# Master of Science Mathematics

## (M.Sc. Mathematics)

## MMT-201

## Applied Mechanics

## Semester-II

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# Suresh Gyanvihar University
## Department of Mathematics
## School of Science

**M.Sc., Mathematics - Syllabus–I year–IISemester (ODL Mode)**

**COURSE TITLE** : APPLIED MECHANICS

**COURSE CODE** : **MMT-201**

**COURSE CREDIT** : **4**

**COURSE OBJECTIVES**

While studying the **APPLIED MECHANICS**, the Learner shall be able to:

Tostudythe fundamentals of wave mechanics.

CO 1: Discuss the kinetic energy of a rigid body with respect to a fixed point.

CO 2: Review about the principles of Angular momentum

CO 3: Represent analytical Method to describe the motion.

CO4: Predict the non-holonomic system with moving constraints through Lagrange'sequation.

CO 5: Describe Hamilton's principle of applied mechanics.

**COURSE LEARNING OUTCOMES**

After completion of the **APPLIED MECHANICS**, the Learner will be able to:

CLO 1: Enable to find the angular momentum of a rigid body.

CLO 2: Describe the general motion of a rigid body.

CLO 3: Enable to describe the steady precession of a spinning top.

CLO 4: Demonstrate an understanding of intermediate applied mechanics topics such as conservative system, ignorable coordinates, Lagrange's mechanics and Hamilton's mechanics

CLO 5: Evaluate the motion of macroscopic objects from projectiles to the pass of machinery as well as astronomical objects on the qualitative structure of phase space

**BLOCK I:KINEMATICS**

Kinematics of a particle and a rigid body - Moments and products of inertia - Kinetic energy -Angular momentum.

**BLOCK II:METHODS OF DYNAMICS IN SPACE**

Motion of a particle - Motion of a system - Motion of a rigid body.

**BLOCK III:APPLICATIONS OF DYNAMICS IN SPACE**

Motion of a rigid body with a fixed point under no forces - Spinning top - General motion of top.

**BLOCK IV:EQUATIONS OF LAGRANGE AND HAMILTON**

Lagrange's equation for a particle - Simple dynamical system - Hamilton's equations.

**BLOCK V:HAMILTONIAN METHODS**

Natural Motions - Space of events - Action - Hamilton's principle - Phase space - Liouville's theorem.

**REFERENCE BOOKS :**


1. Synge L. and Griffith B.A., "Principles of Mechanics", Tata McGraw Hill, 1984.
2. Rana N.C. and Joag P.S., "Classical Mechanics", Tata McGraw Hill, 1991.
3. Berger V.D. and Olsson M.G., "Classical Mechanics - a modern perspective", Tata McGraw Hill International,1995.
4. Bhatia V.B., "Classical Mechanics with introduction to non-linear oscillations and chaos",Narosa Publishing House, 1997.
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# BLOCK - I



# BLOCK - II



# BLOCK - III



# BLOCK - IV

# BLOCK - V

Unit - 13: Natural Motions.

Unit - 14: Phase Space.

Unit - 15: Poisson Brackets.

# Contents

# BLOCK-I

# UNIT 1

# Kinematics of a Particle



**Objectives**

Upon completion of this Unit, the student is exposed to

- the notions of velocity and acceleration in cylindrical coordinates.
- composition of velocities and acceleration.

## 1.1 Introduction

Let $OX, OY, OZ$ be the rectangular axes fixed in a frame of reference and $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along them. We define the following vectors for any particle $P(x, y, z)$.

Position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$

$$\text{Acceleration} \quad \vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}. \tag{1.1.1}$$

It is required that the components of velocity and acceleration be expressed in directions other than $\vec{i}\ \vec{j}, \vec{k}$. In the following section, we consider two cases of components of velocity and acceleration.

**Velocity and Acceleration in Tangential and Normal Components**

Let $C$ be the path of a moving particle $P$ and let $Q$ be a fixed point on $C$. The arc length $QP$ is denoted by $s$. From (1.1.1) the vector $\frac{d\vec{r}}{dt}$ has components $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ along $\vec{i}, \vec{j}, \vec{k}$. It is the unit tangent vector to $C$ at $P$ and will be denoted by $\vec{T}$.

For the velocity of $P$,

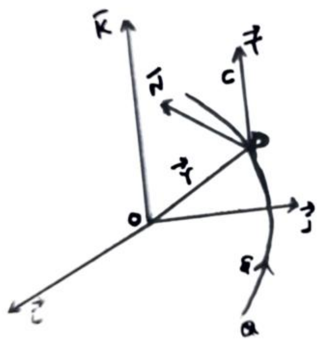
$$\vec{v} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \dot{s}\vec{T} \tag{1.1.2}$$

"The velocity of a particle is directed along the tangent to its path, and has magnitude $\dot{s}$."

For acceleration, we have

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} \\ &= \ddot{s}\vec{T} + \dot{s}\frac{d\vec{T}}{dt} \\ \vec{a} &= \ddot{s}\vec{T} + \dot{s}^2\frac{d\vec{T}}{ds} \end{aligned}$$

But $\frac{d\vec{T}}{ds} = \frac{\vec{N}}{\rho}$, where $\vec{N}$ is the unit principal normal vector and $\rho$ is the radius of curvature of



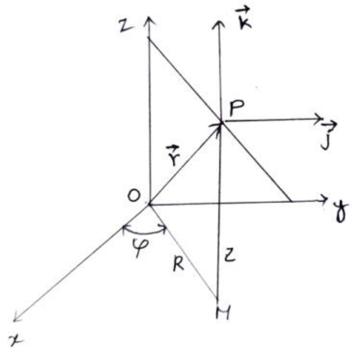
$C$ at $P$.

$$\therefore \vec{a} = \ddot{s}\vec{T} + \frac{\dot{s}^2}{\rho}\vec{N}. \tag{1.1.3}$$

Hence "the acceleration of a particle lies in the osculating plane to its path", the components in the directions of the tangent and principal normal are $\ddot{s}$ and $\frac{\dot{s}^2}{\rho}$ respectively.

## 1.2 Components of Velocity and Acceleration in cylindrical co-ordinates

Let $P$ be the position of a particle at time $t$ and $M$ be the foot of the perpendicular from $P$ on the plane $OXY$. The polar co-ordinates $(R, \psi, z)$ of $M$ together with the $z-$ co-ordinate of $P$ are the cylindrical co-ordinates $(R, \psi, z)$ of $P$. Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors at $P$ in the




directions of the parametric lines of these co-ordinates.

To find the components of velocity and acceleration of $P$ along $\vec{i}, \vec{j}, \vec{k}$. The vector $\vec{k}$ is constant in magnitude and direction. The $\vec{i}$ and $\vec{j}$ directions do not depend on $R$ and $z$, however they are dependent on $\psi$. We have

$$\frac{d\vec{i}}{d\psi} = \vec{j} \quad and \quad \frac{d\vec{j}}{d\psi} = -\vec{i} \tag{1.2.1}$$

Now $\vec{r} = \overrightarrow{OP} = R\vec{i} + z\vec{k}$.

Differentiating $\vec{r}$ with respect to $t$, we have

$$\begin{aligned} \bar{v} &= \frac{d\vec{r}}{dt} \\ &= \dot{R}\vec{i} + R\frac{d\vec{i}}{dt} + \dot{z}\vec{k} \\ &= \dot{R}\vec{i} + R\frac{d}{d\psi}\frac{d\psi}{dt}\vec{i} + \dot{z}\vec{k} \end{aligned}$$

$$\vec{v} = \dot{R}\vec{i} + R\dot{\psi}\vec{j} + \dot{z}\vec{k} \tag{1.2.2}$$

This equation provide the components of velocity in the directions of $\vec{i}, \vec{j}, \vec{k}$.

