$\Rightarrow -Hv'\theta' = U + F_1 - \frac{E_2}{2} - \frac{FE_1}{2E}\,u'v' + \left(\frac{G_1}{2} - \frac{FE_2}{2E}\right)v'^2$$

$$-\theta' = \frac{U}{Hv'} + \left(\frac{2EF_1 - EE_2 - FE_1}{2EH}\right)u' + \left(\frac{EG_1 - FE_2}{2EH}\right)v'$$

$$\Rightarrow -\theta' = -\kappa_g + \left(\frac{2EF_1 - EE_2 - FE_1}{2EH}\right)u' + \frac{EG_1 - FE_2}{2EH}v'$$

$$\kappa_g = \theta' + \left(\frac{2EF_1 - EE_2 - FE_1}{2EH}\right)u' + \frac{EG_1 - FE_2}{2EH}v'$$

$$\kappa_g = \theta' + Pu' + Qv'$$

where $P = \dfrac{2EF_1 - EE_2 - FE_1}{2EH}$; $Q = \dfrac{EG_1 - FE_2}{2EH}$

Let Us Sum Up:

In this unit, the students acquired knowledge to

the Convex region and simple.

the Geodesic polars and Geodesic parallels .

derive the expression for Geodesic curvature.

Check Your Progress:

1. Derive the Liouville's formula for $_g$.
2. Derive the formula for geodesic curvature for $_g$.
3. Prove that for a geodesic, the geodesic curvature is zero.

Choose the correct or more suitable answer:

1. Orthogonal trajectories are called : : : : : :
   - (a) geodesic polars.
   - (b) geodesic parallels.
   - (c) geodesic curvature.
   - (d) geodesic coordinates.
2. The geodesic curvature vector of any curve is : : : : : : to the curve.
   - (a) tangent
   - (b) orthogonal
   - (c) parallel
   - (d) none of these.

Answer:

(1) b (2) b

Suggested Readings:

1. T.J. Willmore, An Introduction to Di erential Geometry , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, Di erential Geometry of Three Dimensions , University Press, Cambridge, 1930.

Block-III

UNIT-9

GEODESICS-III

Structure

Objective

Overview

9. 1 Gauss-Bonnet Theorem

9. 2 Gaussian Curvature

9. 3 Surfaces of constant curvature

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the concept of Gauss-Bonnet Theorem.
- understand the concept of Gaussian Curvature.
- derive Minding's Theorem.

Overview

In this unit, we will illustrate the derivation of Gauss Bonnet theorem and Minding theorem.

9.1. Gauss-Bonnet Theorem:

De nition 9.1 (Simply Connected Regions). If every curve lying in a region R can be contracted continuously in to a point without leaving R then R is said to be simply connected.

For Example: In a plane interior of a circle is simply connected, but the region between two concentric circles is not simply connected.

Theorem 9.1 (Gauss-Bonnet Theorem).

For any curve C enclosing a simply connected region R; the excess of C is equal to the total curvature of R:

Proof. Let us consider a surface $\tilde{r}(u; v)$ and a simply connected region R of the surface bounded by a closed curve C:

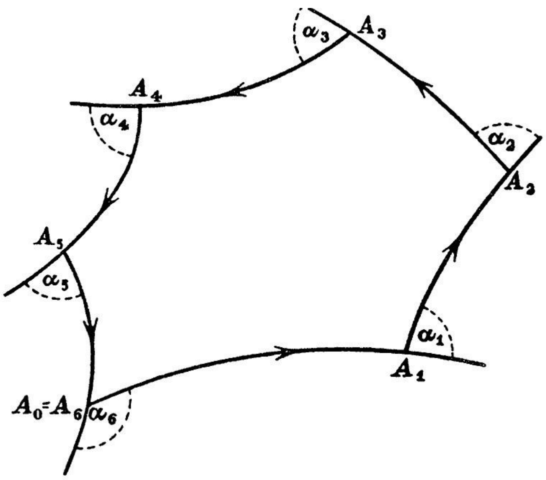


Figure 9.1

Let C consists of n smooth arcs $A_0A_1; A_1A_2; \quad A_{n-2}A_{n-1}; A_{n-1}A_n$ $(A_n = A_0)$ where n is nite and each arc is positively described.

At the vertex $A_i\,(i = 1; 2; \quad ; n):$ Let $\ _i$ be the angle between the tangents to the arcs $A_{i\ 1}A_i$ and A_iA_{i+1} measured with usual convection at vertices A_i so that $\ < \ _i < \ :$ If C is taken to be curvilinear polygon then $\ _i$ are the exterior angles at the vertices $A_i\,(i = 1; 2; \quad ; n):$

The geodesic curvature $\ _g$ exists at each point of C except possibly at the vertices $A_i\,(i = 1; 2; \quad ; n):$

Now, we de ne the excess of the curve C as

$$\text{ex}(C) = 2 \quad \sum_{i=1}^{n} \ _i \quad \int_C \ _g ds \tag{9.1}$$

From Liouville's formula for $\ _g;$

$$\text{we have} \quad _g = \ ^{0} + Pu^{0} + Qv^{0}$$

$$_g = \frac{d}{ds} + P\frac{du}{ds} + Q\frac{dv}{ds} \tag{9.2}$$

where is the angle made by the curve C with the parametric curve $v =$ constant and $P; Q$ are functions of $u; v:$

Since the curve $v =$ constant form a family in the region R enclosed by $C;$ the tangent to C turns through $2 \ $ relative to these curves, $i:e:;$ we have

$$\int_C d \ + \sum_{i=1}^{n} \ _i = 2 \tag{9.3}$$

Using equations (9.2) and (9.3) in (9.1), we get

$$\begin{aligned}
\text{ex}(C) &= \int_C d \ + \sum_{i=1}^{n} \ _i \quad \sum_{i=1}^{n} \ _i \quad \int_C \left[\frac{d}{ds} + P\frac{du}{ds} + Q\frac{dv}{ds} \right] ds \\
&= \int_C (Pdu + Qdv) \\
&= \iint_R \left(\frac{\partial Q}{\partial u} \quad \frac{\partial P}{\partial v} \right) dudv \qquad * \text{ by Green's theorem} \\
&= \iint_R \left(\frac{\partial Q}{\partial u} \quad \frac{\partial P}{\partial v} \right) \frac{ds}{H} \qquad [* \text{ surface element } ds = Hdudv] \\
&= \iint_R \frac{1}{H} \left(\frac{\partial Q}{\partial u} \quad \frac{\partial P}{\partial v} \right) ds
\end{aligned}$$

$$\Rightarrow \text{ex}(C) = \int_R K(u; v)\, ds \tag{9.4}$$

$$\text{where } K = \frac{1}{H} \left(\frac{\partial Q}{\partial u} \quad \frac{\partial P}{\partial v} \right) \tag{9.5}$$

$$\Rightarrow \text{excess of C} = \text{total curvature of R} \tag{9.6}$$

Now, we shall show that the function K is uniquely determined. If possible there is another function K_1 such that $ex(C) = \iint_R K_1(u,v)\,ds.$

$$\text{Now, } \iint_R (K_1 - K)\,ds = \iint_R K_1 ds - \iint_R K ds = ex(C) - ex(C) = 0$$

$$\Rightarrow \iint_R (K_1 - K)\,ds = 0 \qquad (9.7)$$

If $K_1 \neq K$ at some points $P;$ let $K_1 > K$ (for definiteness). Then, since $K_1 - K > 0$ is continuous there exists a region R containing P such that $\iint_R (K_1 - K)\,ds > 0;$ contradicting (9.7). Therefore $K_1 \leq K.$ Similarly we can prove that $K_1 \geq K.$

Thus, $K_1 = K$ at each point. $i.e.,$ K is unique.

Note 9.1. 1. $\iint_s K ds$ is called the total curvature of $R.$

2. When K is uniquely determined, then K is an intrinsic geometrical invariant. It is called the Gaussian Curvature.

3. For a geodesic triangle $ABC;$ having arms as geodesic arcs $AB; BC; CA$ and bounded by a simply connected region $R;$ we have

$$\begin{aligned} ex(C) &= 2\pi - \sum_{i=1}^{n} \alpha_i - \int_C \kappa_g ds \qquad \left(\because \int_C \kappa_g ds = 0\right) \\ &= 2\pi - \text{sum of exterior angles} \\ &= 2\pi - (\pi - A + \pi - B + \pi - C) \\ &= 2\pi - [3\pi - (A + B + C)] = A + B + C - \pi \end{aligned}$$

When $A; B; C$ are the exterior angles of the $\triangle ABC.$

Thus, Total curvature $= A + B + C - \pi = ex(C).$

4. For a geodesic polygon of n sides.

Total curvature $= exc(C) = 2\pi - (\text{sum of exterior angles}).$

5. The formula for K in terms of $E; F$ and G is given by equation (9.7). Hence at any point and in any parameter system,

$$\begin{aligned} K &= \frac{1}{H}\left(\frac{\partial P}{\partial v} - \frac{\partial Q}{\partial u}\right) \\ &= \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2 - EG_1}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right) \end{aligned}$$

When the parametric curves are orthogonal, $F = 0$ and the formula for K can be written in the simplified form is

$$K = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right] \quad \text{where } H = \sqrt{EG}$$

9.2. Gaussian Curvature:

An historical definition of Gaussian curvature follows from Gauss-Bonnet theorem for a geodesic triangle. If P is a given point and Δ the area of a geodesic triangle ABC which contains P, then at P,

$$K = \lim \frac{A + B + C - \pi}{\Delta}$$

Example 9.1. Find the Gaussian curvature of the surface $x = u + v$; $y = u - v$; $z = uv$ at $u = v = 1$.

Solution: Given surface be $\tilde{r}(u, v)$.

$$\begin{aligned}
\text{i.e.,}\quad \tilde{r} &= x\tilde{i} + y\tilde{j} + z\tilde{k} \\
\Rightarrow \tilde{r} &= (u + v)\tilde{i} + (u - v)\tilde{j} + uv\tilde{k} \\
\tilde{r}_1 &= (1 + 0)\tilde{i} + (1 - 0)\tilde{j} + v\tilde{k} = \tilde{i} + \tilde{j} + v\tilde{k} \\
\tilde{r}_2 &= \tilde{i} - \tilde{j} + u\tilde{k} \\
\text{Now, } E &= \tilde{r}_1 \cdot \tilde{r}_1 = 1^2 + 1^2 + v^2 = v^2 + 2 \\
F &= \tilde{r}_1 \cdot \tilde{r}_2 = 1 - 1 + uv = uv \\
G &= \tilde{r}_2 \cdot \tilde{r}_2 = u^2 + 2 \\
\text{and } H^2 &= EG - F^2 = 2\left(u^2 + v^2 + 2\right) \\
\text{i.e., } H &= \sqrt{2}\sqrt{u^2 + v^2 + 2} \\
\text{Now } E_1 &= \frac{\partial E}{\partial u} = 0; \quad E_2 = \frac{\partial E}{\partial v} = 2v \\
F_1 &= \frac{\partial F}{\partial u} = v; \quad F_2 = \frac{\partial F}{\partial v} = u \\
G_1 &= \frac{\partial G}{\partial u} = 2u \\
P &= \frac{2EF_1 - FE_1 - EE_2}{2HE} = 0 \\
Q &= \frac{EG_1 - FE_2}{2HE} = \frac{\sqrt{2}\,u}{\left(v^2 + 2\right)\sqrt{u^2 + v^2 + 2}}
\end{aligned}$$

Thus, the Gaussian curvature K is given by

$$K = \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$$

$$= \frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{2u}{\sqrt{v^2+2}\,\sqrt{u^2+v^2+2}}\right) - \frac{\partial}{\partial v}(0)\right] = \frac{1}{(u^2+v^2+2)^2}$$

Hence, at $u = 1;\ v = 1;$ the Gaussian curvature is $K = \frac{1}{16}.$

Example 9.2. Find the Gaussian curvature at the point (u, v) of a sphere of radius $a.$

Solution: Equation of the sphere with centre at O and radius a is

$$\tilde{r} = a \sin u \cos v\tilde{i} + a \sin u \sin v\tilde{j} + a \cos u\tilde{k};$$

$$\text{where } 0 \le u \le \pi;\ 0 \le v \le 2\pi$$

$$\tilde{r}_1 = a \cos u \sin v\tilde{i} + a \cos u \sin v\tilde{j} - a \sin u\tilde{k}$$

$$\tilde{r}_2 = -a \sin u \sin v\tilde{i} + a \sin u \cos v\tilde{j} + 0\tilde{k}$$

$$E = \tilde{r}_1 \cdot \tilde{r}_1 = a^2 \cos^2 u + a^2 \sin^2 u = a^2$$

$$F = \tilde{r}_1 \cdot \tilde{r}_2 = 0; \quad G = \tilde{r}_2 \cdot \tilde{r}_2 = a^2 \sin^2 u$$

$$H^2 = EG - F^2 = a^4 \sin^2 u$$

$$\Rightarrow H = a^2 \sin u$$

$$E_1 = \frac{\partial E}{\partial u} = 0; \quad E_2 = \frac{\partial E}{\partial v} = 0$$

$$F_1 = \frac{\partial F}{\partial u} = 0; \quad G_1 = \frac{\partial G}{\partial u} = 2a^2 \sin u \cos u$$

$$P = \frac{2EF_1 - FE_1 - EE_2}{2HE} = 0$$

$$Q = \frac{EG_1 - FE_2}{2HE} = \cos u$$

Thus, we have the Gaussian curvature is

$$K = \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) = \frac{1}{a^2 \sin u}[-\sin u] = \frac{1}{a^2}$$

Example 9.3. Find the Gaussian curvature of the anchor ring and show that the total curvature of the whole surface is zero.

Solution: The equation of anchor ring is

$$\tilde{r} = (b + a \cos u) \cos v\tilde{i} + (b + a \cos u) \sin v\tilde{j} + a \sin u\tilde{k}$$

where a, b are constants and $0 \le u \le 2\pi$; $0 \le v \le 2\pi$.

$$\tilde{r}_1 = -a \sin u \cos v\,\tilde{i} - a \sin u \sin v\,\tilde{j} + a \cos u\,\tilde{k}$$

$$\tilde{r}_2 = -(b + 1 \cos u) \sin v\,\tilde{i} + (b + a \cos u) \cos v\,\tilde{j} + 0\tilde{k}$$

$$E = \tilde{r}_1 \cdot \tilde{r}_1 = a^2 \sin^2 u \cos^2 v + a^2 \sin^2 u \sin^2 v + a^2 \cos^2 u = a^2$$

$$F = \tilde{r}_1 \cdot \tilde{r}_2 = 0$$

$$G = \tilde{r}_2 \cdot \tilde{r}_2 = (b + a \cos u)^2$$

$$H^2 = EG - F^2 = a^2 (b + a \cos u)^2$$

$$\Rightarrow H = a (b + a \cos u)$$

Now,

$$E_1 = \frac{\partial E}{\partial u} = 0; \quad E_2 = \frac{\partial E}{\partial v} = 0$$

$$F_1 = \frac{\partial F}{\partial u} = 0$$

$$G_1 = \frac{\partial G}{\partial u} = -2a (b + a \cos u) \sin u$$

$$P = \frac{2EF_1 - FE_1 - EE_2}{2HE} = 0$$

$$Q = \frac{EG_1 - FE_2}{2HE} = -\sin u$$

Thus, the Gaussian curvature is given by

$$K = \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) = \frac{1}{H}[-\cos u] = \frac{-\cos u}{a (b + a \cos u)}$$

Hence, the total curvature of the whole surface is

$$\text{Total curvature} = \iint_S K\,ds = \int_{u=0}^{u=2\pi}\int_{v=0}^{v=2\pi} \frac{-\cos u}{a (b + a \cos u)} H\,du\,dv$$

$$[\because ds = H\,du\,dv]$$

$$\text{i.e., Total curvature} = -\int_{u=0}^{u=2\pi}\int_{v=0}^{v=2\pi} \cos u\,du\,dv = 0$$

Therefore, the total curvature of the whole surface is zero.

Example 9.4. If the parametric curves are at right angles, show that their geodesic curvatures are respectively $-\frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\sqrt{G}$; $\frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\sqrt{E}$.

Solution: The geodesic curvatures of the parametric curves $u =$ constant and $v =$ constant are respectively given by

$$\kappa_u = \frac{2GF_2 - GE_1 - FG_1}{2HG^{3/2}}$$

$$\kappa_v = \frac{2EF_1 - EE_2 - FE_1}{2HE^{3/2}}$$

Since the parametric curves are orthogonal $F = 0$ and $H^2 = EG$.

Thus, the geodesic curvature of the parametric curve $v =$ constant is

$$\kappa_a = \frac{-EE_2}{2\sqrt{EG}\,E^{3/2}} = \frac{-1}{\sqrt{EG}}\left(\frac{E_2}{2E^{1/2}}\right) = \frac{-1}{\sqrt{EG}}\frac{\partial}{\partial v}\sqrt{E}$$

In a similar way, we can prove that the geodesic curvature for the parmaetric curve $u =$ constant is $\kappa_a = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\sqrt{G}$.

Example 9.5. If $\tilde{r} = \tilde{r}(u, v)$ is a set of geodesic curvature on a surface of class 3, such that the parametric curves $v =$ constant are geodesics and u is natural parameter then $K = \frac{-1}{\sqrt{G}}\frac{\partial^2}{\partial u^2}\sqrt{G}$.

Solution:

We know that the Gaussian curvature is given by

$$K = \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$$

$$= \frac{1}{H}\left[\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right) - \frac{\partial}{\partial u}\left(\frac{EG_1 - FE_2}{2HE}\right)\right] \quad (9.8)$$

Also, for $v =$ constant geodesics, we have

$$ds^2 = du^2 + G(u, v)dv^2$$

We get $E = 1$; $F = 0$; $H = \sqrt{G}$.

Thus, the equation (9.8) reduces to

$$K = \frac{1}{\sqrt{G}}\left[-\frac{\partial}{\partial u}\left(\frac{G_1}{2\sqrt{G}}\right)\right] = \frac{-1}{\sqrt{G}}\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\sqrt{G}\right)$$

$$= \frac{-1}{\sqrt{G}}\frac{\partial^2}{\partial u^2}\sqrt{G}$$

Example 9.6. Find the area of geodesic triangle ABC on a sphere of radius a. Also, find the total curvature of the whole space.

Solution: From example (9.2), we see that the Gaussian curvature at any point on the sphere is $\frac{1}{a^2}$.