Again differentiating (1.2.2) with respect to $t$, we get

$$\begin{aligned}
\vec{a} &= \ddot{R}\vec{i} + \dot{R}\frac{d\vec{i}}{dt} + R\dot{\psi}\frac{d\vec{j}}{dt} + \dot{\psi}\dot{R}\vec{j} + R\ddot{\psi}\vec{j} + \ddot{z}\vec{k} \\
&= \ddot{R}\vec{i} + \dot{R}\frac{d}{d\psi}\frac{d\psi}{dt}\vec{i} + R\dot{\psi}\vec{j} + \dot{\psi}\dot{R}\vec{j} + R\ddot{\psi}\vec{j} \\
&= \ddot{R}\vec{i} + \dot{R}\dot{\psi}\vec{j} + R\dot{\psi}^2(-\vec{i}) + \dot{z}\vec{k}, \ from \quad (\textbf{??}) \\
&= (\ddot{R} - R\dot{\psi}^2)\vec{i} + (\dot{R}\dot{\psi} + \dot{\psi}\dot{R} + R\ddot{\psi})\vec{j} + \ddot{z}\vec{k} \\
\vec{a} &= (\ddot{R} - R\dot{\psi}^2)\vec{i} + (\dot{R}\dot{\psi} + \dot{\psi}\dot{R} + R\ddot{\psi})\vec{j} + \ddot{z}\vec{k}
\end{aligned}$$

$$\vec{a} = (\ddot{R} - R\dot{\psi}^2)\vec{i} + \frac{1}{R}\frac{d}{dt}(R^2\dot{\psi})\vec{j} + \ddot{z}\vec{k} \tag{1.2.3}$$

The above equation gives the components of acceleration in the direction of $\vec{i}, \vec{j}, \vec{k}$.

## 1.3 Composition of Velocities and Accelerations

It is often required to connect the velocities (or accelerations) of a particle relative to two different frames of reference $S$ and $S'$. We consider the case where there is no relative rotation of the frames. Let $O$ be a fixed point in $S$ and $O'$ be a point fixed in $S'$. A particle $P$ having position vectors $\vec{r} = \overrightarrow{OP}$ and $\vec{r}' = \overrightarrow{O'P}$ are connected by

$$\vec{r} = \vec{r_0} + \vec{r}'$$

where $\vec{r_0} = \overrightarrow{OO'}$. On differentiation, we have

$$\vec{v} = \vec{v_0} + \vec{v}' \quad and \quad \vec{a} = \vec{a_0} + \vec{a}' \tag{1.3.1}$$

where $\vec{v}$, $\vec{a}$ are the velocity and acceleration of $P$ relative to $S$, $\vec{v}'$, $\vec{a}'$ are the velocity and acceleration of $P$ relative to $S'$ and $\vec{v}_0$, $\vec{a}_0$ are the velocity and acceleration of $S'$ relative to $S$. The equation (1.3.1) give the laws of composition of velocities and accelerations.

# BLOCK-I

# UNIT 2

# Kinematics of a Rigid Body



**Objectives**

Upon completion of this Unit, students will be able to

x identify the motion of a rigid body with a fixed point.

## 2.1 Motion of a Rigid Body with a Fixed Point

Consider a rigid body constrained to rotate about a fixed point $O.$ Let $t_1$ and $t_2$ be two instants of time. The body receives a displacement in the time interval $t_2 - t_1$ which is equivalent to a rotation $\vec{n}$ about $O.$ If we keep $t_1$ fixed and let $t_2$ approach $t_1,$ the direction of $\vec{n}$ will approach some limiting direction, which is denoted by the unit vector $\vec{i}$.

The ratio of the angle of rotation $n$ to the time interval $t_2 - t_1$ will approach a limiting value $\omega.$ The vector $\vec{\omega} = \omega \vec{i}$ is called the angular velocity of the body at the instant $t_1.$ At this instant, the body is rotating about a line through the origin $O$ in the direction of $\omega.$ This line is known as the instantaneous axis of rotation. The rate of turning is $\omega$ radians per unit time and is a rotation in the positive direction about the instantaneous axis.

The body receives an infinitesimal rotation $\omega dt$ in an infinitesimal time $dt$ and hence the displacement of a particle of the body is $\vec{dr} = \omega dt \times \vec{r},$ where $\vec{r}$ is the position vector relative to $O$. The velocity of this particle is

$$\bar{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}.$$

The above relation gives the velocity of any particle of the body in terms of the angular velocity $\vec{\omega}$. Thus $\vec{\omega}$ is a known vector function of time, we can find the velocity of any particle at any time. As the body turns about $O$, the instantaneous axis occupies different positions in the body. This axis passes through $O$ and hence its locus in the body is a cone with vertex $O$, which is called as "Body cone ".

Similarly, the locus of the instantaneous axis in space is another cone with vertex $O$, which is called as the "space cone. "A rigid body moving parallel to a fundamental plane may be considered as a body turning about a point at infinity. In this case, the body and space cones become cylinders. Their intersections with the fundamental plane are called the body and space centrodes respectively.

We know that the body centrode rolls on the space centrode in the motion of a rigid body parallel to the plane. Similarly the body cone rolls on the space cones in the motion of a rigid body with a fixed point. To prove this result, we must show that

(i) The body cone touches the space cone.

(ii) The particles of the body on the line of constant of the cones are instantaneously at rest.

Let $OP$ be the position of the instantaneous axis of rotation at some instant. It is a generator of both the space cone (fixed) and the body cone (moving). After an infinitesimal time $dt$, another generator $OQ$ of the body comes into coincidence with a generator $OQ'$ of the space cone. But the displacement in the time $dt$ is an infinitesimal rotation of magnitude $\omega dt$ about $OP$ and hence the angle between the planes $OPQ$, $OPQ'$ is an infinitesimal angle. Since these planes represent the tangent planes to the two cones along the generator $OP$, it follows that the tangent planes cannot cut at a finite angle. Therefore the cones must touch which satisfies the first condition. Since all particles of the body on the instantaneous axis $OP$ are instantaneously at rest, the second condition is also satisfied, which completes the proof.

## 2.2 General Motion of a Rigid Body

Let us consider a rigid body moving in a general manner. A particle $P$ of the body is selected as a base point and let its velocity be $v_1$. In an infinitesimal time $dt$ the displacement of the body is equivalent to a translation $v_1 dt$ and a rotation $d\vec{n}$ about $P$. The displacement of any particle $Q$ of the body $v_1 dt + d\vec{n} \times \vec{r}$, where $\vec{r} = \overrightarrow{PQ}$. Therefore the velocity of $Q$ is

$$\vec{V} = \vec{V_1} + \vec{\omega} \times \vec{r} \tag{2.2.1}$$

where $\omega = \dfrac{d\vec{n}}{dt}$.

This velocity consists of two parts: (i) the velocity $v_1$ of the base point and (ii) the velocity $\vec{\omega} \times \vec{r}$ of $Q$ relative to $P$. The velocity of $Q$ relative to $P$ is same as if the body were turning out $P$ (as fixed point) with angular velocity $\vec{\omega}$. If the base point $P$ is altered, then the translation $v_1 dt$ is changed, but the rotation $d\vec{n}$ remains the same. The vector $\omega$ pertains to the motion of the body as a whole and hence it is the angular velocity of the body. As $\omega$ does not depend on the choice of the base point it has be considered as a free vector.

The equation (2.2.1) is the velocity of any point of the body when the angular velocity $\vec{\omega}$ and the velocity $\vec{v_1}$ are known. Thus, the two vectors $\vec{\omega}$ and $\vec{v_1}$ completely describe the motion. From (2.2.1), the acceleration $\vec{a}$ can be found by differentiation.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v_1}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt},$$

where $\dfrac{d\vec{v_1}}{dt}$ is the acceleration $\vec{a_1}$ of the base point $P$. It depends only on the motion of $P$ and not on the angular velocity. The term $\dfrac{d\vec{r}}{dt}$ is the velocity of $Q$ relative to $P$. Hence from

$$\vec{V} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

we have

$$\vec{a} = \vec{a}_1 + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{2.2.2}$$

# BLOCK-I

# UNIT 3

# Moments and Products of Inertia



**Objectives**

Upon completion of this Unit, the students will be able to

- find the principal axes and moments of inertia.
- identify the momental ellipse.

## 3.1 Introduction

**Definition 3.1.1.** The moment of Inertia of a particle about a line $L$ is defined as $I = mp^2$, where $m$ is the mass of the particle and $p$ is the perpendicular distance of the particle from the line $L$. For a system of particles, the moment of inertia is defined as the sum of the moments of inertia of the separate variables. Thus if the system of $n$ particles of masses $m_1, m_2, \cdots, m_n$, located at distances $p_1, p_2, \cdots, p_n$ from the line $L$, then the moment of inertia of the system about the line $L$ is

$$I = \sum_{i=1}^{n} m\, p_i^2$$

**Definition 3.1.2.** Let $A, B$ be two planes and let $p, q$ denote the perpendicular distances from them of a particle of mass $m$. The distance is taken positive or negative according as the particle lies on one side or the other side of the corresponding plane. The product of inertia of the particle with respect to the planes $A$, $B$ is defined by the product $mpq$. For a system of particles, the product of inertia is the sum of the products of inertia of several variables.

If $m$ is the mass of a typical particle, with co-ordinates $(x, y, z)$ in the $XYZ$ plane then the moments of inertia of the system of particles about the axes $Ox$, $Oy$ and $Oz$ are respectively given by

$$A = \sum m(y^2 + z^2), \quad B = \sum m(z^2 + x^2), \quad C = \sum m(x^2 + y^2) \tag{3.1.1}$$

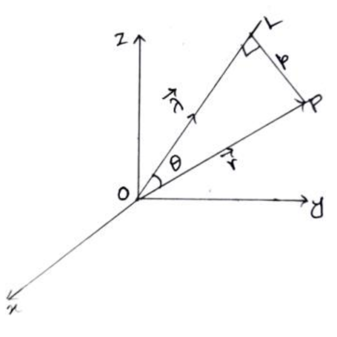
Here the summation extends over all the particles of the system. The product of the inertia with respect to the co-ordinate planes, taken in pairs are

$$F = \sum myz, \quad G = \sum mzx, \quad H = \sum mxy \tag{3.1.2}$$

The summations are replaced by integrations for a continuous distribution of matter. In this case, the mass $m$ is replaced by the mass $\rho dv$ where $\rho$ is the density of a small volume element $dv$.

**Result.1** If the moments of inertia and the products of inertia are known, then we can find the moment of inertia $I$ of the system about any line through $O$.

**Proof.** By definition, $I = \sum mp^2$, where $p$ is the perpendicular distance of a typical particle $P$ (of mass $m$) from the line $L$.




From the figure,

$$\sin\theta = \frac{p}{OP} \Rightarrow p = OP \sin\theta$$

where $\theta$ is the angle between $OP$ and $L$. Thus $p$ equals the magnitude of the vector product $\vec{\lambda} \times \vec{r}$, where $\vec{\lambda}$ is a unit vector along $L$ and $\vec{r} = \overrightarrow{OP}$ The components of $\vec{\lambda}$ are the direction cosines $\alpha, \beta, \gamma,$ of $L$ and the components of $\vec{r}$ are the co-ordinates $x, y, z$ of $P$. Thus the components of $\vec{\lambda} \times \vec{r}$ are

$$\beta z - \gamma y, \quad \gamma x - \alpha z, \quad \alpha y - \beta x.$$

Since $p$ is the magnitude of the vector with these components, we have

$$I = \sum m\left[(\beta z - \gamma y)^2 + (\gamma x - \alpha z)^2 + (\alpha y - \beta x)^2\right] \tag{3.1.3}$$

$$\begin{aligned} I &= \alpha^2 \sum m(y^2 + z^2) + \beta^2 \sum m(z^2 + x^2) + \gamma^2 \sum m(x^2 + y^2) \\ &\quad -2\beta\gamma \sum myz - 2\gamma\alpha \sum mzx - 2\alpha\beta \sum mxy \\ I &= A\alpha^2 + B\beta^2 + C\gamma^2 - 2F\beta\gamma - 2G\gamma\alpha - 2H\alpha\beta. \end{aligned}$$

This gives $I$ in terms of $A, B, C, F, G, H$ and the direction cosines $L$. Hence the moment of inertia of the system about any line through $C$ can be found if $A, B, C, F, G, H$ are known. ■

## 3.2 The Momental Ellipsoid

We obtain the moments of inertia about all lines through $O$ by varying $\alpha, \beta, \gamma$ in (3.1.3). If $I$ is the moment of inertia about the line in consideration, let us measure a distance $OQ = \frac{1}{\sqrt{I}}$ along each line through $O$. The locus of the point $Q$ is

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = 1 \tag{3.2.1}$$

This represents the equation of a quadratic surface with center $O$. Since $I$ does not vanish for any line, the quadratic surface is generally a closed surface. Hence (3.2.1) is the equation of an ellipsoid, which is called the “momental ellipsoid ”at $O$.

By inspecting the coefficients in the equation of momental ellipsoid, we can find the moments and products of inertia with respect to the axes of coordinates(if the equation of momental ellipsoid at point is known). The equation of momental ellipsoid changes from (3.2.1) to the following equation under rotation of axes from $oxyz$ to $ox'y'z$.

$$A'x'^2 + B'y'^2 + C'z^2 - 2F'y'z' - 2G'z'x' - 2Hx'y' = 1 \tag{3.2.2}$$

For the new axes, the moments and products of inertia are given by the coefficients $A'$, $B'$, $C'$, $F'$, $G'$ and $H'$.

**Existence of Principal Axes and Moments of Inertia**

Let $oxyz$ be any axes. Then the moments of inertia $I$ from (3.2.2) attains its maximum value for some line $L_1$. Let this maximum be $I_1$. Let us consider the axes $ox'y'z'$ so that $ox'$ coincides with $L_1$. At present, let the other two axes be unspecified except for the conditions that they shall be perpendicular to $ox'$ and to one another.

Then for any line $L$, the moment of inertia is

$$I = A'\alpha'^2 + B'\beta'^2 + C'\gamma'^2 - 2F'\beta'\gamma' - 2G'\gamma'\alpha' - 2H\alpha'\beta'$$

where $A'$, $B'$, $C'$ $F'$, $G'$, $H'$ are the moments and products of inertia for the axes $Ox'y'z'$ and $\alpha'$, $\beta'$, $\gamma'$ are the direction cosines of $L$ relative to $Ox'y'z'$. If we take $\alpha' = 1$, $\beta' = \gamma' = 0$, then $L$ coincides with $L'$ and hence $A' = I$, the maximum moment of inertia.

We now show that $G' = H' = 0$ is necessary consequence of the fact that $I$ is maximum for $\alpha' = 1$, $\beta' = \gamma' = 0$.