Also, we know that

$$\iint_S K\,ds = \text{exc}(C) \qquad \text{from Gauss Bonnet theorem}$$

$$\text{i.e.,}\quad \frac{1}{a^2}\iint_S ds = A+B+C-\pi$$

$$\Rightarrow \frac{\Delta}{a^2} = A+B+C-\pi;$$

where Δ is the area of the geodesic triangle

$$\Rightarrow \Delta = a^2\left[(A+B+C)-\pi\right]$$

Thus, the total curvature on the whole surface is given by

$$\iint_S K\,ds = \frac{1}{a^2}\iint_S ds = \frac{1}{a^2}4\pi a^2 = 4\pi.$$

9.3. Surfaces of constant curvature:

If K has the same value K_0 at every point of a surface, the surface is said to have constant curvature K_0.

Theorem 9.2 (Minding's Theorem). Two surfaces of the same constant curvature are locally isometric.

Proof. If P is any point of one of these surfaces and P' is any point of the other, then P has a neighbourhood which is isometric with a neighbourhood of P'; the points P and P' being the corresponding points. We prove the theorem by showing that S is a surface with constant curvature K_0; then

1. if $K_0 = 0$; S is isometric with a plane.
2. if $K_0 = \frac{1}{a^2}$; S is isometric with a sphere of radius a.
3. if $K_0 = -\frac{1}{a^2}$; S is isometric with a certain surface of revolution called pseudo sphere determined by the value of a.

In each case a given point of S can be mapped into a prescribed point of the plane, sphere or pseudo sphere.

The theorems for two surfaces S and S' with the same K; then follows by mapping each surface isometrically on to the same plane, or a sphere (or) surface of revolution, so that given points P and P' corresponds to the same point.

Let P be a given point of the surface S of constant curvature K_0; and let C be a geodesic through P. Take as parametric curves the geodesic orthogonal to C together with the orthogonal trajectories.

Let $v = c$ be the geodesic orthogonal to C at a point distance C from P measured along C and let $u = c$ be the orthogonal to the curves $v = c$ and at a distance c from the parallel measured along the geodesic. Then u, v is a parameter system in the neighbourhood of P and the metric of the surface is of the form $du^2 + g^2 dv^2$ for some $g(u, v)$.

Since $u = 0$ is the geodesic C; it follows from the relation

$$GG_1 + FG_2 \quad 2GF_2 = 0; \qquad F = 0; \; G = g^2$$

$$G_1 = \frac{\partial}{\partial u} g^2 = 0 \qquad \text{when } u = 0$$

Also, v is the arcual distance along C. i.e.; $ds = dv$ when $u = 0$; so that $g = 1$ when $u = 0$.

Thus, we have $(g)_{u=0} = 1; \quad (g_1)_{u=0} = 0.$

$K = \frac{g_{11}}{g}$ satisfies the partial differential equation $g_{11} + K_0 g = 0$ with boundary conditions $(g)_{u=0} = 1; \quad (g_1)_{u=0} = 0$ these are sufficient to determine the value of g when K_0 is given.

Case 1: $K_0 = 0$,when $g_{11} = 0$; clearly g_1 is a function of v only and therefore $g_1 = 0$ since $(g_1)_{u=0} = 0$.

Integrating $g_1 = 0$; we get g is a function of v only, since $(g)_{u=0} = 1$ and hence $g = 1$.

Thus the metric becomes $du^2 + dv^2$: when u, v are taken as Cartesian coordinates. Hence the surface S in the neighbourhood of P is isometric with a region in the plane. This implies that K is a satisfactory measure of curvature for a surface since its vanishing is both necessary and sufficient for the surface to be isometric with a plane.

Case 2: $K = \frac{1}{a^2}$.

$$\text{Thus, we have } \; g_{11} + \frac{1}{a^2} = 0$$

solving this partial differential equation, we get

$$g(u, v) = A(v)\sin\frac{u}{a} + B(v)\cos\frac{u}{a}$$

Using the boundary conditions, $(g)_{u=0} = 1; \quad (g_1)_{u=0} = 0$ we get $A = 0; \quad B = 1$.

Therefore $g(u, v) + \cos\frac{u}{a}$ and the metric becomes $du^2 + \cos^2\frac{u}{a}\,dv^2$.

The metric is a sphere of radius a. The surface S in a neighbourhood of P is therefore isometric with a region of a sphere of radius a.

Case 3: $K = \frac{1}{a^2}$:

As in the case (2), we have $g = \cosh \frac{u}{a}$ and the metric becomes $du^2 + \cosh^2 \frac{u}{a}$:

Applying the transformation $u = a\bar{u}$ and $v = \bar{v}$; the metric becomes $a^2 d\bar{u}^2 + a^2 \cosh^2 \bar{u}\, d\bar{v}^2$.

Now the metric of the surface of revolution of the curve $\tilde{r} = (g(u)\cos v;\ g(u)\sin v;\ f(u))$ is $(g_1^2 + f_1^2)\, du^2 + g^2 dv^2$.

Comparing two metrics, we have $g_1^2 + f_1^2 = a^2$; $g = a\cosh u$.

Therefore $f(u) = a \int_0^u \sqrt{1 - \sinh^2 u}\, du$.

Thus the metric is isometric with surface obtained by revolving the curve $x = a\cosh u;\ y = 0;\ z = a \int_0^u \sqrt{1 - \sinh^2 u}\, du$ where $|u| < \log(1 + \sqrt{2})$ above the z-axis.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- derive Gauss-Bonnet Theorem.
- the concept of Gaussian curvature .
- derive Minding's theorem.

Check Your Progress:

1. If two families of geodesics on a surface intersect at a constant angle, prove that the surface has zero Gaussian curvature.
2. State and Prove Gauss-Bonnet Theorem.
3. State and Prove Minding's Theorem.
4. Show that the surface generated by the tangents to any surface curve is a surface of constant zero curvature.

Choose the correct or more suitable answer:

1. Orthogonal trajectories are called : : : : : :

   (a) geodesic polars.

   (b) geodesic parallels.

   (c) geodesic curvature.

   (d) geodesic coordinates.

2. The geodesic curvature vector of any curve is : : : : : : to the curve.

   (a) tangent (b) orthogonal

   (c) parallel (d) none of these

Answer:

(1) b (2) b

Suggested Readings:

1. T.J. Willmore, An Introduction to Di erential Geometry , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, Di erential Geometry of Three Dimensions , University Press, Cambridge, 1930.

Block-IV

Unit-10: The Second Fundamental Form.

Unit-11: Developable Surfaces-I.

Unit-12: Developable Surfaces-II.

Block-IV

UNIT-10

THE SECOND FUNDAMENTAL FORM

Structure

Objective

Overview

10. 1 The second fundamental form

10. 2 Principal curvatures

10. 3 Lines of curvature

10. 3. 1 Dupin indicatrix

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Objectives

After completion of this unit, students will be able to

- F understand the concept of geometrical interpretation of the second fundamental form.
- F explain the concept of principal curvature, principal directions and mean curvature.
- F understand the concept of Umbilic.
- F derive Rodrigue's formula.

Overview

In this unit, we will illustrate the concept of geometrical interpretation of the second fundamental form.

10.1. The second fundamental form:

In the earlier chapter, we discussed essentially with the intrinsic properties of a surface, while this chapter deals with properties of a surface relative to the Euclidean space in which it is embedded.

Bookwork 10.1 (The second fundamental form).

Derive the equation of second fundamental form.

Proof. The normal curvature of a curve at any point on a surface $\tilde{r} = \tilde{r}(u, v)$ is given by the equation

$$ {}_n = \tilde{N}\ \tilde{r}'' \qquad (10.1)$$

Now

$$\tilde{r}' = \tilde{r}_1 u' + \tilde{r}_2 v'$$

$$\begin{aligned}
\tilde{r}'' &= \left(\tilde{r}_1 u' + \tilde{r}_2 v'\right)' = \tilde{r}_1 u'' + \tilde{r}_2 v'' + \tilde{r}_1' u' + \tilde{r}_2' v' \\
&= \tilde{r}_1 u'' + \tilde{r}_2 v'' + \tilde{r}_{11} u'^2 + \tilde{r}_{12} u'v' + \tilde{r}_{21} u'v' + \tilde{r}_{22} v'^2 \\
&= \tilde{r}_1 u'' + \tilde{r}_2 v'' + \tilde{r}_{11} u'^2 + 2\tilde{r}_{12} u'v' + \tilde{r}_{22} v'^2
\end{aligned}$$

Thus the equation (10.1) becomes,

$$
\begin{aligned}
\kappa_n &= \tilde{N}\cdot\left(\tilde{r}_1 u'' + \tilde{r}_2 v''\right) + \tilde{N}\cdot\left(\tilde{r}_{11}u'^2 + 2\tilde{r}_{12}u'v' + \tilde{r}_{22}v'^2\right)\\
&= 0 + \left(\tilde{N}\cdot\tilde{r}_{11}\right)u'^2 + 2\left(\tilde{N}\cdot\tilde{r}_{12}\right)u'v' + \left(\tilde{N}\cdot\tilde{r}_{22}\right)v'^2\\
&= Lu'^2 + 2Mu'v' + Nv'^2\\
&= \frac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2} \qquad (10.2)\\
&= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \qquad (10.3)
\end{aligned}
$$

where L, M, N are defined by the relations

$$
L = \tilde{N}\cdot\tilde{r}_{11}; \quad M = \tilde{N}\cdot\tilde{r}_{12}; \quad N = \tilde{N}\cdot\tilde{r}_{22} \qquad (10.4)
$$

Alternative expression for L, M, N will now be obtained.

By differentiating $\tilde{N}\cdot\tilde{r}_1 = 0$; we get

$$
\begin{aligned}
\tilde{N}_1\cdot\tilde{r}_1 + \tilde{N}\cdot\tilde{r}_{11} &= 0 \qquad (10.5)\\
\tilde{N}_2\cdot\tilde{r}_1 + \tilde{N}\cdot\tilde{r}_{12} &= 0 \qquad (10.6)
\end{aligned}
$$

Similarly differentiating $\tilde{N}\cdot\tilde{r}_2 = 0$; we get

$$
\begin{aligned}
\tilde{N}_2\cdot\tilde{r}_2 + \tilde{N}\cdot\tilde{r}_{22} &= 0 \qquad (10.7)\\
\tilde{N}_1\cdot\tilde{r}_2 + \tilde{N}\cdot\tilde{r}_{21} &= 0 \qquad (10.8)
\end{aligned}
$$

Substitute the equations (10.5), (10.6), (10.7) and (10.8) in (10.4), we get

$$
L = -\tilde{N}_1\cdot\tilde{r}_1; \; M = -\tilde{N}_1\cdot\tilde{r}_2 = -\tilde{N}_2\cdot\tilde{r}_1; \; N = -\tilde{N}_2\cdot\tilde{r}_2
$$

The quadratic $Ldu^2 + 2Mdudv + Ndv^2$ is called the second fundamental form and the functions of u and v denoted by L, M, N are called the second fundamental coefficients.

From equation (10.3), it follows that all curves having the same direction at P have the same normal curvature, hence normal curvature is a property of a surface and a direction at a point on the surface.

Theorem 10.1 (Meusnier's theorem). If θ denotes the angle between the principal normal $\tilde{n}$ to a curve on the surface and the surface normal $\tilde{N}$; then $\kappa_n = \kappa\cos\theta$.

Proof. We know that

$$\tilde{r}^{\,oo} = {}_n\tilde{N} + \tilde{r}_1 + \tilde{r}_2$$

$$) \quad \tilde{N} \;\tilde{r}^{\,oo} = {}_n \qquad * \;\tilde{N} \text{ is normal to both } \tilde{r}_1 \text{ and } \tilde{r}_2$$

$${}_n = \tilde{N} \quad \tilde{n} \qquad * \;\tilde{N} \quad \tilde{n} = 1 \quad 1 \cos$$

$$= \quad \cos$$

Note 10.1. Since the right hand side denominator of equation (10.3) is positive de nite, it follows that the sign of $_n$ depends only upon the sign for the numerator of equation (10.3).

Elliptic, Parabolic and Hyperbolic Points:

If a point P on the surface this form is de nite $(i.e.;$ if $LN \quad M^2 > 0$), then $_n$ maintains the same sign for all directions at $P.$ In this case, the point P is called an elliptic point.

When $LN \quad M^2 = 0;$ then $_n$ retains the same sign for all directions through P except one for which the curvature is zero. Then the point P is called a parabolic point.

When $LN \quad M^2 < 0;$ $_n$ is positive for all directions lying within a certain angle, negative for directions lying outside this angle and zero along the directions which form the angle; then the point P is called a hyperbolic points and the critical directions are called asymptotic directions.

Geometrical Interpretation of the second fundamental form:

Let $P(u; v)$ and $Q(u + h; v + k)$ be near points on the surface and d be the perpendicular distance from a point Q onto the tangent plane to the surface at $P.$

If $\tilde{r}_P$ and $\tilde{r}_Q$ are the position vectors of P and $Q;$ then

$$\begin{aligned} d &= \tilde{r}_Q \quad \tilde{r}_P \quad \tilde{N} \\ &= h\tilde{r}_1 + k\tilde{r}_2 \quad \tilde{N} + \frac{1}{2} \; h^2\tilde{r}_{11} + 2hk\tilde{r}_{12} + k^2\tilde{r}_{22} \quad \tilde{N} + O \;\; h^3; k^3 \\ &= \frac{1}{2} \;\; Lh^2 + 2Mhk + Nk^2 \;\; + O \;\; h^3; k^3 \end{aligned}$$

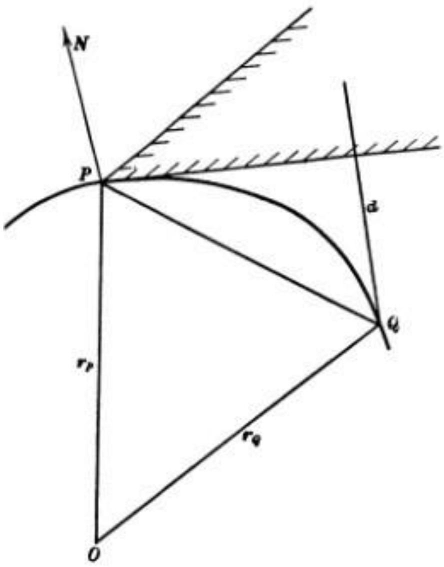


Figure 10.1

Thus the second fundamental form at any point P is equal to twice the length of the perpendicular distance from the neighbouring point Q onto the tangent plane at P.

At an elliptic point d retains the same sign, and this implies that the surface near P lies on entirely to one side of the tangent plane at P.

At a hyperbolic point the surface crosses over the tangent plane, it follows that at any point on an ellipsoidal surface is elliptic, any point on a circular cylinder is parabolic and any point on the hyperboloid is hyperbolic.

10.2. Principal curvatures:

The normal curvature at P in a direction speci ed by direction coe cients $(l; m)$ is given by

$$= Ll^2 + 2Mlm + Nm^2 \quad (10.9)$$

$$\text{where} \quad El^2 + 2Flm + Gm^2 = 1 \quad (10.10)$$

As $l; m$ vary subject to equation (10.10), the normal curvature will vary. Its extreme values may be found by using Lagrange's multipliers.

$$= \quad Ll^2 + 2Mlm + Nm^2 \qquad El^2 + 2Flm + Gm^2 \quad 1$$

then when is stationary,

$$\frac{1}{2}\frac{@}{@l} = Ll + Mm \quad El \quad Fm = 0 \qquad (10.11)$$

$$\frac{1}{2}\frac{@}{@m} = Ml + Nm \quad Fl \quad Gm = 0 \qquad (10.12)$$

Equation (10.11) l + (10.12) m; we get,

$$Ll^2 + 2Mlm + Nm^2 \qquad El^2 + 2Flm + Gm^2 = 0$$

$$= 0 \quad \text{using(10.9) and (10.10)}$$

$$) \quad =$$

Thus, the equations (10.11) and (10.12) will becomes

$$(L \quad E)\, l + (M \quad F)\, m = 0$$

$$(M \quad F)\, l + (N \quad G)\, m = 0$$

Eliminate l and m between these two equations, we get

$$\begin{matrix} L & E & M & F \\ M & & N\ G & \text{I} \end{matrix} = 0$$

On expanding the determinant, we get

$$^2 \quad EG \quad F^2 \qquad (EN + GL \quad 2FM) + \quad LN \quad M^2 = 0$$

This is a quadratic equation in having two roots say $_a$ and $_b$: These two roots are called the principal curvature.

Mean Curvature ( ): Mean curvature is de ned by

$$= \frac{1}{2}\left(_a + \ _b\right) = \frac{EN + GL \quad 2FM}{2\ EG \quad F^{\,2}}$$

Gaussian Curvature (K): The Gaussian curvature K of the surface at any point is de ned by

$$K = \ _a\ _b = \frac{LN\ M^2}{EG\ F^2}$$

Principal Directions:

The principal directions corresponding to principal curvatures are obtained by eliminating from equations (10.11) and (10.12), we get

$$(EM-FL)\,l^2+(EN-GL)\,lm+(FN-GM)\,m^2 = 0 \qquad (10.13)$$

The discriminant of this equation is

$$(EN-GL)^2-4\,(EM-FL)\,(FN-GM)$$

which is identically equal to

$$4\left(\frac{EG-F^2}{E^2}\right)(EM-FL)^2+\left[EN-GL-\frac{2F}{E}(EM-FL)\right]^2$$

We know $EG-F^2>0$ and if E, F, G and L, M, N are not proportional, then the above discriminant is positive and hence the roots of the equation are real and positive.

Umbilic: If E, F, G and L, M, N are proportional.

$$\text{i.e.,}\quad \frac{E}{L}=\frac{F}{M}=\frac{G}{N}$$

then the above discriminant has zero value and therefore the principal directions at the point are indeterminate and the normal curvatures has the same value in all directions. Such a point is called umbilic.

Note 10.2. If the point is not an umbilic, equation (10.13) gives two principal directions which are orthogonal.

If two directions given by $Pdu^2+2Qdudv+Rdv^2=0$ are orthogonal if and only if $ER-2FQ+GP=0.$

Now applying the above conditions in (10.13), we have

$$E\,(FN-GM)-2F\,\frac{(EN-GL)}{2}+G\,(EM-FL) = 0$$

Hence the two directions determined by equation (10.13) are orthogonal.

10.3. Lines of curvature:

Definition 10.1 (line of curvature). A curve on a surface $\tilde{r} = \tilde{r}(u, v)$ whose tangent at each point is along its principal direction is called a line of curvature.

Theorem 10.2 (Rodrigue's Formula). The necessary and sufficient condition that a curve on a surface be a line of curvature at each of its points is $\kappa\, d\tilde{r} + d\tilde{N} = 0$, where κ denotes the normal curvature.

Proof. **The condition is necessary:** Let the curve be a line on the surface $\tilde{r} = \tilde{r}(u, v)$. Now, we shall prove that $\kappa\, d\tilde{r} + d\tilde{N} = 0$.

The line of curvature are given by [(10.11), (10.12)]

$$\left.\begin{aligned} (L - \kappa E)\,du + (M - \kappa F)\,dv &= 0 \\ (M - \kappa F)\,du + (N - \kappa G)\,dv &= 0 \end{aligned}\right\} \qquad (10.14)$$

κ being one of the principal curvature.

Substituting the values of E, F, G, L, M, N by their expressions in terms of derivatives of $\tilde{r}$ and $\tilde{N}$, i.e.,

$$\begin{aligned} E &= \tilde{r}_1^{\,2};\ F = \tilde{r}_1 \cdot \tilde{r}_2;\ G = \tilde{r}_2^{\,2} \\ L &= -\tilde{N}_1 \cdot \tilde{r}_1;\ M = -\tilde{N}_2 \cdot \tilde{r}_1 = -\tilde{N}_1 \cdot \tilde{r}_2;\ N = -\tilde{N}_2 \cdot \tilde{r}_2 \end{aligned}$$

Thus, the equation (10.14) becomes

$$\left.\begin{aligned} \left(\kappa \tilde{r}_1^{\,2} + \tilde{N}_1 \cdot \tilde{r}_1\right) du + \left(\kappa \tilde{r}_1 \cdot \tilde{r}_2 + \tilde{N}_2 \cdot \tilde{r}_1\right) dv &= 0 \\ \text{and} \quad \left(\kappa \tilde{r}_1 \cdot \tilde{r}_2 + \tilde{N}_1 \cdot \tilde{r}_2\right) du + \left(\kappa \tilde{r}_2^{\,2} + \tilde{N}_2 \cdot \tilde{r}_2\right) dv &= 0 \end{aligned}\right\} \qquad (10.15)$$

$$\left.\begin{aligned} \text{i.e.,} \quad \left[\kappa\left(\tilde{r}_1 du + \tilde{r}_2 dv\right) + \left(\tilde{N}_1 du + \tilde{N}_2 dv\right)\right] \cdot \tilde{r}_1 &= 0 \\ \text{and} \quad \left[\kappa\left(\tilde{r}_1 du + \tilde{r}_2 dv\right) + \left(\tilde{N}_1 du + \tilde{N}_2 dv\right)\right] \cdot \tilde{r}_2 &= 0 \end{aligned}\right\} \qquad (10.16)$$

$$\left.\begin{aligned} \text{i.e.,} \quad \left(\kappa\, d\tilde{r} + d\tilde{N}\right) \cdot \tilde{r}_1 &= 0 \\ \text{and} \quad \left(\kappa\, d\tilde{r} + d\tilde{N}\right) \cdot \tilde{r}_2 &= 0 \end{aligned}\right\} \qquad (10.17)$$

Since the vector $d\tilde{N} + \kappa\, d\tilde{r}$ is tangential to the surface, therefore in order to satisfy the equations (10.17), we must have

$$d\tilde{N} + \kappa\, d\tilde{r} = 0$$

The condition is su cient: Assume that the relation $d\tilde{N} + \ d\tilde{r} = 0$ holds along a curve for any function ; then equations (10.14) follows and thus curve is a line of curvature.

Note 10.3. The necessary and su cient condition for the lines of curvature to be parametric curves is $F = 0; M = 0:$

Proof. **The condition is necessary:** Let the equation of curve be $\tilde{r} = \tilde{r}(u; v)$ The di erential equation of line of curvature is

$$(EM \quad FL)\, l^2 + (EN \quad GL)\, lm + (FN \quad GM)\, m^2 = 0 \qquad (10.18)$$

I f the line of curvature be taken as parametric curves, then $F = 0;$ since the principal directions are orthogonal.

Again $u =$ constant and $v =$ constant are the equations of parametric curves and therefore combined di erential equation must reduce to

$$lm = 0 \qquad i:e:; dudv = 0 \qquad (10.19)$$

In order that the line of curvatures are parametric curves equation (10.18) and (10.19) are equivalent. Hence $M = 0:$

Therefore, $F = 0; M = 0$ are necessary condition for the lines of curvature to be parametric curves.

The condition is su cient: Assume that $F = 0; M = 0$ then the equation of line of curvature (10.18) becomes

$$(EN \quad Gl)\, lm = 0 \Rightarrow EN \quad Gl = 0 \quad \text{or } lm = 0$$

$$\text{But } EN \quad Gl \neq 0$$

$$* \; EN \neq Gl \Rightarrow \frac{E}{l} \quad \frac{G}{N} \quad \text{condition for umbilic point}$$

$$\Rightarrow lm = 0 \quad i:e:; \quad dudv = 0$$

which gives $u =$ constant and $v =$ constant.

This is the di erential equation for parametric curves.

Theorem 10.3 (Euler's Theorem). If is the normal curvature in a direction $(l; m)$ making an angle with the principal direction $v =$ constant then $= \ _a \cos^2 \ + \ _b \sin^2$ where $_a$ and $_b$ are the principal curvatures at that point.

Proof. Consider the line of curvatures as the parametric curves. Then we have $F = 0; \ M = 0$ and hence the normal curvature in a direction (l, m) is

$$= Ll^2 + Nm^2$$

The direction coe cients of the parametric curves $v =$ constant and $u =$ constant are $\left(\frac{1}{\sqrt{E}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{G}}\right)$.

$$\begin{aligned}
{}_a &= \text{normal curvature along } v = \text{constant} \\
&= L\left(\frac{1}{E}\right) + N(0) = \frac{L}{E} \\
\text{and } {}_b &= \text{normal curvature along } u = \text{constant} \\
&= L(0) + N\left(\frac{1}{E}\right) = \frac{N}{G} \\
\text{i.e., } {}_a &= \frac{L}{E}; \quad {}_b = \frac{N}{G}
\end{aligned}$$

Now is the angle between the direction (l, m) and the principal direction $v =$ constant.

$$\cos = Ell_0 + F(lm_0 + l_0m) + Gmm_0$$

$$\begin{aligned}
\Rightarrow \cos &= E\left(l\frac{1}{\sqrt{E}}\right) + 0 + G(m)(0) = l\sqrt{E} \\
\text{and } \cos(90 \ ) &= E(l)(0) + G(m)\left(\frac{1}{\sqrt{G}}\right) \\
\text{i.e., } \sin &= m\sqrt{G} \\
\text{Thus, } &= Ll^2 + Nm^2 \\
&= L\left(\frac{\cos}{\sqrt{E}}\right)^2 + N\left(\frac{\sin}{\sqrt{G}}\right)^2 \\
&= \frac{L}{E}\cos^2 + \frac{N}{G}\sin^2 \\
&= {}_a\cos^2 + {}_b\sin^2
\end{aligned}$$

10.3.1. Dupin indicatrix:

The section of a surface by a plane parallel to the tangent plane at any point O on it and at a small distance from it is called Dupin indicatrix at O.

Let P be a point on the Dupin indicatrix at O and let h be the perpendicular distance of P from the tangent plane at O. Then from the Geometrical interpretation of the second fundamental form

$$2h = Ldu^2 + 2Mdudv + Ndv^2 \qquad (10.20)$$

neglecting higher order in nitesimals.

If we choose the line of curvatures as the parametric curves the n $F = 0$ and $M = 0$ so that the equation (10.20) reduces to

$$2h = Ldu^2 + Ndv^2$$

Also, the principal curvatures $_a$ and $_b$ are given by

$$_a = \frac{L}{E} \quad \text{and} \quad _b = \frac{N}{G}$$

$$) \quad 2h = {}_aEdu^2 + {}_bGdv^2$$

Also, the metric along the parametric curves are

$$ds_1^2 = Edu^2; \quad ds_2^2 = Gdv^2$$

$$\text{Thus;} \quad 2h = {}_ads_1^2 + {}_bds_2^2$$

Choose O as origin, OX and OY along the principal directions at O and OZ along the normal to the surface at O:

If the coordinates of the point P on the Dupin indicatrix be $(x; y; z)$ then $x = ds_1;\ y = ds_2;\ z = h$:

Hence the equation to the Dupin indicatrix are $x^2 {}_a + y^2 {}_b = 2h; z = h$: (or) $\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h; z = h$: where $R_a = \frac{1}{_a}$; $R_b = \frac{1}{_b}$:

Thus Dupin indicatrix is a conic section.

Note 10.4. Three cases arise according to the sign of $_a; \ _b$:

Case 1: If $_a$ and $_b$ have the same sign, then Gaussian curvature is positive (i:e:;) $K = {}_a {}_b$; then the points on the surface are called elliptic points.

Case 2: If $_a$ and $_b$ have di erent sign, the indicatrix is one of the two conjugate hyperbolic. The points on the surface $_a; \ _b$ have opposite signs (i:e:;) $K = {}_a \ {}_b < 0$ are called hyperbolic points.

Case 3: If one of $_a$ and $_b$ is zero then $K = 0$; then the indicatrix is a point of straight lines. The points are called parabolic points.

De nition 10.2 (Conjugate Directions). Two directions at P are said to be conjugate if the corresponding diameters of the Dupin Indicatrix are conjugate.

De nition 10.3 (Asymptotic line). An asymptotic line is a curve whose direction at every point is asymptotic. The equation of such a line $\frac{d\tilde{r}}{ds}\cdot\frac{d\tilde{N}}{ds} = 0$ i.e., $Ldu^2 + 2Mdudv + Ndv^2 = 0$ from which it follows that asymptotic lines are self-conjugate.

Example 10.1. Show that Gaussian curvature of the surface given by the Monge's form $z = (x, y)$ is $(rt - s^2)/(1 + p^2 + q^2)^2$.

Solution: The equation of the surface is given by $z = f(x, y)$.

$$p = \frac{\partial z}{\partial x};\quad q = \frac{\partial z}{\partial y};\quad r = \frac{\partial^2 z}{\partial x^2};\quad s = \frac{\partial^2 z}{\partial x \partial y};\quad t = \frac{\partial^2 z}{\partial y^2}$$

If x, y be taken as parameters, then the position vector $\tilde{r}$ of any point on the given surface is given by

$$\begin{aligned}
\tilde{r} &= x\tilde{i} + y\tilde{j} + z\tilde{k} = x\tilde{i} + y\tilde{j} + f(x, y)\tilde{k}\\
\tilde{r}_1 &= \tilde{i} + p\tilde{k};\quad \tilde{r}_2 = \tilde{j} + q\tilde{k}\\
\tilde{r}_{11} &= \frac{\partial^2 \tilde{r}}{\partial x^2} = r\tilde{k};\quad \tilde{r}_{12} = \frac{\partial^2 \tilde{r}}{\partial x \partial y} = s\tilde{k};\quad \tilde{r}_{22} = \frac{\partial^2 \tilde{r}}{\partial y^2} = t\tilde{k}\\
E &= \tilde{r}_1 \cdot \tilde{r}_1 = 1 + p^2;\quad F = \tilde{r}_1 \cdot \tilde{r}_2 = pq;\quad G = \tilde{r}_2 \cdot \tilde{r}_2 = 1 + q^2\\
H^2 &= EG - F^2 = 1 + p^2 + q^2\\
\tilde{N} &= \frac{\tilde{r}_1 \times \tilde{r}_2}{|\tilde{r}_1 \times \tilde{r}_2|} = \frac{(\tilde{i} + p\tilde{k}) \times (\tilde{j} + q\tilde{k})}{\sqrt{1 + p^2 + q^2}} = \frac{-p\tilde{i} - q\tilde{j} + \tilde{k}}{\sqrt{1 + p^2 + q^2}}\\
L &= \tilde{N} \cdot \tilde{r}_{11} = \frac{-p\tilde{i} - q\tilde{j} + \tilde{k}}{\sqrt{1 + p^2 + q^2}} \cdot r\tilde{k} = \frac{r}{\sqrt{1 + p^2 + q^2}}\\
M &= \tilde{N} \cdot \tilde{r}_{12} = \frac{-p\tilde{i} - q\tilde{j} + \tilde{k}}{\sqrt{1 + p^2 + q^2}} \cdot s\tilde{k} = \frac{s}{\sqrt{1 + p^2 + q^2}}\\
N &= \tilde{N} \cdot \tilde{r}_{22} = \frac{t}{\sqrt{1 + p^2 + q^2}}
\end{aligned}$$

Thus, the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

Example 10.2. Obtain the differential equation of the lines of curvature on the surface $z = f(x, y)$ and deduce that at an umbilic $\frac{1 + p^2}{r} = \frac{1 + q^2}{t} = \frac{pq}{s}$.

Solution: From the example 10.1, we have

$$E = 1 + p^2; \quad F = pq; \quad G = 1 + q^2$$

$$L = \frac{r}{\sqrt{1 + p^2 + q^2}}$$

$$M = \frac{s}{\sqrt{1 + p^2 + q^2}}$$

$$N = \frac{t}{\sqrt{1 + p^2 + q^2}}$$

The differential equation of the lines of curvatures is

$$(EM - FL)\, du^2 + (EN - GL)\, dudv + (FN - GM)\, dv^2 = 0$$

$$i.e., \left[\left(1 + p^2\right)s - pqr\right] dx^2 + \left[\left(1 + p^2\right)t - \left(1 + q^2\right)r\right] dxdy + \left[pqt - \left(1 + q^2\right)s\right] dy^2 = 0$$

At an umbilic, we have $\frac{E}{L} = \frac{F}{M} = \frac{G}{N}$

$$i.e., \quad \frac{1 + p^2}{t} = \frac{pq}{s} = \frac{1 + q^2}{t}$$

Let Us Sum Up:

In this unit, the students acquired knowledge to

- derive Rodrigue's formula.
- the concept of Umbilic .
- the concept of Dupin indicatrix.

Check Your Progress:

1. Derive the second fundamental form.
2. Define elliptic points and hyperbolic points..
3. State and Prove Rodrigue's Theorem.
4. Derive the equation Dupin's Indicatrix.
5. Define Mean Curvature.

Choose the correct or more suitable answer:

1. The Gaussain curvaitre K of the surface at any point is de ned by : : : : : :

   (a) $K = \frac{LN + M^2}{EG \quad F^2}$.

   (b) $K = \frac{LN \quad M^2}{EG + F^2}$.

   (c) $K = \frac{LN \quad M^2}{EG \quad F^2}$.

   (d) $K = \frac{LN + M^2}{EG + F^2}$.

2. The necessary and su cient condition for the lines of curvature to be parametric curves is.

   (a) $F = 0; M \neq 0$ (b) $F \quad 0; M = 0$

   (c) $F \neq 0; M \neq 0$ (d) $F = 0; M = 0$

Answer:

(1) c (2) d

Glossaries:

Dupin Indicatrix: In di erential geometry, the Dupin indicatrix is a method for characterising the local shape of a surface.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-IV

UNIT-11

DEVELOPABLE SURFACES-I

Structure

Objective

Overview

11. 1 Developables

Let us Sum Up

Check Your Progress

Suggested Readings

Objectives

After completion of this unit, students will be able to

- understand the concept of developable surfaces.
- understand the concept of characteristic line and characteristic point.

Overview

In this unit, we explained the concept of Edge of regression.

11.1. Developables:

Definition 11.1 (Developable surface).

A developable is a surface enveloped by a one parameter family of planes.

$$\tilde{r}\ \tilde{a} = p$$

where $\tilde{a}$ and p are functions of a real parameter $u.$

Definition 11.2 (Characteristic line).

As we are familiarising with the concept of two planes intersect along a straight line. Based on this idea, now we are going to define the characteristic lines.

If $f(u) = \tilde{r}\ \tilde{a}(u) \quad p(u);$ the equation of these lines are $f(u) = 0$ and $f(v) = 0.$ From Rolle's theorem, it follows that there is a value $u_1 \quad u < u_1 < v$ such that $f(u_1) = 0.$

As $v \,!\, u; \ u_1 \,!\, u$ and the equations of the limiting position of the line becomes

$$\left.\begin{aligned} \tilde{r}\ \tilde{a} &= p \\ \tilde{r}\ \tilde{a} &= p \end{aligned}\right\} \tag{11.1}$$

This line is called the characteristic line corresponding to the plane $u.$

Definition 11.3 (Characteristic point).

The ultimate intersection of consecutive characteristic lines is called a characteristic point. The characteristic point is obtained from the equations.

$$\left.\begin{aligned} \tilde{r}\ \tilde{a} &= p \\ \tilde{r}\ \tilde{a} &= p \\ \tilde{r}\ \tilde{a} &= p \end{aligned}\right\}$$

If $\tilde{a};\ \tilde{a};\ \tilde{a}$ are linearly dependent, these equations have no solution or else the solution is indeterminate.

Note 11.1. The above definition can be restated as the ultimate intersection of three consecutive planes is called the characteristic point. The limiting position of this point $v \,!\, u$ and $w \,!\, u$ independently is called the characteristic point corresponding to $u.$ By Rolle's theorem, the equations which determine the characteristic points are

$$\begin{aligned} \tilde{r}\ \tilde{a} &= p \\ \tilde{r}\ \tilde{a} &= p \\ \tilde{r}\ \tilde{a} &= p \end{aligned} \qquad (11.2)$$

De nition 11.4 (Edge of Regression).

The locus of ultimate intersection of consecutive characteristic lines are called the edge of regression which is a curve lying on the developable.

In other words, the edge of regression is the locus of the characteristic point. It is given by equations (11.2) with $\tilde{r}$ regarded as a function of u:

Bookwork 11.1. Show that the tangents to the edge of regression are the characteristic lines.

Proof. The edge of regression is given by

$$\tilde{r}\ \tilde{a} = p \qquad (11.3)$$

$$\tilde{r}\ \tilde{a} = p \qquad (11.4)$$

$$\tilde{r}\ \tilde{a} = p \qquad (11.5)$$

where $\tilde{r}; \tilde{a}; p$ are all functions of the parameter u:

Now, di erentiating (11.3) and (11.4) with respect to the parameter u; we get

$$s\tilde{t}\ \tilde{a} + \tilde{r}\ \tilde{a} = p \qquad (11.6)$$

$$s\tilde{t}\ \tilde{a} + \tilde{r}\ \tilde{a} = p \qquad (11.7)$$

Using equation (11.4) in (11.6), we get

$$\tilde{t}\ \tilde{a} = 0 \qquad (11.8)$$

Similarly using equation (11.5) in (11.7), we get

$$\tilde{t}\ \tilde{a} = 0 \qquad (11.9)$$

Thus, the equation (11.8) and (11.9) show that the tangent to the edge of the regression is perpendicular to both $\tilde{a}$ and $\tilde{a}$ and hence it is parallel to $\tilde{a}\ \ \tilde{a}$:

But the characteristic line through the point is also parallel to $\tilde{a}\ \ \tilde{a}$: Thus, we have that the tangent to the edge of regression is the characteristic line.