Since $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$, we have

$$I - I_1 = -2\alpha'(G'\gamma' + H'\beta') + (B' - I_1)\beta'^2 + (C' - I_1)\gamma'^2 - 2F'\beta'\gamma' \tag{3.2.3}$$

If we take a line $L$ near $L_1$, $\alpha'$ will be approximately one and $\beta'$, $\gamma'$ are negligibly small.

If at least one of $G'$ $H'$ is different from zero, we choose $\beta'\gamma'$, so that $(G'\gamma' + H'\beta')$ is negative. But since $\beta'$ and $\gamma'$ are small, the sign of the right hand side of (3.2.3) is determined by the first term. Hence $I - I_1$ may be made positive. But this is not possible, since $I_1$ is the maximum of $I$. Therefore, the assertion, we made about $G'$ and $H'$ is false, and we conclude that $G' = H' = 0$.

Therefore, for any line $L$, we have

$$I = I_1\alpha^2 + B'\beta'^2 + C'\gamma'^2 - 2F'\beta'\gamma'.$$

This is true for all axes $o'x'y'z'$, with $ox'$ coinciding on $L_1$.

Now let us consider $oy'$ subject to the condition that, of all lines perpendicular to $ox'$, $oy'$ has the maximum moment of inertia say $I_2$. Then $B' = I_2$ and so for any line $L$.

$$I = I_1\alpha'^2 + I_2\beta'^2 + C'\gamma'^2 - 2F'\beta'\gamma'$$

For any line $L$ perpendicular to $ox'$, $\alpha' = 0$, $\beta'^2 + \gamma'^2 = 1$ and hence

$$I - I_2 = -2F'\beta'\gamma' + (C' - I_2)\gamma'^2$$

If we consider a line near $oy'$, then $\beta'$ will be approximately one and $\gamma'$ will be negligibly small. It is evident that if $F'$ does not vanish, we can make $I$ greater than $I_2$ which is again impossible, since $I_2$ is a maximum. Hence $F' = 0$ and for any line $L$

$$I = I_1\alpha'^2 + I_2\beta'^2 + C'\gamma'^2.$$

From, $\gamma'^2 = 1 - \alpha'^2 - \beta'^2$, we have

$$I - C' = (I_1 - C')\alpha'^2 + (I_2 - C')\beta'^2 \geq 0$$

and so the third moment of inertia $C'$ is the least of all moments of inertia for lines through $o$.

We summarise as "It is always possible to choose rectangular axes $oxyz$ such that the moment of inertia $I$ of a system about a line $L$ through $o$ is given by

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 \tag{3.2.4}$$

where $\alpha, \beta, \gamma$ are the direction cosines of $L$ relative to $oxyz$. ""These axes are called *principal axes* of inertia at $o$, and the moments of inertia $A, B, C$ about them are called *principal moments* *of inertia*". The planes defined by the principal axes are called *principal planes*.

From (3.2.4), there are no terms of $E$, $G$ and $H$ and hence the product of inertia vanishes for any pair of principal planes. The equation of the momental ellipsoid for principal axes is $Ax^2 + By^2 + Cz^2 = 1$.

## 3.3 Method of Finding Principal Axes and Moments of Inertia

Let $ox'y'z'$ be the principal axes and $A', B', C'$ be the principal moments of inertia. Any point $P$ has two sets of coordinates $(x, y, z)$ and $(x', y', z')$ according to the axes. One set of coordinates are the linear functions of the other such that

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

for every $P$. Also

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = A'x'^2 + B'y'^2 + C'z'^2$$

for every $p$. Therefore

$$Ax^2+By^2Cz^2-2Fy^2-2Gzx-2Hxy-K(x^2+y^2+z^2) = A'x'^2+B'y'^2+C'z'^2-K(x'^2+y'^2+z'^2) \quad (3.3.1)$$

is true for any $K$.

Let the above identity be denoted by $\Phi$. Consider the equations

$$\frac{\partial\Phi}{\partial x'} = 0, \quad \frac{\partial\Phi}{\partial y'} = 0 \quad and \quad \frac{\partial\Phi}{\partial z'} = 0. \qquad (3.3.2)$$

Then, we have

$$(A' - K)x' = 0, \quad (B' - K)y' = 0, \quad (C' - K)z' = 0 \Rightarrow x' = y' = z' = 0 \quad or \quad A' = B' = C' = K.$$

Rejecting the trivial solutions, $x' = y' = z' = 0$, we have the following solutions,

$$K = A', \ x' - arbitrary, \quad y' = 0, \ z' = 0.$$

$$K = B', \ x' = 0 \quad y' - arbitrary, \quad z' = 0.$$

$$K = C', \ x' = 0, \quad y' = 0, \quad z' - arbitrary.$$

Thus (3.3.2) have non-trivial solutions with $K$ equal to one of the principal moments on inertia; the corresponding values of $x', y', z'$ gives the principal axes.

Now

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial x'}.\frac{\partial x'}{\partial x} + \frac{\partial\Phi}{\partial y'}.\frac{\partial y'}{\partial x} + \frac{\partial\Phi}{\partial z'}.\frac{\partial z'}{\partial x}$$

Similarly, we can write for $\frac{\partial\Phi}{\partial y}$ and $\frac{\partial\Phi}{\partial z}$.

From (3.3.2),

$$\frac{\partial\Phi}{\partial x} = 0, \quad \frac{\partial\Phi}{\partial y} = 0 \quad and \frac{\partial\Phi}{\partial z} = 0. \tag{3.3.3}$$

If $K, x', y', z'$ are chosen as above, (3.3.2) is satisfied. Hence (3.3.3) gives

$$\begin{aligned} (A - K)x - Hy - Gz &= 0 \\ -Hx + (B - K)y - Fz &= 0 \\ -Gx - Fy + (C - K)z &= 0 \end{aligned} \tag{3.3.4}$$

These equations have a solution other than $x = y = z = 0$. Hence

$$\begin{vmatrix} A - K & -H & -G \\ -H & B - K & -F \\ -G & -F & C - K \end{vmatrix} = 0$$

The above equation is a cubic equation in $K$. It's three roots are the three principal moments of inertia. Let us sum up "starting with general axes $oxyz$ with moments and products on inertia $A, B, C, F, G, H$. The three principal moments of inertia at $O$ are the values of $K$ satisfying the above cubic equation and the directions of the three principal axes are given by the ratios $x : y : z$

determined by (3.3.4), when the above values of $K$ are substituted.

## 3.4 Method of Symmetry

If we rotate a body of resolution about it's axis through any angle, we do not alter the distribution of matter - the whole body appears exactly as before. In a similar manner, if we turn a three - bladed propeller about it's axis through an angle $\frac{2\pi}{3}$, the final distribution of matter is the same which we started. These rotations are examples of covering operations.

A covering operation for a body is a transformation which does not alter the distribution of matter as a whole, although the individual particles are moved. In the case of a curve or surface, where the distribution of matter is involved, a covering operation is a transformation that bears the curve or surface unchanged as a whole.

We shall consider the following covering operations:

(i) a rotation about a line or axis and (ii) a reflection in a plane. A body is said to possess symmetry whenever there exists a covering operation for that body.

(i) If the operation is a rotation through an angle $\frac{2\pi}{n}$ about an axis (where $n$ is a positive integer other than one), then the axis is called an axis of $n$*-gonal symmetry* for $n = 2, 3, 4,$ the symmetry is diagonal, trigonal and tetragonal respectively. Thus for a $2-$ bladed propeller, the axis is of diagonal symmetry; the axis is of diagonal symmetry; for a $3-$ bladed propeller, the axis is of trigonal symmetry. (ii) If the operation is a reflection in a plane, then that plane is a plane of symmetry for the body.