Bookwork 11.2. Prove that the osculating plane of the edge of regression at any point is the tangent plane to the developable at that point.

Proof. Differentiating equation (11.8) with respect to the parameter u; we get

$$\frac{d\tilde{t}}{ds}\frac{ds'}{du}\,\tilde{a}+\tilde{t}\,\tilde{a} = 0$$

$$\text{i.e.,}\quad s\,\tilde{n}\,\tilde{a}+\tilde{t}\,\tilde{a} = 0 \quad \text{(using Serret Frenet formule)}$$

$$\Rightarrow s\,\tilde{n}\,\tilde{a} = 0 \quad \text{[using (11.9)]}$$

$$\text{i.e.,}\quad \tilde{n}\,\tilde{a} = 0 \tag{11.10}$$

From equations (11.8) and (11.10), we see that $\tilde{a}$ is perpendicular to both $\tilde{t}$ and $\tilde{n}$ and hence it is parallel to the binormal of the edge of regression.

Thus, we conclude that the osculating plane of the edge of regression at any point is the tangent plane to the developable at that point.

Bookwork 11.3. Prove that the two sheets of the developable are tangent to the edge of regression along a sharp edge.

Proof. Let O be the point $s = 0$ on the edge of regression C and let $Ox; Oy; Oz$ be a set of rectangular Cartesian axes chosen respectively along $\tilde{t}; \tilde{n}$ and $\tilde{b}$ at O: Then at any point on the developable has position vector given by

$$\tilde{R} = \tilde{r} + v\tilde{t}$$

Expanding $\tilde{R}$ in terms of s; we get

$$\tilde{R} = \tilde{r} + v\tilde{t}(s) = s\tilde{t} + \frac{1}{2}s^2\,\tilde{n} + \frac{s^3}{6}\left({}^{0}\tilde{n} + \tilde{b}\;{}^{2}\tilde{t}\right) + O\left(s^4\right)$$

$$+v\left(\tilde{t} + s\,\tilde{n} + \frac{1}{2}s^2\left({}^{0}\tilde{n} + \tilde{b}\;{}^{2}\tilde{t}\right) + O\left(s^3\right)\right)$$

The normal plane $x = 0$ meets the surface where

$$v = s\;\frac{1}{3}\,s^3\,{}^2 + O\left(s^4\right)$$

Using this in the above expansion, we get

$$y = \frac{1}{2}s^2 + O\left(s^3\right)$$

$$z = \frac{1}{3}s^3 + O\left(s^4\right)$$

Upon eliminating s; we get

$$z^2 = \frac{8}{9}\,{}^{2}\,y^3$$

from which, we say that the developable cuts the normal plane to the edge of regression in a cusp whose tangent is along the principal normal. Thus, the two sheets of the developable are tangents to the edge of regression along a sharp edge.

Let Us Sum Up:

In this unit, the students acquired knowledge to

Characteristic lines and Characteristic points.

Edge of regression.

Check Your Progress:

1. De ne developable surface.
2. De ne Edge of regression.
3. De ne Characteristic line and Characteristic point.
4. Show that the tangents to the edge of regression are the characteristic lines.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).
2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-IV

UNIT-12

DEVELOPABLE SURFACES-II

Structure

Objective

Overview

12. 1 Developables associated with space curves

12. 1. 1 Osculating developable

12. 1. 2 Polar developable

12. 2 Rectifying developable

12. 3 Developables associated with curves on surfaces

12. 4 Minimal surfaces

12. 5 Ruled surfaces

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Glossaries

Suggested Readings

Overview

In this unit, we will illustrate the concept of minimal surface and ruled surface.

Objectives

After completion of this unit, students will be able to

- know to explain the concept of osculating developable.
- know to explain the concept of rectifying developable.

12.1. Developable associated with space curves:

At each point of a curve we have three planes, namely osulating plane, normal plane and the rectifying plane. Each of these planes contains only one parameter i:e:; the arc lengths. The envelope of these planes are respectively called , osculating developable, polar developable and rectifying developable.

12.1.1. Osculating developable:

The family of osculating plane of a space curve is osculating developable. Its characteristic lines are the tangents to the curve and hence this developable is also referred to as the tangential developable.

Bookwork 12.1. Prove that the edge of the regression of the osculating developable is the curve itself.

Proof. Consider the osculating plane at any point P with position vector $\tilde{r}$ on a space curve $\tilde{r} = \tilde{r}(s)$:

If $\tilde{R}$ is the position vector of any point on the osculating plane, then $\tilde{R} - \tilde{r}$ lies in the osculating plane. Hence the family of osculating plane has equation

$$\left(\tilde{R} - \tilde{r}(s)\right) \cdot \tilde{b}(s) = 0 \qquad (12.1)$$

Differentiating both sides with respect to arc length s, we get

$$\tilde{r}' \cdot \tilde{b} + \left(\tilde{R} - \tilde{r}(s)\right) \cdot \tilde{b}' = 0$$

$$\text{i.e.,} \quad \tilde{t} \cdot \tilde{b} + \left(\tilde{R} - \tilde{r}(s)\right) \cdot (-\tau \tilde{n}) = 0$$

$$\text{i.e.,} \quad \left(\tilde{R} - \tilde{r}(s)\right) \cdot \tilde{n} = 0 \qquad (12.2)$$

The characteristic lines are the intersection of equations (12.1) and (12.2), which represent the osculating plane and rectifying plane respectively and hence their intersection is the tangent to the curve at $P(\tilde{r})$.

Differentiating both sides of (12.2) with respect to s, we get

$$\tilde{t} \cdot \tilde{n} + \left(\tilde{R} - \tilde{r}(s)\right) \cdot \left(-\kappa \tilde{t} + \tau \tilde{b}\right) = 0$$

$$\text{i.e.,} \quad \left(\tilde{R} - \tilde{r}(s)\right) \cdot \tilde{t} = 0 \quad \text{(using (12.1))} \qquad (12.3)$$

Thus, from (12.1), (12.2) and (12.3) we have

$$\tilde{R} - \tilde{r} = 0$$

$$\Rightarrow \tilde{R} = \tilde{r}$$

Thus the characteristic point which is the intersection of (12.1), (12.2) and (12.3) is $P(\tilde{r})$ itself. The edge of regression which is the locus of the characteristic point is therefore the curve itself.

12.1.2. Polar developable:

This is the surface enveloped by the normal plane of a space curve.

Bookwork 12.2. Show that the edge of regression of the polar developable is the locus of centres of spherical curvature of the given curve.

Proof. The equation of normal plane at $\tilde{r} = \tilde{r}(s)$ is

$$\left(\tilde{R} - \tilde{r}(s)\right) \cdot \tilde{t} = 0 \qquad (12.4)$$

Differentiating both sides of equation (12.4) with respect to s, we get

$$-\tilde{t} \cdot \tilde{t} + \left(\tilde{R} - \tilde{r}\right) \cdot \kappa \tilde{n} = 0$$

$$\Rightarrow \left(\tilde{R} - \tilde{r}\right) \cdot \tilde{n} = \frac{1}{\kappa} = \rho \qquad (12.5)$$

Di erentiating equation (12.5) with respect to s; we get

$$\tilde{t}\ \tilde{n} + \tilde{R}\quad \tilde{r}\qquad \tilde{t} + \tilde{b} = {}^{o}$$

$$\Rightarrow 0\quad \tilde{R}\quad \tilde{r}\quad \tilde{t} + \tilde{R}\quad \tilde{r}\ ,\ \tilde{b} = {}^{o}$$

$$\Rightarrow \tilde{R}\quad \tilde{r}\quad \tilde{b} = \frac{1}{\ }\ {}^{o} = {}^{o} \qquad \text{(using (12.4))} \qquad (12.6)$$

From equations (12.4), (12.5) and (12.6) we nd that the characteristic point is $\tilde{R} = \tilde{r} + \ \tilde{n} + \ {}^{o}\tilde{b}$:

But this is the centre of osculating sphere. Thus the edge of regression of the polar developable is the locus of spherical curvature of the given curve.

12.2. Rectifying developable:

The rectifying developable of a space curve of the rectifying planes of the space curve.

Bookwork 12.3.

Show that the edge of regression of the rectifying developable has equation $\tilde{R} \cong r + \frac{\ \tilde{t} + \ \tilde{b}}{{}^{o}\quad{}^{o}}$:

Proof. The position vector $\tilde{R}$ of any point on the rectifying developable is given by

$$\tilde{R}\quad \tilde{r}\quad \tilde{n} = 0 \qquad (12.7)$$

Di erentiate (12.7) with respect to s; we get

$$\tilde{t}\ \tilde{n} + \tilde{R}\quad \tilde{r}\qquad \tilde{t} + \tilde{b} = 0$$

$$\Rightarrow \tilde{R}\quad \tilde{r}\qquad \tilde{t} + \tilde{b} = 0 \qquad (12.8)$$

Di erentiating (12.8) with respect to s; we get

$$\tilde{t}\qquad \tilde{t} + \tilde{b} + \tilde{R}\quad \tilde{r}\qquad {}^{2}\tilde{n}\quad {}^{o}\tilde{t}\quad {}^{2}\tilde{n} + {}^{o}\tilde{b} = 0$$

$$\Rightarrow \quad + \tilde{R}\quad \tilde{r}\qquad {}^{o}\tilde{t} + {}^{o}\tilde{b} = 0 \qquad (12.9)$$

The point of intersection of the planes (12.7), (12.8) and (12.9) is the characteristic point whose locus is the edge of regression.

From (12.7) and (12.8) we see that $\tilde{R} \quad \tilde{r}$ is perpendicular to both $\tilde{n}$ and $\tilde{t} + \quad \tilde{b}$ and hence it is parallel to $\tilde{n} \quad\quad \tilde{t} + \quad \tilde{b} \quad i.e.; \quad \tilde{b} + \quad \tilde{t}.$

Thus, we can write it as

$$\tilde{R} \quad \tilde{r} \quad = \quad\quad \tilde{t} + \quad \tilde{b} \tag{12.10}$$

Now, our wish is to nd the value of :

For this, using the equation (12.10) in (12.9), we get

$$+ \quad\quad \tilde{t} + \quad \tilde{b} \quad\quad {}^{0}\tilde{t} + \quad {}^{0} \quad = \quad 0$$

$$i.e.; \quad\quad + \quad {}^{0} \quad\quad + \quad {}^{0} \quad\quad = \quad 0 \quad \Rightarrow \quad \overline{\quad {}_{0} = \quad\quad {}_{0}}$$

Hence, the equation to the edge of regression of the rectifying developable is given by $\tilde{R} = \tilde{r} + \frac{\tilde{t} + \quad \tilde{b}}{{}_{0} \quad\quad {}_{0}}.$

Bookwork 12.4. A necessary and su cient condition for a surface to be developable is that its Gaussian curvature shall be zero.

Proof. If the developable is a cylinder or cone, then evidently the Gaussian curvature is zero. If we excluded these cases, the developable may be regarded as the osculating developable of its edge of regression and its equation may be written as $\tilde{R} = \tilde{r}(s) + v\tilde{t}(s).$

Di erentiation with respect to the parameters s and v are denoted by su xes 1 and 2 respectively. Then, we have

$$\tilde{R}_1 \quad = \quad \tilde{t} + v \quad \tilde{n}$$

$$\tilde{R}_2 \quad = \quad \tilde{n}$$

$$\tilde{R}_{11} \quad = \quad \tilde{n} + v \ {}^{0}\tilde{n} + v \quad\quad \tilde{t} + \quad \tilde{b}$$

$$\tilde{R}_{12} \quad = \quad \tilde{0}$$

$$\tilde{N} \quad = \quad \frac{\tilde{R}_1 \quad \tilde{R}_2}{\tilde{R}_1 \quad \tilde{R}_2} \quad = \quad \frac{v \ \tilde{b}}{v} \quad = \quad \tilde{b}$$

$$L \quad = \quad \tilde{N} \quad \tilde{R}_{11} = \quad v^2 \quad = \quad v$$

$$M \quad = \quad \tilde{N} \quad \tilde{R}_{12} = 0; \quad N = \tilde{N} \quad \tilde{R}_{22} = 0$$

Thus, the Gaussian curvature $K \quad = \quad \dfrac{LN \quad M^2}{EG \quad F^2} = 0$

Hence $K = 0$ is the necessary condition for a surface to be developable.

Now, it remains to prove the sufficient part. Let $K = 0$ for a surface $\tilde{r} = \tilde{r}(u, v)$.

Hence $LN - M^2 = 0$

Since $L = -\tilde{r}_1 \cdot \tilde{N}_1$; $M = -\tilde{r}_1 \cdot \tilde{N}_2$; $N = -\tilde{r}_2 \cdot \tilde{N}_2$.

We obtain

$$
\begin{aligned}
LN - M^2 &= (\tilde{r}_1 \cdot \tilde{N}_1)(\tilde{r}_2 \cdot \tilde{N}_2) - (\tilde{r}_1 \cdot \tilde{N}_2)(\tilde{r}_2 \cdot \tilde{N}_1) \\
&= \begin{vmatrix} \tilde{r}_1 \cdot \tilde{N}_1 & \tilde{r}_1 \cdot \tilde{N}_2 \\ \tilde{r}_2 \cdot \tilde{N}_1 & \tilde{r}_2 \cdot \tilde{N}_2 \end{vmatrix} \\
&= (\tilde{r}_1 \times \tilde{r}_2) \cdot (\tilde{N}_1 \times \tilde{N}_2) = H\tilde{N} \cdot (\tilde{N}_1 \times \tilde{N}_2) = H\left[\tilde{N}, \tilde{N}_1, \tilde{N}_2\right] \\
K &= 0 \Rightarrow LN - M^2 = 0 \\
\Rightarrow \left[\tilde{N}, \tilde{N}_1, \tilde{N}_2\right] &= 0 \\
&\Rightarrow \tilde{N}, \tilde{N}_1, \tilde{N}_2 \text{ are coplanar.}
\end{aligned}
$$

Since $\tilde{N} \cdot \tilde{N} = 1$. Differentiating with respect to u and v, we get $\tilde{N} \cdot \tilde{N}_1 = 0$; $\tilde{N} \cdot \tilde{N}_2 = 0$.

Thus $\tilde{N}$ is perpendicular to both $\tilde{N}_1$ and $\tilde{N}_2$.

If $\tilde{N}_1$ and $\tilde{N}_2$ are non-zero vectors then three vectors cannot be coplanar unless $\tilde{N}_1$ and $\tilde{N}_2$ are parallel.

Thus, we have the following three possibilities:

(i) $\tilde{N}_2 = 0$; (ii) $\tilde{N}_1 = 0$; $\tilde{N}_1 = \lambda \tilde{N}_2$.

Case (i): $\tilde{N}_2 = 0$. The equation to the tangent plane at a point on the surface is $(\tilde{R} - \tilde{r}) \cdot \tilde{N} = 0$.

Now, $\frac{\partial}{\partial v}\left[(\tilde{R} - \tilde{r}) \cdot \tilde{N}\right] = (\tilde{R} - \tilde{r}) \cdot \tilde{N}_2 - \tilde{r}_2 \cdot \tilde{N} = 0 \quad * \; \tilde{N}_2 = 0$ and $\tilde{r}_2 \cdot \tilde{N} = 0$ ($\tilde{r}_2$ being a vector in the tangent plane).

Thus $(\tilde{R} - \tilde{r}) \cdot \tilde{N}$ is independent of v and therefore we find that the equation to the tangent plane contains only one parameter u. Hence the surface is the envelope of a one-parameter family (i.e., a developable.

Case (ii): As in the previous case, the tangent plane will contain only one parameter v and hence the surface will be developable.

Case (iii): $\tilde{N}_1 = \lambda \tilde{N}_2$. Transform the parameters u, v to u', v' by the transformation $u = u' + v'$, $v' = u' - \lambda v'$, we obtain

$$\tilde{N}_1' = \frac{\partial \tilde{N}}{\partial u'} = \frac{\partial \tilde{N}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \tilde{N}}{\partial v}\frac{\partial v}{\partial u'} = \tilde{N}_1 + \tilde{N}_2$$

$$\tilde{N}_2' = \frac{\partial \tilde{N}}{\partial v'} = \frac{\partial \tilde{N}}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial \tilde{N}}{\partial v}\frac{\partial v}{\partial v'} = \tilde{N}_1 \qquad \tilde{N}_2 = 0$$

This shows that the surface normal $\tilde{N}_2$ is independent of v and hence depends on only one parameter.

Thus, the surface is developable.

12.3. Developables associated with curves on surfaces:

The following theorem due to Monge characterise lines of curvature on a surface.

Theorem 12.1 (Monge's theorem).

A necessary and su cient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Proof. Consider the surface formed by the normals along the curve $\tilde{r} = \tilde{r}(s)$: Any point on this surface will have the position vector

$$\tilde{R} = \tilde{r}(s) + v\tilde{N}(s) \tag{12.11}$$

Di erentiation with respect to s and v are denoted by su x 1 and 2 respectively. Thus, we have

$$\begin{aligned}
\tilde{R}_1 &= \tilde{t} + v\tilde{N}' \\
\tilde{R}_2 &= \tilde{N} \\
\tilde{R}_{11} &= \tilde{t}' + v\tilde{N}'' \\
\tilde{R}_{12} &= \tilde{R}_{21} = \tilde{N}' \\
\tilde{R}_{22} &= 0
\end{aligned}$$

Surface normal $\tilde{N} = \dfrac{\tilde{R}_1 \times \tilde{R}_2}{|\tilde{R}_1 \times \tilde{R}_2|} = \dfrac{\tilde{t} \times \tilde{N} + v\,\tilde{N}' \times \tilde{N}}{H}$ $\quad * \; H = |\tilde{R}_1 \times \tilde{R}_2|$

$$\text{Thus,}\quad M = \tilde{N}\cdot\tilde{R}_{12} = \frac{1}{H}\left[\tilde{t}\times\tilde{N} + v\,\tilde{N}\times\tilde{N}'\right]\cdot\tilde{N}'$$

$$= \frac{1}{H}\left[\tilde{t},\ \tilde{N},\ \tilde{N}'\right]$$

$$\text{Also,}\quad N = \tilde{N}\cdot\tilde{R}_{22} = 0$$

Hence the Gaussian curvature $K = \dfrac{LN - M^2}{EG - F^2} = 0$ of the surface will be zero if and only if $LN - M^2 = 0$ i.e., $M = 0$, if and only if $\left[\tilde{t},\ \tilde{N},\ \tilde{N}'\right] = 0$.