**Theorem 3.4.1.** *A covering operation for a body, which leaves a point O of the body unchanged, is a covering for the momental ellipsoid at O.*

**Proof.** The proof of the theorem depends on the following points:

(a) When a body rotated about a line, the momental ellipsoid at any point on the on the line turns with the body.

(b) When a body is reflected in a plane, the momental ellipsoid at any point on the plane is also reflected in this plane.

When the rotation (or reflection) is a covering operation for the body, the distribution of matter is unaltered and the momental ellipsoid at a point on the axis of rotation (or in the plane of reflection) is the same as before. The rotation (or reflection) is hence a covering operation for the momental ellipsoid also, which proves the theorem. ■

From the geometry of ellipsoid, we know that, there are only very special covering operations, in the case where the axes are unequal. They are

(i) a rotation through an angle $\pi$ about a principal axis and (ii) a reflection in a principal plane. If the ellipsoid has more general covering operations, it must necessarily be of revolution or in particular, a sphere.

For example, if the covering operation is a rotation through an angle $\frac{2\pi}{3}$, then the ellipsoid must be of revolution. If the covering operation is a rotation through an angle $\pi$ about a line $L$, then $L$ must be a principal axis. If the covering operation is a reflection in a plane $P$, then $P$ must be a principal plane.

Let us now apply the above facts to the momental ellipsoid. Then the following facts are obvious.

(a) An axis of $n-$ gonal symmetry is a principal axis of inertia at any point of itself (Eg. - a 2-bladed propeller).

(b) At any point on an axis of trigonal or tetragonal symmetry, the momental ellipsoid has this axis for the axis of revolution and two of the principal moments of inertia are equal (Eg. - a $3-$ or $4$ bladed propeller)

(c) The normal to a plane of symmetry is a principal axis of inertia at a point where it cuts the plane of symmetry. (Eg. - the hull of a ship).

## 3.5 The Momental Ellipse

We now consider a distribution of matter in a plane $P$ and let $ox$, $oy$ be the rectangular axes in the plane. Since plane $P$ is a plane of symmetry, its normal at $O$ is a principal axis of inertia and the section of the momental ellipsoid at $O$ by the plane $P$ is a principal section. This is called the *momental ellipse* at $O$. Let $A$ and $B$ be the moments of inertia about $ox$ and $oy$ respectively and $H$ be the products of inertia with respect to planes through $ox$, $oy$ perpendicular to $P$. Then the equation of this ellipse is

$$Ax^2 - 2Hxy + By^2 = 1 \tag{3.5.1}$$

(We obtain this by introducing the third axis $oz$ and putting $z = 0$ in the equation of momental ellipsoid). It is clear that the principal axes of this ellipse are the principal axes of this ellipse are the principal axes of inertia at $O$. We proceed in the following way to find them. Let $ox'$, $oy'$ be the new axes, $ox'$ making an angle $\theta$ with $ox$. If $(x', y')$ and $(x, y)$ are the co-ordinates of a point referred to the axes $ox'y'$ and $oxy$ respectively, then

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned} \tag{3.5.2}$$

The equation of ellipse (3.5.1) referred to the axes $ox'$, $oy'$ is

$$\begin{aligned} A(x' \cos\theta - y' \sin\theta)^2 - 2H(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta) & \\ +B(x' \sin\theta + y' \cos\theta)^2 &= 1 \\ A(x'^2 \cos^2\theta + y'^2 \sin^2\theta - 2x'y' \cos\theta \sin\theta) & \\ -2H(x'^2 \sin\theta \cos\theta - x'y' \sin^2\theta + x'y' \cos^2\theta - y'^2 \sin\theta \cos\theta) & \\ +B(x'^2 \sin^2\theta + y'^2 \cos^2\theta + 2x'y' \sin\theta \cos\theta) &= 1 \end{aligned}$$

or

$$A'x'^2 - 2H'x'y' + B'y'^2 = 1, \tag{3.5.3}$$

where

$$\begin{aligned} A' &= A \cos^2\theta - 2H \sin\theta \cos\theta + B \sin^2\theta \\ H' &= (A - B) \sin\theta \cos\theta + H(\cos^2\theta - \sin^2\theta) \\ B' &= A \sin^2\theta + 2H \sin\theta \cos\theta + B \cos^2\theta \end{aligned} \tag{3.5.4}$$

If $H' = 0$, then (3.5.3) represents the equation of an ellipse referred to principal axes at it's center.
$H' = 0 \Rightarrow$

$$(A - B)\frac{\sin 2\theta}{2} + H(\cos 2\theta) \tag{3.5.5}$$

(From (3.5.5) and using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
Dividing (3.5.5) by $\cos 2\theta$ on both sides, we get

$$\tan 2\theta = \frac{2H}{B - A} \tag{3.5.6}$$

Hence $ox'$ and $oy'$ are the principal axes of inertia at $O$ if (3.5.6) is satisfied.

## 3.6 Moments of Inertia of Some Simple Bodies

Let us consider the equation of an ellipsoid $E$ with semi axes $a, b, c$ referred to as principal axes at its center. The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The moment of inertia about the $x-$ axis is

$$\frac{A}{\rho} = \iiint_C (y^2 + z^2)\,dx\,dy\,dz$$

where $\rho$ is the density. Let $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$. Then the above equation becomes

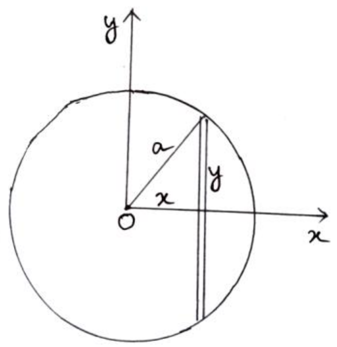
$$\frac{A}{\rho} = \iiint_S (b^2 y'^2 + c^2 z'^2)(a\,dx')(b\,dy')(c\,dz'), \tag{3.6.1}$$

where $S$ is a unit sphere and range of integration is the interior of $S$. From the symmetry of $S$, we have

$$\iiint_S y'^2\,dx'\,dy'\,dz' = \iiint_S z'^2\,dx'\,dy'\,dz'$$

$$\iiint_S y'^2\,dx'\,dy'\,dz' = \frac{1}{2}\iiint_S (y'^2 + z'^2)\,dx'\,dy'\,dz' \tag{3.6.2}$$

The last integral is the moment of inertia of s sphere (of unit radius and density) about a diameter. To find the moment of inertia of a solid sphere of mass $m$ and radius $a$ about a diameter, we imagine that it is split into thin circular disks by planes perpendicular to the diameter.




The above figure shows the section of the sphere by a plane through the diameter ($ox$) about which the moment is to be calculated. If $\rho$ is the density of the material, the mass of the disk between planes at distances $x$, $x+dx$ from the center is $\rho\pi y^2 dx$, where $y$ is the radius of the disk. The moment of inertia of the disk is

$$\begin{aligned} dI &= \frac{1}{2}\pi\rho y^4 dx \\ I &= \frac{1}{2}\pi\rho \int_{-a}^{a} (a^2 - x^2)^2 dx, \qquad (\because y^2 = a^2 - x^2) \\ I &= \frac{8}{15}\pi\rho a^5 \end{aligned}$$

As the sphere is of unit radius and density,

$$I = \frac{8\pi}{15} \tag{3.6.3}$$

Hence

$$
\begin{aligned}
\frac{A}{\rho} &= abc \times \frac{8\pi}{2\times 15}(b^2+c^2) \quad (\textit{using} \quad (3.6.2) \quad \textit{and} \quad (3.6.3) \quad \textit{in} \quad (3.6.1))\\
A &= \frac{4\pi\rho abc}{15}(b^2+c^2)\\
\textit{But} \quad m &= \frac{4}{3}\pi\rho a^3 \quad (\textit{here a is the radius o f the sphere})\\
m &= \frac{4}{3}\pi\rho\\
\therefore A &= \frac{m}{5}(b^2+c^2)\\
S\,\textit{imilarly,}\ B &= \frac{m}{5}(c^2+a^2)\\
B &= \frac{m}{5}(a^2+b^2)
\end{aligned}
$$

The following table gives the principal axes and moment of inertia at the mass center of some simple bodies. In all cases, the bodies possess constant density.

| Body(mass= $m$) | Principal axes (at mass center $O$) | Principal moments of inertia |
|---|---|---|
| Rectangular plate (edges $2a$, $2b$) | $Ox$, $Oy$ parallel to edges $2a$, $2b$ respectively; $Oz$ perpendicular to plate | $A = \frac{1}{3}mb^2$, $B = \frac{1}{3}ma^2$, $C = \frac{1}{3}m(a^2 + b^2)$ |
| Solid rectangular cuboid (edges $2a$, $2b$, $2c$) | $Ox$, $Oy$, $Oz$ parallel edges $2a$, $2b$, $2c$, respectively | $A = \frac{1}{3}m(b^2 + c^2)$, $B = \frac{1}{3}m(c^2 + a^2)$, $C = \frac{1}{3}m(a^2 + b^2)$ |
| Circular plate (semi axes $a$, $b$) | $Ox$, $Oy$ in plane of plate; $Oz$ perpendicular to plate | $A = B = \frac{1}{4}ma^2$, $C = \frac{1}{2}ma^2$, |
| Elliptical plate (semi axes $a$, $b$) | $Ox$, $Oy$ along semi axes, $a$, $b$ respectively; $Oz$ perpendicular to plate | $A = \frac{1}{4}mb^2$, $B = \frac{1}{4}ma^2$, $C = \frac{1}{4}m(a^2 + b^2)$ |
| Solid circular cylinder (radius $a$, length $2l$) | $Ox$, $Oy$ perpendicular to axis; $Oz$ along axis | $A = B = \frac{1}{12}m(3a^2 + 4l^2)$, $C = \frac{1}{2}ma^2$ |
| Solid elliptical cylinder (semi axes $a$, $b$; length $2l$) | $Ox$, $Oy$ along semi axes $a$, $b$ of section, respectively; $Oz$ along axis of cylinder | $A = \frac{1}{12}m(3b^2 + 4l^2)$, $B = \frac{1}{12}m(3a^2 + 4l^2)$, $C = \frac{1}{4}m(a^2 + b^2)$, |
| Sphere (radius $a$) | $Ox$, $Oy$, $Oz$ any three perpendicular line | $A = B = C = \frac{2}{5}ma^2$ |
| Solid ellipsoid (semi axes $a$, $b$, $c$) | $Ox$, $Oy$, $Oz$ along semi axes $a$, $b$, $c$ respectively | $A = \frac{1}{5}m(b^2 + c^2)$, $B = \frac{1}{5}m(c^2 + a^2)$, $C = \frac{1}{5}m(a^2 + b^2)$, |

The most of the results given in the table provide us the following rule known as *Routh's rule*:

**Routh's Rule:**

For solid bodies of the cuboid, elliptical cylindrical and ellipsoidal types, the moment of inertia about a principal axis through the center is equal to $\dfrac{m(a^2 + b^2)}{n}$, where $m$ is the mass of the body, $a$, $b$ are the semi axes perpendicular to the principal axis in consideration, and $n$ = 3, 4 or 5 according as the body belongs to the cuboid, elliptical cylindrical or ellipsoidal type.

**The Methods of Decomposition and Differentiation** The method of decomposition to calculate the moment of inertia of a body consists of dividing the body into a number of parts for each which the moment of inertia known. By adding the moments of inertia of these parts, we get the moment of inertia of the whole body.

The *method of differentiation* can be used to find the moment of inertia of a shell when the corresponding moment of inertia for a similar solid is known.

**Equimomental Systems**

Two distribution of matter which have the same total mass and the same principal moments of inertia at the mass center are said to be *equimomental systems.*

**Example 1.** A hoop of mass $m$ and radius $\frac{a}{\sqrt{2}}$ is equimomental with a circular plate of mass $m$ and radius $a$. Two rigid bodies are equimomental have the same dynamical behavior. That is, two such bodies will behave in the same way, when acted on by identical force systems.

# BLOCK-I

# UNIT 4

# Kinetic Energy

**Objectives**

**4.1 The Kinetic Energy of a Rigid body with respect to a Fixed Point**

**4.2 The Kinetic Energy of a Rigid Body in General**

**4.3 Angular Momentum**

**4.4 Angular Momentum of a Rigid Body**

**4.5 Worked Examples**

**Objectives**

Upon completion of this Unit, the students are expected to

x understand the concept of kinetic energy of a rigid body.

x identify angular momentum of a rigid body.

x find moment of inertia, magnitude of velocity and acceleration.

## 4.1 The Kinetic Energy of a Rigid with respect to a Fixed Point

Consider a rigid body turning about a fixed point $O$ with angular velocity $\omega$. A particle $P$ of this body with velocity $\vec{v}$ and mass $\delta m$ has kinetic energy $\frac{1}{2}\delta m v^2$.

The kinetic energy of the body is

$$T = \frac{1}{2}\sum \delta m v^2 \tag{4.1.1}$$

where the summation extends over all particles of the body. Let us find an alternative expression for $T$, involving the angular velocity $\omega$ and the principal moments of inertia at $O$. Let $oxyz$ be the rectangular axes with origin $o$ and $\vec{i}$, $\vec{j}$, $\vec{k}$ be the unit vectors along them. Then

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$, where $\vec{r} = \overrightarrow{OP}$.

For the velocity $\vec{v}$ of $P$,

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{4.1.2}$$

$$i.e., \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\vec{i} + (\omega_3 x - \omega_1 z)\vec{j} + (\omega_1 y - \omega_2 x)\vec{k}$$

Using (4.1.2) in (4.1.1), we get

$$2T = \sum \delta m \left[(\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2\right]$$

$$2T = \omega_1^2 \sum \delta m(y^2 + z^2) + \omega_2^2 \sum \delta m(z^2 + x^2) + \omega_3^2 \sum \delta m(x^2 + y^2)$$
$$- 2\omega_2\omega_3 \sum \delta myz - 2\omega_3\omega_1 \sum \delta mzx - 2\omega_1\omega_2 \sum \delta mxy$$

$$i.e., T = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_2\omega_3 - 2G\omega_3\omega_1 - 2H\omega_1\omega_2\right) \tag{4.1.3}$$

where $A$, $B$, $C$ are the moments and products of inertia for $oxyz$. If the axes are the principal axes of inertia, then $F = G = H = 0$. Therefore (4.1.3) reduces to

$$T = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) \tag{4.1.4}$$

where $A$, $B$, $C$ are now principal moments of inertia.

**Remark.** For a rigid body turning about a fixed line $L$ through $O$, the expression (4.1.3) simplifies to

$$T = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2 = \frac{1}{2}C\omega^2$$

where $C$ is the moment of inertia about the line $L$ [we have to take $oz$ along $L$, which gives $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$].

## 4.2 The Kinetic Energy of a Rigid Body in General

Let us find the kinetic energy $T$ of a rigid body moving in space. From Konig's theorem, we have

$$T = \frac{1}{2}mv_0^2 + T_1 \tag{4.2.1}$$

where $m =$ mass of the body

$v_0 =$ speed of mass center

$T_1 =$ kinetic energy of motion relative to mass center.