According to the previous theorem, the surface normals along the curve form a developable if and only if $\left[\tilde{t},\ \tilde{N},\ \tilde{N}'\right] = 0$.

Now, our wish is to prove that this condition is satisfied if and only if the curve is a line of curvature.

Since $\tilde{t}\times\tilde{N}'$ is normal to the given surface, the equations $\left[\tilde{t},\ \tilde{N},\ \tilde{N}'\right] = 0$ implies that $\tilde{t}\times\tilde{N}' = 0$.

$$\begin{aligned} \text{i.e.,}\quad \tilde{N}' &= k\tilde{t} \quad \text{for some function } k \\ \frac{d\tilde{N}}{ds} &= k\frac{d\tilde{r}}{ds} \\ \Rightarrow\quad d\tilde{N} + k\,d\tilde{r} &= 0 \end{aligned}$$

Hence, by Rodrigue's formula, the curve is a line of curvature.

$$\begin{aligned} \text{Conversely, if}\quad d\tilde{N} + k\,d\tilde{r} &= 0 \\ \text{i.e.,}\quad \frac{d\tilde{N}}{ds} &= k\frac{d\tilde{r}}{ds} \\ \text{i.e.,}\quad \tilde{N}' &= k\tilde{t} \\ \Rightarrow\quad \left[\tilde{t},\ \tilde{N},\ \tilde{N}'\right] &= 0 \end{aligned}$$

This completes the proof of the theorem.

Note 12.1. Now we obtain an alternative interpretations of the conjugate diameters defined in section (10.3).

Theorem 12.2. Let C be a curve lying on a surface and let P be any point on C. Then the characteristic line at P of the tangential developable of C is in the direction conjugate to that of the tangent to C at P.

Proof. The tangent planes at points on a curve C lying on a surface form a developable, and now we prove that the characteristic line of the developable at any point P on C is in a direction conjugate to that of the tangent to C at P.

The equation of family of tangent planes is

$$\tilde{R} \quad \tilde{r} \quad \tilde{N} = 0 \qquad (12.12)$$

Di erentiating equation (12.12), we get

$$\tilde{t} \quad \tilde{N} + \tilde{R} \quad \tilde{r} \frac{d\tilde{N}}{ds} = 0$$

$$\text{i.e.,} \quad \tilde{R} \quad \tilde{r} \frac{d\tilde{N}}{ds} = 0$$

$$\text{i.e.,} \quad \tilde{R} \quad \tilde{r} \quad \tilde{N}_1 u' + \tilde{N}_2' v' = 0 \qquad (12.13)$$

The characteristic lines is the intersection of equations (12.12) and (12.13).

If (l, m) are the direction coe cients of the characteristic line at a point P, then

$$\tilde{R} \quad \tilde{r} = l\tilde{r}_1 + m\tilde{r}_2 \qquad (12.14)$$

Using equation (12.15) in equation (12.13), we get

$$l\tilde{r}_1 + m\tilde{r}_2 \quad \tilde{N}_1 u' + \tilde{N}_2 v' = 0$$

$$\text{i.e.,} \quad \tilde{N}_1 \quad \tilde{r}_1 \quad lu' + \tilde{N}_2 \quad \tilde{r}_1 \quad lv' + \tilde{N}_1 \quad \tilde{r}_2 \quad mu' + \tilde{N}_2 \quad \tilde{r}_2 \quad mv' = 0$$

$$\text{i.e.,} \quad Llu' + M \; lv' + mu' + Nmv' = 0$$

But this is exactly the condition that the direction (l, m) is conjugate to the direction (u', v') of the tangent at P. This completes the proof of the theorem.

12.4. Minimal surfaces:

De nition 12.1 (Minimal surfaces).

Surfaces whose mean curvature is zero at all points are called minimal surfaces.

Note 12.2. The mean curvature is given by

$$= \frac{EN + GL \quad 2FM}{2 \; EG \quad F^2} = \frac{EN + GL \quad 2FM}{2H^2}$$

The condition for minimal curvature is $= 0.$

Thus, we have $EN + GL \quad 2FM = 0.$

Theorem 12.3. If there is a surface of minimum area passing through a closed space curve, it is necessarily a minimal surface *i.e.,* a surface of zero mean curvature.

Proof. Let Σ be a surface bounded by a closed curve C; and let Σ' be another surface derived from Σ by a small displacement $\lambda(u, v)$ in the direction of the normal. We assume that $\lambda_1 = \frac{\partial\lambda}{\partial u}$ and $\lambda_2 = \frac{\partial\lambda}{\partial v}$ are both small and more precisely $\lambda_1 = O(\epsilon)$; $\lambda_2 = O(\epsilon)$ as $\epsilon \to 0$.

The position vector of the displaced surface is noted by $\tilde{R}$.

$$\text{Thus, we have } \tilde{R} = \tilde{r} + \lambda\tilde{N} \tag{12.15}$$

Differentiating equation (12.15) with respect to u and v; we get

$$\begin{aligned}
\tilde{R}_1 &= \tilde{r}_1 + \lambda_1\tilde{N} + \lambda\tilde{N}_1 \\
\tilde{R}_2 &= \tilde{r}_2 + \lambda_2\tilde{N} + \lambda\tilde{N}_2
\end{aligned}$$

Let E', F', G' denote the first fundamental coefficients of Σ'. Then

$$\begin{aligned}
E' &= \tilde{R}_1^2 = \left(\tilde{r}_1 + \lambda_1\tilde{N} + \lambda\tilde{N}_1\right)^2 \\
&= \tilde{r}_1^2 + 2\lambda_1\tilde{r}_1\cdot\tilde{N} + 2\lambda\tilde{r}_1\cdot\tilde{N}_1 + O\left(\epsilon^2\right) \\
&= E - 2\lambda L + O\left(\epsilon^2\right) \\
F' &= \tilde{R}_1\cdot\tilde{R}_2 = \left(\tilde{r}_1 + \lambda_1\tilde{N} + \lambda\tilde{N}_1\right)\cdot\left(\tilde{r}_2 + \lambda_2\tilde{N} + \lambda\tilde{N}_2\right) \\
&= F - 2\lambda M + O\left(\epsilon^2\right) \\
G' &= \tilde{R}_2^{\,2} = \tilde{R}_2\cdot\tilde{R}_2 \\
&= \left(\tilde{r}_2 + \lambda_2\tilde{N} + \lambda\tilde{N}_2\right)\cdot\left(\tilde{r}_2 + \lambda_2\tilde{N} + \lambda\tilde{N}_2\right) \\
&= G - 2\lambda N + O\left(\epsilon^2\right) \text{ as } \epsilon \to 0 \\
\text{Now, } H'^2 &= E'G' - F'^2 \\
&= \left[E - 2\lambda L + O\left(\epsilon^2\right)\right]\left[G - 2\lambda N + O\left(\epsilon^2\right)\right] - \left[F - 2\lambda M + O\left(\epsilon^2\right)\right]^2 \\
&= EG - F^2 - 2\lambda(EN + GL - 2FM) + O\left(\epsilon^2\right) \\
&= H^2 - 4\lambda H^2\mu + O\left(\epsilon^2\right) \\
H' &= H(1 - 4\lambda\mu)^{1/2} + O\left(\epsilon^2\right) \\
&= H(1 - 2\lambda\mu) + O\left(\epsilon^2\right) \qquad \text{(using binomial expansion)}
\end{aligned}$$

$$\text{Let } A = \iint_P H\,dudv$$

$$A = \iint_{P_o} H\,dudv = \iint_P H(1 \quad 2 \quad)\,dudv + O^2$$

$$= \iint Hdudv \quad 2 \quad Hdudv + O^2$$

$$= A \quad \int 2 \quad dudv + O^2$$

If A is stationary, then clearly $= 0$ which shows that the surface is necessarily of zero mean curvature.

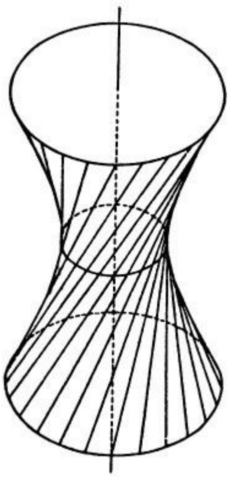
This completes the proof of the theorem.

12.5. Ruled surfaces:

A ruled surface is generated by the motion of a straight line with one degree of freedom, the various positions of the line being called generators.

The developable surfaces discussed in section (11.1) belong to the family of ruled surfaces, are very special and have properties not characteristic of ruled surfaces in general.

An example of ruled surface which is not developable is hyperboloid of revolution.



Let C be any curve on a ruled surface having the property that it meets each generator precisely once. Such a curve will be called a base curve. It is clear that such a curve is by no means uniquely determined. Then the surface is determined by any base curve C and the direction of the generators at each point of C:

Theorem 12.4. Show that the Gaussian curvature K for a ruled surface is given by $K = \frac{p^2 \tilde{g}^2}{H^4}$ where g is the unit vector along the generator conclude that a developable surface is a ruled surface for which the parameter of distribution is identically zero.

Proof. Let $\tilde{r}(u)$ be the position vector of a current point P on C and let $\tilde{g}(u)$ be a unit vector along the generator at P: Then the position vector of a general point Q on the ruled surface is given by

$$\tilde{R} = \tilde{r} + v\tilde{g} \qquad (12.16)$$

where v is the parameter which measures directed along the generator from C:

Di erentiating (12.16) with respect to the parameters u and v; we get

$$\tilde{R}_1 = \tilde{r} + v\tilde{g} \qquad (* \ \tilde{r} \text{ and } \tilde{g} \text{ are functions of } u \text{ alone})$$

$$\tilde{R}_2 = \tilde{g}$$

$$\tilde{R}_{11} = \tilde{r} + v\tilde{g}$$

$$\tilde{R}_{12} = \tilde{g}; \quad \tilde{R}_{22} = 0$$

The rst fundamental coe cients are

$$E = \tilde{R}_1^{\,2} = \tilde{r}^{\,2} + 2v\tilde{g} \ \ \tilde{r} + v^2\tilde{g}^2$$

$$F = \tilde{R}_1 \ \ \tilde{R}_2 = \tilde{g} \ \ \tilde{r}$$

$$G = \tilde{R}_2^{\,2} = \tilde{g}^{\,2} = 1$$

The metric is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$= \tilde{r}^{\,2} + 2v\tilde{g} \ \ \tilde{r} + v^2\tilde{g}^2 \ \ du^2 + 2\tilde{g} \ \ \tilde{r}dudv + dv^2 \qquad (12.17)$$

The unit normal vector $\tilde{N}$ is given by

$$H\tilde{N} = \tilde{R}_1 \ \ \tilde{R}_2 = \tilde{r} + v\tilde{g} \quad \tilde{g} \qquad (12.18)$$

unless $\left[\tilde{r}; \tilde{g}; \tilde{g}\right] = 0$: Thus, the tangent plane to the surface varies at points on the some generator

The second fundamental coe cients of the surface are given by

$$\left.\begin{aligned} HL &= H\tilde{N} \cdot \tilde{R}_{11} = [\tilde{r}, \tilde{r}, \tilde{g}] + [\tilde{r}, \tilde{g}, \tilde{g}]\, v + [\tilde{g}, \tilde{r}, \tilde{g}]\, v + [\tilde{g}, \tilde{g}, g]\, v^2 \\ HM &= H\tilde{N} \cdot \tilde{R}_{12} = [\tilde{r}, \tilde{g}, \tilde{g}] \\ HN &= 0 \end{aligned}\right\} \tag{12.19}$$

The asymptotic lines are given by du [Ldu + 2Mdv] = 0 from which it follows that the generators are asymptotic lines. The other family of asymptotic lines is given by an equation of the form

$$\frac{dv}{du} = A + Bv + cv^2$$

Where A, B, C are functions of u alone. This is a Riccati type di erential equation, and the most general solution of the form

$$v = \frac{cP + Q}{cR + S} \tag{12.20}$$

where P, Q, R, S are functions of u and c is an arbitrary constant.

Let the four asymptotic lines of this family be speci ed by the values c_1, c_2, c_3, c_4 and let these lines be met by the generator $u = u_0$ in four points where v parameter has values v_1, v_2, v_3, v_4. From the equation (12.20), it follows that the cross-ratio (v_1, v_2, v_3, v_4) is equal to the cross ratio (c_1, c_2, c_3, c_4) and is independent of u_0. Thus the cross-ratio of the four points in which four given asymptotic lines are met by any generator is the same for all generators.

From equation (12.19), the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{[\tilde{r}, \tilde{g}, \tilde{g}]^2}{H^4}$$

It is convenient to de ne a function $p(u)$ called the parameters of the distribution by writing $p(u) = \dfrac{[\tilde{r}, \tilde{g}, \tilde{g}]}{\tilde{g}^2}$.

This is independent of the particular base curve chosen and also independent of the parameter u.

In terms of p the Gaussian curvature is given by

$$K = \frac{p^2 \tilde{g}^2}{H^4} \tag{12.21}$$

So K is always negative except along those generators where $p = 0$. Since $K = 0$ for a developable, it follows that developable surface is a ruled surface for which the parameter of distribution is identically zero.

De nition 12.2 (Central Point). On each generator of the general ruled surface there is a special point called critical point of the generator. This is determined as follows:

Let $P; Q$ be two given points on some base curve C and let the common perpendicular to the generating line through $P; Q$ meet these generators in $P_1; Q_1$ respectively.

As Q tends to $P;$ the point P_1 will tend to some point called the critical point of the generator.

Bookwork 12.5. Derive the formula to determine the position of the central point on each generator.

Proof. The limiting direction of the vector $\widetilde{P_1 Q_1}$ must lie in the surface and hence be perpendicular to $\tilde{N};$ also it must be perpendicular to the generator through P and hence parallel to the vector $\tilde{g} \quad \tilde{N}:$

This direction must be perpendicular to the generators through P and Q and proceeding to the limit as $Q \to P$ we have $\tilde{g} \quad \tilde{g} \quad \tilde{N} = 0$ or $\tilde{g} \quad \tilde{g} \quad \tilde{N} = 0:$

$$\text{But } H\tilde{N} = \tilde{r} \quad v\tilde{g} \quad \tilde{g}$$

$$\Rightarrow \quad \tilde{g} \quad \tilde{g} \quad \tilde{r} + v\tilde{g} \quad \tilde{g} \quad = \quad 0 \qquad (\because H \neq 0)$$

$$i:e:; \quad \tilde{g} \quad \tilde{r} + v\tilde{g}^{\,2} \quad = \quad 0 \tag{12.22}$$

from which v is uniquely determined provided $\tilde{g}^{\,2} \neq 0:$

De nition 12.3 (Line of Striction). The central points of all the generators form a locus called the line of striction, which is a well determined curve naturally associated with the ruled surface.

Theorem 12.5. Show that the tangent of the angle through which the normal $\tilde{N}$ rotates as the point P moves along a generator varies directly with the distance moved from the central point.

Proof. If we choose the line of striction as base curve, then it follows from the equation (12.22) that $\tilde{g} \quad \tilde{r} = 0 \quad (\because v = 0):$

Also, in addition $\tilde{g} \quad \tilde{g} = 0;$ thus we have the vector $\tilde{r} \quad \tilde{g}$ must be parallel to $\tilde{g}:$

Thus, we can write $\tilde{r} \quad \tilde{g} = \quad \tilde{g}$ for some function :

Then scalar multiplication by $\tilde{g}$ implies

$$[\tilde r', \tilde g, \tilde g'] = \lambda \tilde g'^2 = p\tilde g'^2$$

so, we have $\lambda = p$. Thus $\tilde r' \times \tilde g = p\tilde g'$.

$\Rightarrow$ Equation (12.18) can be rewritten as

$$H\tilde N = p\tilde g' + v\tilde g' \times \tilde g$$

From equation (12.17) with $\tilde g' \cdot \tilde r' = 0$, we have

$$H\tilde N = v^2 (\tilde g' \times \tilde g)^2 + \tilde r'^2 - (\tilde g \cdot \tilde r')^2$$

$$\text{i.e., } H\tilde N = v^2 \tilde g'^2 \tilde g^2 + \tilde r'^2 \tilde g^2 - (\tilde g \cdot \tilde r')^2 \qquad \because \tilde g^2 = 1$$

$$\Rightarrow H^2 = v^2 \tilde g'^2 + (\tilde r' \times \tilde g)^2$$

$$\Rightarrow H^2 = (p^2 + v^2)\tilde g'^2 \qquad \because \tilde r' \times \tilde g = p\tilde g' \tag{12.23}$$

$$\text{Thus, } \tilde N = \frac{p}{(p^2+v^2)^{1/2}}\tilde a + \frac{v}{(p^2+v^2)^{1/2}}\tilde a \times \tilde g$$

where $\tilde a$ is the unit vector along $\tilde g'$.

Let ϕ denote the angle between the directions of $\tilde N$ at points on a generator distant v and O from the central point.

Then if $p \neq 0$, we have

$$\left.\begin{aligned} \sin\phi &= \frac{v}{(p^2+v^2)^{1/2}} \\ \cos\phi &= \frac{p}{(p^2+v^2)^{1/2}} \\ \tan\phi &= \frac{v}{p} \end{aligned}\right\} \tag{12.24}$$

Thus the tangent of the angle through which the normal $\tilde N$ rotates as the point P moves along a generator varies directly with the distance moved from the central point.

Note 12.3. As v increases from $-\infty$ to ∞, the angle ϕ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ if $p > 0$ and decreases from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$ if $p < 0$.

When the central point is reached the normal has rotated through an angle $\frac{\pi}{2}$, and this fact justifies the word central.

Thus, equation (12.21) and (12.23) provides the simple formula to determine Gaussian curvature at the point distant v from the central point on a generator of parameter p is

$$K = -\frac{p^2}{\left(p^2+v^2\right)^{1/2}}$$

Bookwork 12.6. Find the necessary and sufficient condition that the surface $z = f(x, y)$ should represent a developable.

Proof. The equation of the tangent plane at a point (x, y, z) is

$$\begin{aligned} (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + (Z-z) &= 0 \\ \text{i.e.,}\quad p(X-x) + q(Y-y) + Z - z &= 0 \end{aligned}$$

In case the surface is developable surface the equation of tangent plane should be in terms of single parameter and hence there a relation between p and q denoted by $p = \phi(q)$. Thus, we have

$$\begin{aligned} \frac{\partial p}{\partial x} &= \phi'(q)\frac{\partial q}{\partial x} \\ \frac{\partial p}{\partial y} &= \phi'(q)\frac{\partial q}{\partial y} \end{aligned}$$

Eliminating ϕ' between the above two equations, we get

$$\begin{aligned} \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} &= \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} \\ \text{i.e.,}\quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 f}{\partial y\partial x}\frac{\partial^2 f}{\partial x\partial y} \\ \text{i.e.,}\quad rt &= s^2 \end{aligned}$$

Thus $rt - s^2 = 0$ is the required condition for a surface to be developable.

Conversely,

$$\begin{aligned} \text{if } rt - s^2 &= 0 \\ \Rightarrow \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} &= 0 \\ \text{i.e.,}\quad \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial x} \end{vmatrix} &= 0 \\ \text{i.e.,}\quad \frac{\partial(p, q)}{\partial(x, y)} &= 0 \end{aligned}$$

Thus, the functions p and q must depend on the single parameter, so shall do the tangent plane, therefore the surface is developable.

Example 12.1. Show that the surface $xy = (z-c)^2$ is developable.

Solution:

$$
\begin{aligned}
(z-c) &= \sqrt{xy} \\
\Rightarrow z &= c + \sqrt{xy} \\
\frac{\partial z}{\partial x} &= p = \frac{1}{2}\sqrt{\frac{y}{x}}; \quad \frac{\partial z}{\partial y} = q = \frac{1}{2}\sqrt{\frac{x}{y}} \\
r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} = -\frac{1}{4}y^{1/2}x^{-3/2}; \quad \frac{\partial^2 z}{\partial y^2} t = = \frac{\partial q}{\partial y} = -\frac{1}{4}x^{1/2}y^{-3/2} \\
s &= \frac{\partial p}{\partial y} = \frac{1}{4}x^{-1/2}y^{-1/2} \\
rt - s^2 &= \frac{1}{16}\frac{1}{xy} - \frac{1}{16}\frac{1}{xy} = 0
\end{aligned}
$$

Hence the given surface is developable.

Example 12.2. Show that the surface $e^z \cos x = \cos y$ is minimal.

Solution:

$$
\begin{aligned}
e^z \cos x &= \cos y \\
z + \log(\cos x) &= \log \cos y \\
\text{i.e.,} \quad z &= \log \cos y - \log(\cos x) \\
E &= 1 + p^2 = 1 + \left(\frac{\partial z}{\partial x}\right)^2 = 1 + \tan^2 x = \sec^2 x \\
G &= 1 + q^2 = 1 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \tan^2 y = \sec^2 y \\
F &= pq = -\tan x \tan y \\
L &= \frac{r}{H} = \frac{\sec^2 x}{H}; \quad M = \frac{s}{H}; \quad N = \frac{t}{H} = -\frac{\sec^2 y}{H} \\
EN - 2FM + GL &= -\frac{\sec^2 x \sec^2 y}{H} - 0 + \frac{\sec^2 x \sec^2 y}{H} = 0
\end{aligned}
$$

Thus, the condition for the surface to be minimal $EN - 2FM + GL = 0$ is satisfied.

Hence the given surface is minimal.

Example 12.3. Find the equation to the developable which has the curve $x = 6t;\ y = 3t^2;\ z = 2t^3$ for its edge of regression.

Solution:

The equation to the edge of regression is $\tilde{r} = (6t, 3t^2, 2t^3)$.

Now, the developable can be considered as the tangential developable of the edge of regression.

If $\tilde{R}$ is the position vector of any point on the developable then

$$\tilde{R}(t, v) = \tilde{r} + v\dot{\tilde{r}} \quad \text{where } \dot{\tilde{r}} = \frac{d\tilde{r}}{dt} = (6, 6t, 6t^2)$$

$$\text{Thus, } (x, y, z) = (6t, 3t^2, 2t^3) + 6v(1, t, t^2)$$

$$\text{i.e., } (x - 6t, y - 3t^2, z - 2t^3) = 6v(1, t, t^2)$$

$$\Rightarrow \frac{x - 6t}{1} = \frac{y - 3t^2}{t} = \frac{z - 2t^3}{t^2} = 6v$$

Consider the first two ratios and last two ratios, we get

$$xt - y = 3t^2; \quad yt - z = t^3$$

$$\Rightarrow t(xt - y) = 3t^3 = 3(yt - z)$$

$$\Rightarrow xt^2 - 4yt + 3z = 0$$

$$\text{Also, } 3t^2 - xt + y = 0$$

Solving the last two equations, we get

$$\frac{t^2}{3xz - 4y^2} = \frac{t}{9z - xy} = \frac{1}{12y - x^2}$$

$$\text{i.e., } (3xz - 4y^2)(12y - x^2) = (9z - xy)^2$$

This is the required developable.

Example 12.4. Show that the ruled surface generated by the binormals of a space curve has the curve itself as the line of striction.

Solution:

Consider the given space curve C as the base curve, then the equation to the ruled surface can be written as

$$\tilde{R}(s, v) = \tilde{r}(s) + v\tilde{g}(s) \tag{12.25}$$

where $\tilde{r}(s)$ is the position vector of the point P on C and $\tilde{g}(s)$ is the unit vector along the generator at P.

Since the ruled surface is generated by the binormals to C, we have $\tilde{g} = \tilde{b}$.

Let v be the distance from P of the central point of the generator at P.

Then from equation $\tilde{g}^{o} \; \tilde{r}^{o} + v\tilde{g}^{2} = 0$; where we have used notation primes instead of dots since the parameter of the curve is taken as the arc length s;

$$\text{Thus}; \quad \tilde{b}^{o} \; \tilde{t} + v\tilde{b}^{o\,2} = 0$$

$$\tilde{n} \; \tilde{t} + v \quad \tilde{n}^{2} = 0$$

$$^{2}v = 0 \quad \text{(or)} \quad v = 0$$

This shows that the central point on the generator at P is P itself. Thus the given curve itself is the line of striction of the ruled surface.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- Osculating developable, Polar developable and Rectifying developable.
- the Minimal surfaces and Ruled surfaces.
- derive Monge's theorem.

Check Your Progress:

1. De ne osculating developable, polar developable and rectifying developable.
2. State and prove a necessary and su cient condition for a surface to be developable.
3. Show that $e^{x} \cos x = \cos y$ is minimal.
4. Prove that the Gaussian curvature is the same at two points of a generator which are equidistant from the central point.

Choose the correct or more suitable answer:

1. : : : : : : is the surface enveloped by the normal plane of a space curve.

   (a) Osculating developable.

   (b) Polar developable.

(c) Rectifying developable.

(d) none of these.

2. The condition for minimal curvature is : : : : : :

   (a) $EN + 2GL \quad FM = 0$.

   (b) $EN + GL \quad 2FM = 0$.

   (c) $EN + 2GL \quad 2FM = 0$.

   (d) $EN \quad 2GL \quad FM = 0$.

Answer:

(1) b (2) b

Glossaries:

Polar Developable: The polar developable of a curve is the envelope of its normal planes.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-V

Unit-13: Compact Surfaces.

Unit-14: Complete Surfaces.

Unit-15: Hilbert's theorem.

Block-V

UNIT-13

COMPACT SURFACES

Structure

Objective

Overview

13. 1 Introduction

13. 2 Compact surfaces whose points are umblices

13. 3 Hilbert's lemma

Check Your Progress

Let us Sum Up

Suggested Readings

Objectives

After completion of this unit, students will be able to

- know the concept of Compact surfaces.
- derive Hilbert's lemma.

Overview

In this unit, we will explained in detail about the compact surfaces.

13.1. Introduction:

In the previous unit, we were discussed the properties of a region of a surface de ned by suitably restricting the parameters u and v: These are essentially local properties, the word local indicating that in order to obtain the property at a point P it is necessary to have information about the surface only in the neighbourhood of P:

In the present unit, we shall be concerned with properties involving the surface as a whole. For example, whether like a spherical cap it has a boundary or whether it is compact like a sphere. Di erential geometry of surface in the large is the study of relations between the local and global properties of surfaces.

13.2. Compact surfaces whose points are umblics:

For proving the rst few theorems of this unit, we shall use the de nition of surface given in the earlier unit and assume that each point has a neighbourhood (homeomorphic to an open 2 -cell) which can be determined by parametric equations $\tilde{r} = \tilde{r}(u; v)$:

Theorem 13.1. The only compact surfaces of class 2 for which every point is an umbilic are spheres.

Proof. By way of local geometry developed in the earlier chapters we shall prove that in the neighbourhood of any point the surface is either spherical or plane, then by use the property of compactness to reject one of alternative. Hence we show that the surface must be a sphere.

Let S be a compact surface of class 2 for which every point is an umbilic. Let P be any point on S; and let V be a coordinate neighbourhood of S containing P; in which part of S is represented parametrically by $\tilde{r} = \tilde{r}(u; v)$:

Since every point of V is an umbilic, it follows that every curve lying in V must be a line of curvature. Hence from Rodrigue's formula, at all points of V;

$$d\tilde{N} + \;\; d\tilde{r} = 0 \qquad (13.1)$$

where is the normal curvature of S in the director $d\tilde{r}$.

$$\Rightarrow \; d\tilde{N} = \;\; d\tilde{r}$$

$$i.e., \; \tilde{N}_1 = \;\; \tilde{r}_1 \quad \text{and} \quad \tilde{N}_2 = \;\; \tilde{r}_2$$

Using the identity $\tilde{N}_{12} = \tilde{N}_{21}$; in the above equations, we get $_2\tilde{r}_1 \;\; _1\tilde{r}_2 = 0$.

Since $\tilde{r}_1, \tilde{r}_2$ are linearly independent we obtain $_1 = \; _2 = 0$; so that is a constant.

Integrating equation (13.1), we get

$$\tilde{r} = \tilde{a} \;\; ^{1}\tilde{N} \qquad (13.2)$$

for $\neq 0$ showing that V lies on the surface of a sphere of centre $\tilde{a}$ and radius ^{1}.

When $= 0$; equation (13.1) gives $\tilde{N} = \tilde{b}$ showing that the V lies on a plane.

This completes the local part of the theorem i.e., so far all we have proved is that in the neighbourhood of any point the surface is spherical or plane.

Associate with each point P on the surface a neighbourhood V; having the above said property. The set of all neighbourhoods V_P covers S and from the compactness, we conclude that S is covered by a nite sub-cover formed by $V_j (j = 1, 2, \ldots, N)$. Consider two over lapping neighbourhoods V_i, V_j. From the previous local argument it follows that is constant in V_i and also in V_j. By considering the values of at the points in $V_i \cap V_j$ we nd that has the same value over the whole of the surface. Moreover, this value cannot be zero. Otherwise the surface would contain a straight line and would not be compact.

Hence the surface must be a sphere and hence the theorem is proved.

13.3. Hilbert's lemma:

Lemma 13.1. In a closed region R of a surface of constant positive Gaussian curvature without umbilics, the principle of curvature take their extreme values at the boundary.

This lemma is purely concerned local in character and results of earlier chapters can be used to prove it.

W.F. Newm suggested the above lemma can be restated in a slightly di erent form.

If a point P_0 of any surface, the principal curvatures $_a$ and $_b$ are such that either (i) $_a > _b;$ $_a$ has a maximum at P_0 or (ii) $_a < _b;$ $_a$ has minimum at $P_0;$ $_b$ and has a maximum at $P_0;$ then the Gaussian curvature K cannot be positive at $P_0.$

Proof. Now, we shall prove the lemma by contradiction.

Assume that the lemma is false. Then there is a point P_0 at which the principal curvature have distinct extreme values, one maximum and the other minimum.

Consider the lines of curvatures as parametric curves, then principal curvatures are

$$_a = \frac{L}{E}; \quad _b = \frac{N}{G}. \tag{13.3}$$

The Codazzi equations are

$$\left.\begin{aligned} L_2 &= \frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right) \\ N_1 &= \frac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right) \end{aligned}\right\}, \tag{13.4}$$

$$\begin{aligned} \frac{@_a}{@v} &= \frac{EL_2 \; LE_2}{E^2} \\ &= \frac{E\frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right) LE_2}{E^2} \\ &= \frac{\frac{1}{2}EE_2\frac{N}{G} \; \frac{1}{2}LE_2}{E^2} \end{aligned}$$

$$= \frac{1}{2}\frac{E_2}{E}\left({}_b \quad {}_a\right)$$

$$\text{Similarly,} \quad \frac{@ {}_b}{@u} = \frac{1}{2}\frac{G_1}{G}\left({}_a \; {}_b\right)$$

Since the principal curvatures have extrema, the $L.H.S.$ members vanishes at P_0. It follows that at P_0:

$$E_2 = G_1 = 0 \quad \text{and hence at } P_0$$

$$\frac{@^2 {}_a}{@v^2} = \frac{1}{2}\frac{E_{22}}{E}\left({}_b \quad {}_a\right)$$

$$\frac{@^2 {}_b}{@u^2} = \frac{1}{2}\frac{G_{11}}{G}\left({}_a \quad {}_b\right) \qquad (13.5)$$

Now, there are two possibilities arises:

either (i) ${}_a$ has a maximum:

$$\text{In this case} \quad {}_a \quad {}_b > 0; \quad \frac{@^2 {}_a}{@v^2} \quad 0; \quad \frac{@^2 {}_b}{@u^2} \quad 0 \qquad (13.6)$$

or (ii) ${}_b$ has a minimum:

$$\text{Then} \quad {}_a \quad {}_b < 0; \quad \frac{@^2 {}_a}{@v^2} \quad 0; \quad \frac{@^2 {}_b}{@u^2} \quad 0 \qquad (13.7)$$

In either case E_{22} 0 and G_{11} 0 (Note that the signs of ${}_a$; ${}_b$ are irrelevant).

But this contradicts the fact that the Gaussian curvature K satis es

$$K = \frac{1}{2EG}(E_{22} + G_1)$$

$$K = \frac{1}{2H}\frac{@}{@u}\frac{G^1}{H} + \frac{@}{@v}\frac{E_2}{H}$$

Since the R.H.S of the above expression is zero or negative, while K is assumed strictly positive. Thus contradiction arises.

This completes the proof of the lemma.

Let Us Sum Up:

In this unit, the students acquired knowledge to

the compact surface.

derive Hilbert's lemma.

Check Your Progress:

1. Show that the only compact surfaces of class 2 for which every point is an umblic are spheres.
2. State and Prove Hilbert's lemma.

Suggested Readings:

1. T.J. Willmore, An Introduction to Di erential Geometry , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).
2. C.E.Weatherburn, Di erential Geometry of Three Dimensions , University Press, Cambridge, 1930.

Block-V

UNIT-14

COMPLETE SURFACES

Structure

Objective

Overview

14. 1 Compact surfaces of constant Gaussian or mean curvature

14. 2 Complete Surfaces

14. 3 Characterization of complete surfaces

Let us Sum Up

Check Your Progress

Answers to Check Your Progress

Suggested Readings

Objectives

After completion of this unit, students will be able to

- explain the concept of Complete surfaces.

Overview

In this unit, we will illustrate the characterization of complete surfaces.

14.1. Compact surfaces of constant Gaussian or mean curvature:

We note that a compact surface must possess a highest point and at this point the curvature is necessarily non-negative. Moreover, a compact surface cannot have constant zero curvature, for otherwise it would contain straight lines which would contradict the compactness.

Theorem 14.1. The only compact surfaces with constant Gaussian curvature are spheres.

Proof. Let S be a compact surface with constant positive Gaussian curvature $K:$ Since S is compact, there is a point P_0 at which attains the maximum value of the principal curvature (i:e:; the Gaussian curvature) is constant.

Hence the principal curvatures have respectively a maximum and minimum value at P_0 with the maximum value not less than the minimum.

From Hilbert's lemma, it follows that the two principal curvatures must be equal i:e:; at no points does either principal curvature exceeds $\sqrt{K}:$ Hence every point of S is an umbilic.

Hence by theorem (13.1), only compact surfaces with constant Gaussian curvature are spheres.

Theorem 14.2. The only compact surfaces whose Gaussian curvature is positive and mean curvature constant are spheres.

Proof. Let S be a compact surfaces of positive Gaussian curvature and constant mean curvature, and it is denoted respectively by $_a$ (larger principal curvature), $_b$ (smallest principal curvature).

Since $_a$ is continuous and S is compact there is a point P_0 at which $_a$ attains its maximum value. Also the mean curvature $_b$ is constant hand hence it follows that $_b$ attains its minimum value at $P_0:$

Thus, we have $_a$ $_b$ every where. Suppose if $_a > _b$ at $P_0;$ then by

Hilbert's lemma, we can conclude that the Gaussian curvature K is negative which contradicts our hypothesis.

Thus, we must have $_a = _b =$ at the point P_0 and hence everywhere on S:

This completes the proof of the theorem.

14.2. Complete Surfaces:

In the previous section, we restrict the surfaces to be compact. But this restriction may exclude for example, developable surfaces and many common surfaces like paraboloids.

Definition 14.1 (Metric Spaces).

A set of points S carries the structure of a metric space when there is a real valued function $: S\ S \to R_1$ with the properties:

(i) $(A; B) = 0 \Leftrightarrow A = B$

(ii) $(A; B) = (B; A)$ (symmetry)

(iii) $(A; C) \quad (A; B) + (B; C)$ (triangle inequality)

for all points $A;\ B;\ C$ of S

Definition 14.2 (Length of the segment).

Let us assume that the surface S is connected so that any two points can be joined by arc-wise connected paths.

If is any path joining A to B then this path can be divided into a finite number of segments so that each segment lies entirely in one coordinate neighbourhood overlap.

The length of the segment whose equation relative to a coordinate neighbourhood is $u = u(t);\ v = v(t)$ is given by $\int \sqrt{Eu^2 + 2Fuv + Gv^2}\, dt$ taken between the appropriate limits.

The length of is defined as the sum of the length of its segments.

Definition 14.3 (Distance function).

Distance function is defined by

$(A; B)$ is the greatest lower bound $(l.u.b)$ of all the lengths of all arc-wise connected C^1 paths joining A to B.