The mass center may be regraded as a base point in the body and hence the motion relative to the mass center is the motion of a rigid body turning about a fixed point. Thus

$$T_1 = \frac{1}{2}A\omega_1^2 + B\omega_2^2 + C\omega_3^2$$

Hence

$$T = \frac{1}{2}mv_0^2 + \frac{1}{2}\ A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \tag{4.2.2}$$

where $A,\ B,\ C$ are the principal moments of inertia at the mass center and $\omega_1,\ \omega_2,\ \omega_3$ are the components of the angular velocity $\omega$ in the directions of principal axes of inertia at the mass center.

## 4.3 Angular Momentum

The angular momentum of a particle about a line is the moment of the linear momentum vector about the line in consideration.

**Angular Momentum of a Particle**

Let us consider a particle of mass $m$ moving with velocity $\vec{v}$ relative to same frame of reference $S$. The linear momentum is the product of mass and velocity *i.e.,* $m\vec{v}$. The angular momentum

$H$ about any point $O$ is defined as the moment of $m\vec{v}$ about $O$.

$$\vec{H} = \vec{r} \times m\vec{v} \tag{4.3.1}$$

where $\vec{r}$ is the position vector of the particle with respect to $O$.

**Angular Momentum of a System of Particles**

The angular momentum of a system of particles is the vector sum of the angular momenta of the several particles. Let $m_i$ — be the mass of the $i^{th}$ particle, $\vec{r}_i$, the position vector of the particle relative to a point $O$, and $\vec{v}_i$ the velocity of the $i^{th}$ particle. The angular momentum about $O$ is

$$\vec{H} = \sum_{i=1}^{m} (\vec{r}_i \times m_i \vec{v}_i) \tag{4.3.2}$$

where $n$ is the number of particles in the system. If $O$ is fixed in the frame of reference, then

$$\vec{v}_1 = \dot{\vec{r}}_i = \dot{x}_i \vec{i} + \dot{y}_i \vec{j} + \dot{z}_i \vec{k}.$$

From (4.3.2), the components of $H$ along the rectangular axes fixed in the frame are

$$\sum_{i=1}^{n} m_i(y_i\dot{z}_i - z_i\dot{y}_i), \quad \sum_{i=1}^{n} (z_i\dot{x}_i - x_i\dot{z}_i) \quad and \quad \sum_{i=1}^{n} m_i(x_i\dot{y}_i - y_i\dot{x}_i) \tag{4.3.3}$$

If we change the frame of reference to a new frame say $S'$, with velocity $\vec{v}_1$ of transition relative to $S$, then the velocities of a particle $\vec{v}_i$ and $\vec{v}_i'$ relative to $S$ and $S'$ are connected by

$$\vec{v}_i = \vec{v}_1 + \vec{v}_i' \tag{4.3.4}$$

Then the angular momenta about $O$ are

$$\vec{H} = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i)$$

for $S$ and

$$\vec{H}' = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i')$$

for $S'$ Substituting (4.3.4) in the above expression for $\vec{H}$, we have

$$H = \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_1 + H \qquad (4.3.5)$$

If $O$ is the mass center, then

$$\sum_{i=1}^{n} m_i \vec{r}_i = 0 \qquad H = H$$

i.e., angular momentum about the mass center is the same for all frames of reference in relative transitional motion.

## 4.4 Angular Momentum of a Rigid Body

The angular momentum about $O$ can be written as

$$\vec{H} = \sum (\vec{r} \times \delta m \, . \, \vec{v}) \qquad (4.4.1)$$

where $\delta m$ is the mass of a typical particle $\vec{r}$, the position vector of the particle with respect to $O$ and $\vec{v}$, the velocity of the particle. Since $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{\omega}$ is the angular velocity of the body, we have

$$\vec{H} = \sum \delta m \left[\vec{r} \times (\vec{\omega} \times \vec{r})\right]$$

$$\vec{H} = \sum \delta m \, . \left[\omega r^2 - \vec{r}\,(\vec{\omega} \, . \, \vec{r})\right] \qquad (4.4.2)$$

using vector triple products.

Let $\vec{i}, \vec{j}, \vec{k}$ represent an orthogonal triad about $O$. Then

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad and \qquad \vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$$

Let $A, B, C, F, G, H$ be the moments and products of inertia with respect to the triad $\vec{i}, \vec{j}, \vec{k}$. The $\vec{H}$ component in $\vec{i}$ direction is

$$
\begin{aligned}
H_1 &= \sum \delta m \left[\omega_1(x^2 + y^2 + z^2) - x(\omega_1 x + \omega_2 y + \omega_3 z)\right] \\
H_1 &= \omega_1 \sum \delta m(y^2 + z^2) - \omega_2 \sum \delta mxy - \omega_3 \sum \delta mzx
\end{aligned}
$$

$$H_1 = A\omega_1 - H\omega_2 - G\omega_3 \tag{4.4.3}$$

When $A, B, C, F, G, H$ are known for any set of rectangular axes through the mass center, the moment of inertia $I$ of the system about any line $L$ can be found in the following manner:

*(i)* Find the moment of inertia about a line through the mass center parallel to $L$.

*(ii)* Apply the theorem of parallel axes to find $I$.

If $A, B, C, F, G, H$ are known for a point other than the mass center, then $I$ can be found in a similar manner by the application of the theorem of parallel axes.

## 4.5 Worked Examples

**Example 2.** Find the moment of inertia of a spherical shell about a diameter.

**Solution.** Let us consider uniform solid sphere of density $\rho$ and radius $a$. Its moment of inertia about a diameter is $\frac{8\pi}{15}\rho a^5$. If the radius of the sphere is increased to $a + da$, the moment of inertia is increased by $dI = \frac{8\pi}{15}\rho 5a^4 da$. That is

$$dI = \frac{8\pi}{3}\rho a^4 da$$

This is the moment of inertia of a spherical shell of radius $a$, thickness $da$ and mass $4\pi\rho a^2 da$. Hence the moment of inertia of a spherical shell of radius $a$ and mass $m$, about a diameter is $\frac{2}{3}ma^2$. $\frac{8\pi}{3}\rho a^4 da = 4\pi\rho a^2 da \times \frac{2}{3}a^2$

**Example 3.** At time $t$, the position of a moving particle relative to axes $Oxyz$ is given by $x = 5\cos 2t$, $y = 5\sin 2t$, $z = 4t$. Find the magnitude of velocity and acceleration at $t = 2$.

**Solution.** The position vector of the particle is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = 5\cos 2t\,\vec{i} + 5\sin 2t\,\vec{j} + 4t\,\vec{k}$$

$$\text{velocity } \vec{v} = \dot{\vec{r}}$$

$$\vec{v} = -10\sin 2t\,\vec{i} + 10\cos 2t\,\vec{j} + 4\,\vec{k}$$

$$\text{Magnitude of velocity} = \sqrt{10^2\sin^2 2t + 10^2\cos^2 2t + 4^2}$$

$$= \sqrt{116} \text{ units.}$$

$$\text{Acceleration } \vec{a} = \ddot{\vec{r}}$$

$$\vec{a} = -20\cos 2t\,\vec{i} - 20\sin 2t\,\vec{j}$$

$$\text{Magnitude of acceleration} = \sqrt{20^2\cos^2 2t + 20^2\sin^2 2t}$$

$$= 20 \text{ units.}$$

**Example 4.** What is the kinetic energy of a circular cylinder of mass $m$ and radius $a$ rolling on a plane with linear velocity $v$.