Note 14.1. It is clear that the distance defined as above satisfied conditions (ii) and (iii) of the metric space axioms while condition (i) is satisfied because the first fundamental form of the surface is positive definite.

Definition 14.4 (Cauchy Sequence).

A sequence of points $\{x_n\}$ on the surface is said to form a Cauchy sequence when given $\epsilon > 0$, there exists an integer n_0 such that $\rho\{x_n, x_m\} < \epsilon$ when $m, n > n_0$. Clearly if $\{x_n\}$ converges to limit x then the sequence $\{x_n\}$ is a Cauchy sequence.

Note 14.2. If the surface is such that every Cauchy sequence converges, then the metric space is said to be complete

The following example shows that not all surfaces are complete.

Consider the surface formed by the two-dimensional Cartesian plane of pairs of real numbers (x, y) when the origin is removed.

Distance ρ is defined by

$$\rho(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

where $(x_A, y_A), (x_B, y_B)$ are the rectangular Cartesian coordinates of A and B. We can easily seen that the sequence of points $\left(\frac{1}{n}, 0\right)$ is a Cauchy sequence which does not converge in the surface and so the surface is not complete.

14.3. Characterization of complete surfaces:

Now, we are going to discuss three important properties which will be used to characterize complete surface and they are:

(a) Every Cauchy sequence of points of S is convergent.

(b) Every geodesic can be prolonged indefinitely in either direction, or else it forms a closed curve.

(c) Every bounded set of points of S is relatively compact.

Now, we shall prove that the above three properties are equivalent.

Property (c) implies property (a) is quite obvious.

Now we shall prove that the property (a) implies property (b):

If be a closed curve, then the condition (b) is obviously satis ed. If is not a closed curve and if P(x) is some point on then there is some number l such that can be prolonged for distances (measured along) less than l; but cannot be prolonged for distance greater than l:

Consider the sequence of points $\{x_n\}$ lying on at distance from P lying is given by $l\left(1 - \frac{1}{n}\right)$:

Clearly $\{x_n\}$ is a Cauchy sequence and hence by condition (a) converges to some point Q on whose distance from P is exactly l:

If x'_n is another Cauchy sequence such that $x_1; x'_n \to l$; then x'_n tends to some limit Q'

Now the sequence $x_1; x'_1; x_2; x'_2; \ldots$ is also a Cauchy sequence tending to both Q and Q': Hence $Q = Q'$; and there exists a unique end point Q at a distance l from P along :

Consider a coordinate neighbourhood of S which contains Q: At Q there is uniquely determined a direction $\tilde{t}$ which is the direction of the geodesic which starts at Q:

In this coordinate neighbourhood there is a unique geoedesic at Q which has the direction $\tilde{t}$ and this gives a continuation of beyond Q; contrary to the hypothesis.

Thus, must satisfy condition (b):

Next, we have to prove that the condition (b) implies (c) so that all the three conditions are equivalent.

Assume that the condition (b) holds good for S:

Consider a point of S; and geodesic which start at a: Now we de ne the initial vector of a geodesic are starting at a to be the tangent to this arc a which has the same sense as the geodesic and whose length is equal to the length of the geodesic arc. Since property (b) is holds good for S; it follows that every tangent vector to S at a; whatever its length, is the initial vector of some geodesic arc starting at a which is uniquely determined. This arc may cut itself or if it forms part of a closed geodesic, may even cover part of itself.

Let $S_r = \{x \in S : \rho(x, a) \le r\}$ and E_r be the set of points x of S_r, which can be joined to a by a geodesic arc whose length is equal to $\rho(x, a)$.

Now our claim is to prove that E_r is compact.

For this, let $\{x_h\}_{n=1}^{\infty}$ be a sequence of points of E_r.

Let $\tilde{T}_h$ be the initial vector of a geodesic arc of length $\rho(a, x_h)$ joining a to x_n, then the sequence of vectors $\left(\tilde{T}_h\right)$ regarded as a sequence of points in two dimensional Euclidean space, admits at least one vector of accumulation $\tilde{T}$. Moreover, this vector $\tilde{T}$ is the initial vector of a geodesic arc whose extremely belongs to E_r and is an accumulation point of $\{x_n\}$. This proves that E_r is compact.

Next, our aim is to prove that $E_r = S_r$.

It is easily seen that $E_r = S_r$ is true for $r = 0$. Also it is true for $r = R > 0$, then it is certainly true for $r < R$.

Now, we shall prove that conversely if $E_r = S_r$ is true for $r < R$ then it is still true for $r = R$.

Now, every point of S_R is the limit point of sequence of points whose distance from a is less than R. By hypothesis these points belong to E_R, and since E_R is closed, it follows that their limit belongs to E_r. Thus, $E_r = S_r$ is true for $r = R$.

In order to prove $E_r = S_r$ is completely, it is necessary to show that if it holds for $r = R$, then it still holds for $r = R + s$, $s > 0$.

This follows because it would then be possible to extend the range of validity of $E_r = S_r$ to an arbitrary extent by an appropriate number of extensions of the range by an amount s.

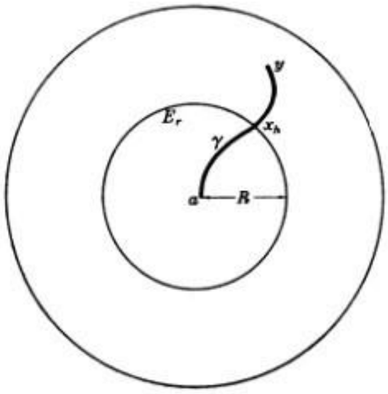


Figure 14.1

Next, we have to prove that to any point y such that $(a; y) > R$ there is a point x such that

$$(a; y) = R \qquad (14.1)$$

$$\text{and} \quad (a; y) = R + (y; x) \qquad (14.2)$$

Since $(a; y)$ has been de ned as the lower bound of the lengths of arcs from a to y; it follows that we can join a to y by a curve whose length is less than $(a; y) + h^{1}$ for any integer h:

Let $\{x_h\}$ be the last point of this curve belonging to $E_R = S_R$:

Now we have

$$(a; y) \quad (a; x_h) + (x_h; y)$$

$$i:e:; \quad R + (x_h; y)$$

$$i:e:; \quad (x_h; y) \quad (a; y) \quad R \qquad (14.3)$$

Since the arc length of from a to y is the sum of the arc lengths from a to x_h and from x_h to y; we have

$$(x_h; y) \quad \text{arc}\,(x_h; y)$$

$$(x_h; y) \quad \text{arc}\,(a; y) \quad \text{arc}\,(a; x_h)$$

$$(a; y) + h^{1} \quad \text{arc}\,(a; x_h)$$

$$(a; y) + h^{1} \quad R$$

Now, let $h \; ! \; 1$ will have at least one point of accumulation x with the property

$$(x; y) \quad (a; y) \quad R \qquad (14.4)$$

Comparing equations (14.3) and (14.4), shows at this point $(a; y) = R + (y; x):$

Thus we have proved that the existence of a point x satisfying equations (14.1) and (14.2).

We have seen earlier that provided two points $x; y$ are not too far apart then the point y is the extremity of one and only one geodesic arc of origin s and of length $(x; y):$ More precisely there exists a continuous function $s(x) > 0$ such that if $(x; y) < s(x);$ the point y is the extremity of the unique geodesic arc of length $(x; y)$ joining x to y: Further the continuous function $s(x)$ attains a positive minimum value on the compact set E_R and we take s to be this minimum.

if $E_r = S_r$ is true for $r = R$ and if $R < r(a; y) \quad R + s$ there exists an $x \in E_r$ such that $(a; x) = R$ and $(x; y) = (a; y) \quad R \quad s$: Consequently there exists a geodesic arc L' of length $(a; x)$ joining a to x and a geodesic by L' and L'' joins a to y and has its length $(a; y)$: This composite arc is a geodesic arc and y is thus joined to a by a geodesic arc whose length is equal to the distance of y from a:

Hence $y \in E_R$; and the range of validity of $E_r = S_r$ is thus extended from E_R to E_{R+s}: We have proved incidentally that hypothesis (c) implies that any two points of S can be joined by a geodesic arc whose length is equal to their distances.

Suppose we are now given a bounded set of points of M on S: Clearly we can nd some R such that M is contained in S_R and since $S_R (= E_R)$ is compact, it follows that M is relatively compact.

Thus, we have proved that the condition (b) implies (c) and hence all the three conditions are equivalent.

Theorem 14.3. On a complete surface any two points can be joined by a geodesic arc whose length is equal to their distance.

Let Us Sum Up:

In this unit, the students acquired knowledge to

- the concept of complete surfaces.
- the characterization of complete surfaces.

Check Your Progress:

1. De ne metric spaces.
2. De ne length of the segment.
3. Explain characterization of complete surfaces.

Choose the correct or more suitable answer:

1. The only compact surfaces with constant Gaussian curvature are $\ldots\ldots$

   (a) straight lines.

   (b) circles.

   (c) spheres.

   (d) parabolas.

Answer:

(1) c

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).

2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

Block-V

UNIT-15

HILBERT'S THEOREM

Structure

Objective

Overview

15. 1 Hilbert's theorem

15. 2 Conjugate points on geodesics

Check Your Progress

Let us Sum Up

Suggested Readings

Objectives

After completion of this unit, students will be able to

- derive Hilbert's theorem.
- derive Jacobi's theorem.

Overview

In this unit, we will explain the derivation of Hilbert's theorem and Bonnet theorem.

15.1. Hilbert's theorem:

The following notion of universal covering space of a given space is being used for proving the following theorem:

Let P be a point on the surface $S;$ and let Q be the set of all paths of S which begin at $P.$ Let us divide the set Q into classes, putting into each class the totality of paths that are homotopically equivalent.

Let S^0 denote the set of these classes, so that a point of S^0 is an equivalence class of paths of $S:$

There is a natural mapping of the set S^0 on the space $S:;$ for if A is a point on $S^0;$ then all the equivalent paths in S belonging to A must end in the same point $a;$ and we write $a = (A):$ It is shown that the set of points S^0 can be considered as forming a surface called the universal covering space which has the following properties:

(1) The natural mapping of S^0 on S is a continuous open mapping. Moreover, is locally homeomorphic mapping, i:e:; for every point A of S^0 there exists a neighbourhood U such that the mapping is homeomorphic on the neighbourhood U :

(2) The universal covering of surface S^0 of a surface S is always simply connected.

Property (1) implies that S and S^0 are locally homeomorphic so that all the local properties of the space S are automatically true for $S^0:$ Moreover, the di erential geometric structure on S induces a di erential-geometric structure on $S^0:$

Theorem 15.1. A complete analytic surface free from singularities, with constant negative Gaussian curvature, cannot exist in three dimensional Euclidean space.

Note 15.1. We have already seen that a compact surface with these properties cannot exist, but here the condition of compactness is relaxed to completeness and hence the proof is quite di cult.

Proof. Let us prove the theorem by contradiction. i:e:; Assume that there exists a surface S exists having the required property

Consider an arbitrary geodesic line on the surface S and taken an arbitrary point O on the geodesic as origin.

If s denote the arc length of this geodesic measured from O; since S is complete, the geodesic can be continued in both the direction from $+\infty$ to $-\infty$. It is possible that the geodesic will ultimately cross itself so that the same point on S will have two different s-values.

However, if we consider instead of S its universal covering surface S'; then different values of s will correspond to different point on S'. This follows because on a surface of a negative Gaussian curvature two geodesics arcs cannot enclose a simply connected region.

At each point of parameter s on the given geodesic, consider the orthogonal geodesic line and let its arc length t be chosen as parameter so that the equation of geodesic is $t = 0$.

Now two of these geodesic arc at s_1, s_2 cannot meet on the surface S in order to form with the geodesic arc s_1, s_2 a simply connected region. For if, this were the case, then the sum of the angles of the geodesic triangle so formed would not be less than 2π; which is a contradiction.

Let us denote a point in the covering space S' by the pair of coordinates (s, t) and it can be seen that different pairs (s, t) correspond to different point on S'. Now, we show that every point of S can be represented on the covering surface S' in this manner.

It follows from Minding theorem that the line element of the surface assumes the form $ds^2 + G(s)dt^2$.

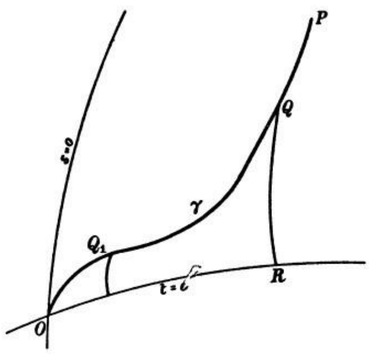


Figure 15.1

Suppose now that a point P on the surface S remained uncovered by our construction (see Fig.(15.1)). Joint P to O $(s = 0, t = 0)$ by some rectifiable curve γ:

Then there must be some point Q on with the property that all points between O and Q can be covered, while points on arbitrarily near Q on the side of Q remote from O cannot be covered. If Q_1 lies on between O and Q it follows from the form of the metric that the length of the curve OQ_1 is greater than or equal to s_{Q_1}; where s_{Q_1} is the s-coordinate of the corresponding point on S^0:

The set of values s_{Q_1} is bounded, and we define s_Q to be the least upper bound of this set.

Let R be the point on the geodesic $t = 0$ distant s_Q from O; and consider the orthogonal geodesics along some interval on the geodesic $t = 0$ which contains R:

These geodesics will cover a strip of the surface which certainly contains the point Q; and the points beyond Q on the curve which gives a contradiction and hence we conclude that every point of the surface S can be covered in this way.

Thus there is a local homeomorphism between points of S and the (s-t) plane, but this correspondence may not be (1-1) in the large. However, the covering space S^0 is homeomorphic with the (s-t) plane.

Consider the asymptotic lines on the surface S:

These lines are given by the differential equation $Lds^2 + 2Mdsdt + Ndt^2 = 0$:

Since $K < 0$; we conclude that $LN \quad M^2 < 0$ and hence that at each point of S; the asymptotic directions are real and different. Hence at each point of S^0 these determine two distinct directions, and similarly at each point of the (s-t) plane.

Since the (s-t) plane is simply connected, the differential equation gives rise to two vector fields which can be continued over the whole plane.

The Lipschitz condition for uniqueness of the solution of the differential equation is satisfied for we have assumed that S is of class w:

Thus throughout the (s-t) plane there are two systems of asymptotic lines with the property that a curve from each system passes through an arbitrary point. Further since S is free singularities, the differential equation has no singularities.

Therefore, from the theorem of Bendixon that each asymptotic lines can be prolonged to an arbitrary extent in both directions and if denotes the arc length.

$$\lim \; s^2 + t^2 = 1; \qquad \lim \; s^2 + t^2 = 1$$

Now, let us prove that each asymptotic line of one system cuts each asymptotic line of the other system in exactly one point. First we prove that two such lines cut in at most one point. Suppose this is not so, then there would be region of the (s-t) plane bounded by two asymptotic lines of di erent systems.

Consider the rst case when the asymptotic lines meet at A and B such that the continuation of the lines does not contain any interior point of the region bounded by the two lines. Let P be a point on one of the lines lying between A and B; and consider the asymptotic line of the second system which passes through P. Because this second line through P cannot intersect the line AB belonging to the same system, it follows that it must intersect the line AB of the opposite system in a further point Q. Moreover, as P moves from A towards the end B; so Q will move from B towards the end A. There must be one point where P and Q coincide, at that point the asymptotic directions will coincide. This contradicts the fact that $K < 0$.

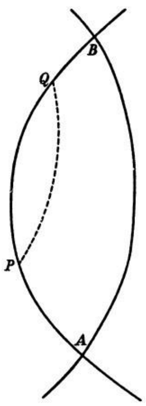


Figure 15.2

Consider now the second case, where by continuation of the asymptotic lines at least one line penetrates the region bounded by the two asymptotic lines (see Fig.((15.3)).

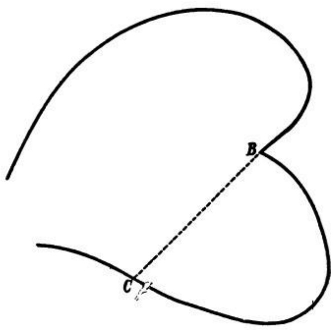


Figure 15.3

Then this asymptotic line will meet the line of the opposite system at a second point C.

Then the continuation BC together with the asymptotic line BC form a system of the type discussed above and again contradiction arises.

Thus, we have proved that each asymptotic line of one system cannot meet each asymptotic line of the other system is more than one point.

In order to prove such lines must meet in atleast one point, it is convenient to refer to the asymptotic lines as parameter lines.

Suppose that N is a neighbourhood of S in which the line of curvature are chosen as parametric lines.

If $_a; \, _b$ denote the principal curvature at a point P on N and if $K = \frac{1}{a^2}$ is the constant negative Gaussian curvature, we can write

$$_a = a^{1} \cot \; ; \quad _b = a^{1} \tan \; ; \quad 0 < \; < \frac{}{2} \qquad (15.1)$$

Using an argument similar to section (15.6). we get

$$\left.\begin{aligned} \frac{@ \; _a}{@v @} &= \frac{1}{2} \frac{E_2}{E} (_b \quad _a) \\ \frac{_b}{@u} &= \frac{1}{2} \frac{G_1}{G} (_a \quad _b) \end{aligned}\right\} , \qquad (15.2)$$

Using equation (15.1), we get

$$\left.\begin{aligned} \frac{E_2}{E} &= 2 \; _2 \cot \\ \frac{G_1}{G} &= 2 \; _1 \tan \end{aligned}\right\} , \qquad (15.3)$$

Upon integration, we get

$$E = U(u) \sin^2 \; ; \quad G = V(v) \cos^2 \qquad (15.4)$$

where $U(u); V(v)$ are certain functions of u and v respectively.

By means of a suitable reparametrization, the function may be taken as unity and the rst fundamental form becomes

$$\sin^2 du^2 + \cos^2 dv^2$$

In terms of the new parameters

$$\begin{aligned} L &= \; _a E = a^{1} \sin \cos \; ; \\ N &= \; _b G = a^{1} \sin \cos \\ M &= 0 \end{aligned}$$

and the asymptotic lines are given by $du^2 \quad dv^2 = 0$:

Choose new parameters ; where $= \frac{1}{2}(v + u)$; $= \frac{1}{2}(v \quad u)$:

Then, the parametric curves = constant, = constant are asymptotic lines.

Moreover, the metric assumes the form

$$d^2 + 2\cos 2 \; d \; d + d^2 \qquad (15.5)$$

and ; measures the arc lengths of the asymptotic lines.

Through O of the (s-t) plane there pass two asymptotic lines.

Through each point on these two lines we draw the asymptotic line of opposite system.

Then we prove that each point of the (s-t) plane lie on one asymptotic line of each system.

Suppose that there is a point P on the plane which cannot be reduced in this way. Join P to O by a continuous curve with the property that each pair of lines from di erent systems cut in a single point in this neighbourhood. Consider a point Q_0 lying in this neighbourhood and let the asymptotic lines through Q_0 cut the coordinate curves $= 0$; $= 0$ in two points $Q_0^{(1)}; Q_0^{(2)}$ respectively.

Let Q_i denote a typical point which lines on becomes Q_0 and Q: Let the asymptotic lines through Q_i meet the coordinate curves at $Q_i^{(1)}; Q_i^{(2)}$ and let these lines meet the lines through Q_0 in $Q_i^{(1)}$ and $Q_i^{(2)}$ (see Fig. (15.6)).

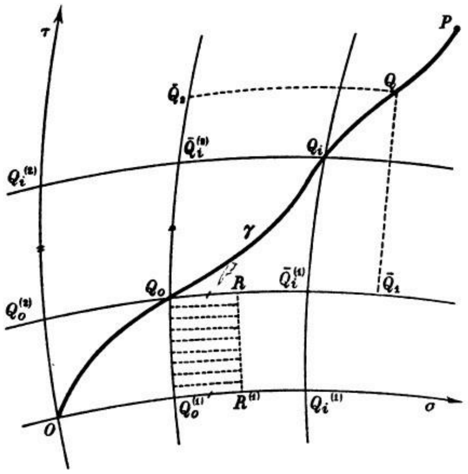


Figure 15.4

Then $\overline{Q_0Q_i}^{(1)} = Q_0^{(1)}Q_i^{(1)}$ and $\overline{Q_0Q_i}^{(2)} = Q_0^{(2)}Q_i^{(2)}$; provided Q_i lies in a neighbourhood of Q_0 where the line element is of the form given by (15.5).

Any asymptotic lines which cuts $Q_0\overline{Q_i}$ lies between Q_0 and $\overline{Q_i}$ which is su ciently close to Q_0 will cut equal lengths from all asymptotic lines which meet $Q_0Q_0^1$:

Suppose if these were not true for all the asymptotic lines meeting $\overline{Q_0Q_i}$ such that all points between Q_0 and R possess this property, but there are points arbitrarily close to R (may be R itself) which does not hold this property. The asymptotic line through R will intersect the coordinate line $= 0$ in the point $R^{(1)}$ such that the lengths $Q_0R;\ Q_0^{(1)}R^{(1)}$ are equal and further all the asymptotic lines between $Q_0^{(l)}$ and Q_0 will have equal lengths intercepted by the asymptotic line through R:

Let us measure o from all these asymptotic lines the length Q_0R in the direction of increasing :

Now, we assert that the end points of these segments form an asymptotic line. This is clearly the case when we consider neighbourhoods of points on the line $RR^{(1)}$ and make use of the net of asymptotic lines in this neighbourhood.

It is true for all asymptotic lines which meet Q_0Q_1 in a neighbourhood of R: In particular it is true for the asymptotic lines through $\overline{Q_1}$ and so for those in a certain neighbourhood of $\overline{Q_1}$; which contradicts to our hypothesis.

Thus the two asymptotic lines through O will cut an arbitrary asymptote line in the plane, and since the point O has been chosen arbitrarily, it follows that each asymptotic line of one system meets every asymptotic line of the other system in exactly one point,. We can take (;) as coordinates for points in the whole plane and the metric is of the form $d\ ^2 + 2\cos 2\ d\ d\ + d\ ^2$:

Let ω be the angle between the parametric curves.

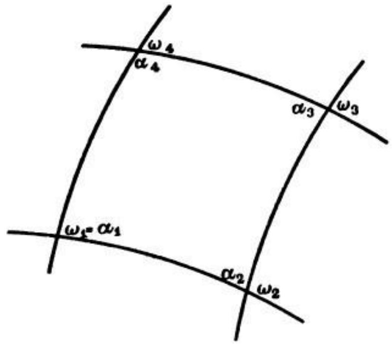


Figure 15.5

Then $\cos\omega = \frac{F}{\sqrt{EG}} = \cos 2\theta$

Here $F = \cos 2\theta$; $E = 1$; $G = 1$

$\Rightarrow \omega = 2\theta$ and hence $0 < \omega < \pi$

Now, using $K = \frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$ for the calculating Gaussian curvature and thus we have $K = \frac{1}{a^2}$:

Also, $\frac{\partial^2\omega}{\partial\alpha\,\partial\beta} = K\sin\omega$:

Consider now the quadrilateral formed by the asymptotic lines $\alpha = \alpha_1, \alpha_2$; $\beta = \beta_1, \beta_2$: (see Fig. (15.5)).

$$\text{Total curvature} = \iint K\,ds = \iint K\sin\omega\,d\alpha\,d\beta$$

$$= \omega_1 - \omega_2 + \omega_3 - \omega_4$$

$$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi$$

Thus, it follows that the absolute magnitude of the total curvature of an arbitrarily large region cannot exceed 2π :

Let us now consider the first form of metric $ds^2 + G(s)dt^2$:

Thus, we have

$$K = -\frac{1}{2\sqrt{G}}\frac{\partial}{\partial s}\left(\frac{G_s}{G}\right) \qquad * \; H = \sqrt{EG} = \sqrt{G} \text{ as } E = 1 \text{ :}$$

and $\sqrt{G} = \cosh\left(\frac{s}{a}\right)$:

The total curvature over a region bounded by parametric lines $x = \pm l$; $t = \pm l$ is

$$\iint K\,ds = \iint K\sqrt{G}\,ds\,dt = \iint -\frac{1}{2}\frac{\partial}{\partial s}\left(\frac{G_2}{\sqrt{G}}\right)ds\,dt$$

$$= -\frac{4l}{a}\sinh\frac{l}{a}$$

But in magnitude this tends to ∞ as $l \to \infty$ which contradicts our earlier assertion that the absolute magnitude of the total curvature cannot exceed 2π :

This completes the proof of the Hilbert's theorem.

15.2. Conjugate points on geodesics:

In earlier chapter, we have studied that if there exists a curve of shortest distance between two points on a surface, then the curve is necessarily a geodesic.

Now, we are going to consider the case whether a given geodesic joining two points is necessarily the shortest distance between them. The following theorem proves that this is the case when the given geodesic can be embedded in a eld of geodesics.

De nition 15.1 (Field of Geodesics). By a eld of geodesics is meant a one-parameter set of geodesics, de ned over a region R of a surface such that through each point of R passes one and only one curve of the set.

Theorem 15.2. If P and Q are two points of a geodesic which can be embedded in a eld of geodesics, then the arc PQ of the geodesic is shorter than any other arc which joints P and Q and lies entirely in that region of the surface covered by the eld.

Proof. Let us choose parameters so that the geodesics of the family are the curves $v =$ constant with $v = v_0$ as the given geodesic. Let the curves $u =$ constant be geodesics parallels orthogonal to them, then the metric reduces to the form $ds^2 = du^2 + {}^2 dv^2$:

If the coordinates of P and Q are $(u_1; v_0); (u_2; v_0)$ with $u_2 > u_1$; the length of the geodesic arc PQ is $(u_2 \; u_1)$:

Let C be an arbitrary curve passing through P and Q given by the equation $v = \;(u)$ where $\;(u_1) = v_0; \quad (u_2) = v_0$:

Then the arc length of C is

$$I = \int_{u_1} \left[1 + {}^2 \left(\frac{d}{du} \right)^2 \right]^{1=2} du$$

Clearly I exceeds $u_2 \; u_1$ unless d fracd du $= 0$ when C is the given geodesic.

Note 15.2. However, it is most unlikely that the region R of the geodesic eld extends over the entire surface S:

In general, the above argument cannot be applied to complete surface.

For instance, the surface of a sphere cannot be covered by a geodesic eld because any two great circles intersect in two points of the sphere. Moreover, if $A; B$ are any two non-antipodal points, the geodesic arc which the longer part of the great circle joining A and B and clearly is not the shortest distance from A to $B.$

Theorem 15.3. When the surface S has negative curvature everywhere, the length of a geodesic which joins any two points $A; B$ is always less than the lengths of the neighbouring curves through A and $B.$

Proof. Let us now consider two systems, one of them is system of parametric curves be the geodesics normal to the given geodesics AB and the other system be the orthogonal trajectories. Let u denote the length of the geodesic normal PQ from P to AB and v denote the length $AQ.$

The lime element of the surface becomes $ds^2 = du^2 + {}^2dv^2;$ where $(0; v) = 1; \quad {}_1(0; v) = 0.$

In terms of these parameters the Gaussian curvature is given by $K = \frac{\ \ _{11}}{}$ i.e.; $\ _{11} = \ K.$

The function can be expanded as a power series in $u;$ we get in the form $= 1 \quad K\frac{u^2}{2} \quad K_1\frac{u^3}{6} + O(u^4)$ where K and K_1 are evaluated with $u = 0.$

A neighbouring curve APB which di ers slightly from AB will have an equation of the form $u = (v)$ where u will be small.

The length of this curve will be

$$I = \int_B^A \left({}^{o\,2} + {}^2 \right)^{1=2} dv = \int_B^A \left[1 + {}^{o\,2} \quad K \ {}^2 \quad \frac{1}{3}K_1 \ {}^3 \right]^{1=2} dv$$

where terms of the forth order are neglected.

Let us assume that ^{o} never becomes in nite and is thus of the same order of smallness as $u.$

Hence $l \quad s = \frac{1}{2}\int_B^A \left[{}^{o\,2} \quad K \ {}^2 \quad \frac{1}{3}K_1 \ {}^3 \right]^{1=2} dv.$

The sign of variation of the arc length will be given by the second order term, provided that these do not vanish identically.

If only these terms are retained, we have

$$l \quad s = \int_B^A \left({}^{o\,2} \quad K \ {}^2 \right) dv \qquad (15.6)$$

Now, if K is always negative, the integrand is always positive and hence we have $l > s$.

This completes the proof of the theorem.

Note 15.3. Now, we shall consider the analogous problem, when K is not always negative.

Lemma 15.1 (Erdmann's lemma). For an extreme value, in addition to the equation of Euler, it is necessary that $f_{+y'} = f_{-y'}$

Proof. Consider the problem of finding a curve $y = y(x)$, which passes through two points $(x_1, y_1), (x_2, y_2)$, has a discontinuity of slope on the line $x = x_1$ and is such that the integral $J = \int_{x_1}^{x_2} f(x, y, y')\, dx$ assumes an extreme values.

$$\text{Let } y'_+ = \lim_{\epsilon \to 0} y'(x_3 + \epsilon)$$
$$y'_- = \lim_{\epsilon \to 0} y'(x_3 - \epsilon) \quad \text{where } \epsilon \text{ is positive.}$$

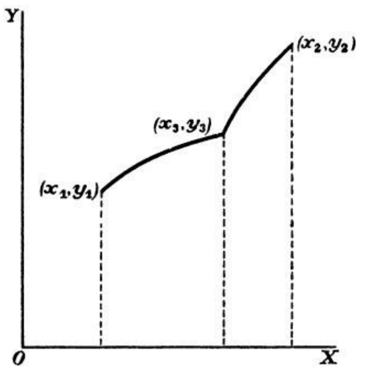


Figure 15.6

The variation of the integral over the curve $y(x)$ and $y + \alpha\eta(x)$, where $\eta(x_1) = 0$, $\eta(x_2) = 0$ is given by

$$J(\alpha) = \int_{x_1}^{x_3} f(x, y + \alpha\eta, y' + \alpha\eta')\, dx + \int_{x_3}^{x_2} f(x, y + \alpha\eta, y' + \alpha\eta')\, dx$$

It is assumed that the corner still moves along the line $x = x_3$.

The necessary condition for extrema is $J'(0) = 0$.

Thus, it reduces to

$$\int_{x_1}^{x_3} \eta\left(f_y - \frac{d}{dx} f_{y'}\right) dx + \int_{x_3}^{x_2} \eta\left(f_y - \frac{d}{dx} f_{y'}\right) dx + \eta_3\left(f_{-y'} - f_{+y'}\right) = 0$$

In addition to Euler's equation $f_y \quad \frac{d}{dx} f_{y'} = 0$, we have the necessary condition is $f_{y'} = f_{+y'}$:

Thus the lemma is proved.

Theorem 15.4 (Jacobi). In order that the geodesic distance AB should be the shortest distance it is necessary and su cient condition that B lies between A and its conjugate point A_1:

Proof. From equation (15.6), it follows that geodesic distance s is a minimum provided that $\;^2(s) = \frac{1}{2}\int_A^B \left(u'^2 \quad Ku^2\right) dv$ is non-negative.

If $\;^2(s) \quad 0$ for all u; then the curve $u = 0$ must make the integral $\int_A^B \left(u'^2 \quad Ku^2\right) dv$ a minimum. It can be easily veri ed that, except for notation, the Euler equation corresponding to this is Jacobi's di erential equation.

Now assume that the geodesic distance AB still gives the shortest distance with B lying beyond A_1 i:e:; $\;^2(s) \quad 0$ and thus a contradiction arise.

By hypothesis there is a solution of Jacobi's di erential equation (and hence for Euler's equation) which vanishes at A and has its next zero at A_1: If $u = \;(v)$ is such a solution, then of course is $u = \;(v)$ for an arbitrary constant :

De ne a new function $\bar{u}$ which coincides with $u = \;(v)$ from A to A_1 and is identically zero from A_1 to B:

Our aim is to prove that such a function $\bar{u}$ is a corner solution of the problem of giving $\;^2(s)$ an extreme value.

$$\text{Since} \int_A^{A_1} uu'' \, dv = \left[uu'\right]_A^{A_1} \quad \int_A^{A_1} u'^2 dv = \quad \int_A^{A_1} u'^2 dv \quad \text{where } u = \;(v)$$

$$\text{It follows that} \int_A^B \left(\bar{u}'^2 \quad K\bar{u}^2\right) dv = \int_A^{A_1} \left(u'^2 \quad Ku^2\right) dv$$

$$= \quad \int_A^{A_1} u\left(u'' + Ku\right) dv$$

$$= 0 \quad * \; u'' + Ku = 0$$

Since $\bar{u}$ satis es the condition $\;^2(s) = 0$ and can be chosen as near to the curve $u = 0$ as we please since is arbitrary it follows that $u = 0$ gives $\;^2(s)$ is minimal value.

Moreover, u must be a corner solution of the problem of nd a minimum $\;^2(s)$:

From Erdmann's lemma, the necessary condition is $u'_+ = u'_-$.

But this is quite impossible because there is no non-trivial solution of the equation $u'' + Ku = 0$ which vanishes simultaneously with its derivative.

This gives the required contradiction and the theorem is completely proved.

Now, we are going to state the Sturm's theorem without proof which will be use to prove the Bonnet theorem.

Theorem 15.5 (Sturm's theorem). Consider the two distinct differential equations $\frac{d^2V}{dx^2} = HV$;
$\frac{d^2V}{dx^2} = H'V$ where for all values of x in the range considered, $H'(x) \geq H(x)$.

Then If $\phi(x)$ is a solution of the first equation having two consecutive zeros at x_0 and x_1, a solution of the second equation which has a zero at x_0 cannot have another zero in the closed interval $[x_0, x_1]$.

Corollary 15.1. If for all values of x in the range considered $H'(x) \leq H(x)$, and if $\phi(s)$ is a solution of the first equation having two consecutive zeros at x_0 and x_1, then any solution of the second equation which has a zero at x_0 must have at least one other zeros in the interval $[x_0, x_1]$.

Theorem 15.6 (Bonnet). If along a geodesic the Gaussian curvature exceeds a positive constant $\frac{1}{a^2}$ then the curve cannot be the shortest distance between its extremities along an arc exceeding πa.

Proof. Consider Jacobi's differential equation $\frac{d^2p}{dv^2} + kp = 0$ which is of the type considered by sturm.

Let p be a solution of the equation and let v_0, v_1 be two consecutive zeros corresponding to the point A and A_1.

Thus, the arc AB will be the shortest distance between A and B if and only if B lies between A and A_1 (by using Jacobi's theorem).

Suppose the Gaussian curvature along the line AA_1 always exceeds the positive constant $\frac{1}{a^2}$, so that $K \geq \frac{1}{a^2}$.

The solution of the equation $\frac{d^2p}{dv^2} = -\frac{p}{a^2}$ which vanishes for $v = v_0$ is $C \sin\left(\frac{v - v_0}{a}\right)$ and its next zero after v_0 is just $v_0 + \pi a$.

Thus, if the arc length AB exceeds $a;$ then B will not lie between A and A_1 and hence the theorem is proved.

Theorem 15.7. If at all points of a geodesic the Gaussian curvature is less than $\frac{1}{b^2};$ then the curve is necessarily of shorter length neighbouring curves along an arc length at least equal to $b:$

Proof. Given that $K \quad \frac{1}{b^2}:$

We know that the interval between consecutive roots of the equation $\frac{d^2p}{dv^2} = \quad \frac{p}{a^2}$ is $ab:$ This cannot be smaller than the interval between consecutive roots of previous equation.

Thus, if the arc length AB is less than $b;$ then B will certainly lie between A and $A_1:$

This completes the proof of the theorem.

Theorem 15.8. If on a compact surface $S;$ the curvature everywhere exceeds $\frac{1}{a^2};$ the maximum distance between any two points cannot exceed $a:$

Proof. Given that the surface S is compact and has the property that $K \quad \frac{1}{a^2}$ everywhere.

Thus, if A and B are any two points on S there is a geodesic joining A to B which is of shorter length than the neighbouring curve.

By Bonnet theorem, the maximum distance between A and B cannot exceed a

Let Us Sum Up:

In this unit, the students acquired knowledge to

derive Hilbert's Theorem.

derive Bonnet's theorem .

derive Erdamann's lemma.

Check Your Progress:

1. State and Prove Hilbert's theorem.
2. De ne eld of Geodesics.
3. State and Prove Erdamnn's lemma.
4. State and Prove Jacobi's theorem.
5. State Sturm's theorem.

Suggested Readings:

1. T.J. Willmore, *An Introduction to Di erential Geometry* , Oxford University press, (17th Impression), New Delhi, 2002. (Indian Print).
2. C.E.Weatherburn, *Di erential Geometry of Three Dimensions* , University Press, Cambridge, 1930.

zzzzz