**Solution.** The kinetic energy of a rotating body is given by

$$T = \frac{1}{2}mv^2\left(1 + \frac{k^2}{a^2}\right)$$

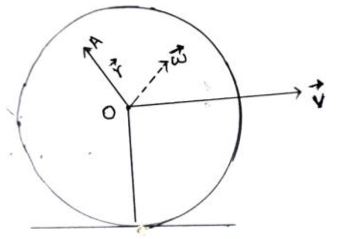
where $\frac{k}{a}$ is the ratio of radius of gyration to the radius of the body $\frac{k^2}{a^2}$ for a solid cylinder is $\frac{1}{2}$.

$$\therefore \text{Kinetic energy } T = \frac{1}{2}mv^2\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{4}mv^2$$

**Example 5.** A circular wheel is rolling with constant speed along a straight level track. Find the acceleration of the wheel.

**Solution.** Let the base point be the center $O$ of the wheel and its velocity be $\vec{v}$. This velocity lies in the plane of the wheel and it is a constant vector acting along the horizontal. The angular velocity $\vec{\omega}$ of the wheel is a vector perpendicular to its plane and it is also a constant vector. Let $A$ be a particle of the wheel. Then its velocity is given by $\vec{v} + \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector of $\vec{A}$, i.e., $\vec{r} = \overrightarrow{OA}$. Now $\omega \times \vec{r}$ is a vector perpendicular to $\vec{a}$. This velocity lies in the plane of the wheel ($\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$). Since $\vec{\omega}$ and $\vec{v}$ are constant




vectors, the acceleration of $A$ is

$$\begin{aligned}\vec{a} &= \vec{\omega} \times (\vec{\omega} \times \vec{v}) \\ &= \vec{\omega}(\vec{\omega}.\vec{r}) - \vec{r}(\vec{\omega}.\vec{\omega}) \\ &= [\vec{\omega}, \vec{\omega}, \vec{\omega}] - \vec{r}\,\omega^2 \\ \vec{a} &= -\vec{r}\,\omega^2\end{aligned}$$

Thus each particle of the wheel has an acceleration of magnitude $r\omega^2$ directed towards the center $O$.

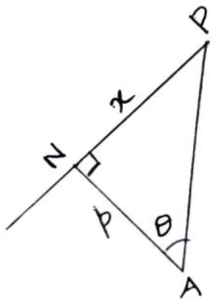
## Check Your Progress

1. Show that the angular velocity about a fixed point $A$ of a particle $P$ moving uniformly in a straight line varies inversely as the square of the distance of the line from the fixed point.
2. A particle moves so that the radial and transverse components of velocity are $ar$ and $b\theta$. Show that the radial and transverse components of its acceleration are $a^2r - \frac{b^2\theta^2}{r}$, $ab\theta + \frac{b^2\theta}{r}$
3. A car is travelling with a velocity of $v\ m/s$ and having a kinetic energy of $12500\,kgm^2s^2$ and it has mass of $250\,kg$. Compute its velocity.
4. The motion of a creature in three dimensions are described by $x(t) = 3t^2 + 5$, $y(t) = -t^2 + 3t - 2$ and $z(t) = 2t + 1$ in the $x$, $y$ and $z$ directions. Find the magnitude of the velocity and acceleration at $t = 0$.

# Answer to Check Your Progress

1. Hint: $x = p\tan\theta \Rightarrow \dot{x} = p\sec^2\theta\dot{\theta} \Rightarrow \dot{\theta} = \frac{k}{p}\cos^2\theta$, where $\dot{x} = k$, a constant.

$$\dot{\theta} = \frac{k}{p}\left(\frac{AN}{AP}\right)^2 = kp\frac{1}{AP^2}$$




2. Hint: $\dot{r} = ar$ and $r\dot{\theta} = b\theta$. Differentiate them with respect to $t$, and obtain $\ddot{r} - t\dot{\theta}^2$ and $2\dot{r}\dot{\theta} + r\ddot{\theta}$ in terms of $r$ and $\theta$.

3. Hint: Kinetic energy = $\frac{1}{2}mv^2 = \frac{1}{2} \times 250 \times v^2 = 12500 \Rightarrow v^2 = 100 \Rightarrow v = 10\,m/s.$

4. Hint: $\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = 6t\vec{i} + (-2t + 3)\vec{j} + 2\vec{k}$. At $t = 0$, $\vec{v} = 3\vec{j} + 2\vec{k}$ and so $|\vec{v}| = \sqrt{3^2 + 2^2}$ units $\vec{a} = 6\vec{i} - 2\vec{j} \Rightarrow |\vec{a}| = \sqrt{6^2 + (-2)^2}$ units.

# BLOCK-II

# UNIT 5

# Motion of a Particle



**Objectives**

Upon completion of this Unit, the students will be able to

- find the equation of motion.
- apply principle of energy.

## 5.1 Equations of Motion

A particle of mass $m$ is acted on by a force $\vec{F}$. Then by the fundamental law, we have

$$\vec{F} = m\vec{a} \tag{5.1.1}$$

where $\vec{a}$ is the acceleration relative to a Newtonian frame of reference. Equation (5.1.1) can also be written in the form

$$\frac{d}{dt}(m\vec{v}) = \vec{F} \quad since \quad \vec{a} = \frac{d}{dt}(\vec{v}) \tag{5.1.2}$$

where $\vec{v}$ is the velocity of the particle. The above result is often referred to as the principle of linear momentum for a particle: "The rate of change of linear momentum of a particle is equal to the applied force. "(linear momentum $p$ = mass × velocity). By resolving the vectors $\vec{a}$ and $\vec{F}$ in the directions of rectangular axes $Oxyz$, fixed in the frame of reference , we get

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z \tag{5.1.3}$$

where $X, Y, Z$ are the components of $\vec{F}$ along the axes. Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along the tangent principal normal and binormal to the path of the particle. Then

$$\vec{a} = \ddot{s}\,\vec{i} + \frac{\dot{s}^2}{\rho}\vec{j}, \quad by(5.1.3) \tag{5.1.4}$$

where $s$ is the arc length along the path and $\rho$ is the radius of curvature. Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ , we get the following intrinsic equations of motion from $\vec{F} = m\vec{a}$.

$$m\ddot{s} = F_1, \quad \frac{m\dot{s}^2}{\rho} = F_2, \quad 0 = F_3 \tag{5.1.5}$$

Let us now take $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors in the directions of the parametric lines of cylindrical coordinates $(R, \phi, z)$. We have from (5.1.3)

$$\vec{a} = (\ddot{R} - R\dot{\psi}^2)\vec{i} + \frac{1}{R}\frac{d}{dt}(R^2\dot{\psi})\vec{j} + \ddot{z}\,\vec{k} \tag{5.1.6}$$

Thus if $\vec{F} = F_R\vec{i} + F_\psi\vec{j} + F_z\vec{k}$, then we get the following equations of motion in cylindrical polar coordinates.

$$m(\ddot{R} - R\dot{\psi}^2) = F_R, \quad m\frac{1}{R}\frac{d}{dt}(R^2\dot{\psi}) = F_\psi \quad and \quad m\ddot{z} = F_z. \tag{5.1.7}$$

## 5.2 Principle of Angular Momentum

By (5.3.1), the angular momentum of a particle about a fixed point $O$ is

$$\vec{H} = \vec{r} \times m\vec{v} \tag{5.2.1}$$

Let us find the rate of change of $\vec{H}$. Differentiating (5.2.1), we have

$$\begin{aligned} \dot{\vec{H}} &= \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\dot{\vec{v}} \\ &= \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} \\ \dot{\vec{H}} &= 0 + \vec{r} \times \vec{F} \end{aligned}$$

$$\dot{\vec{H}} = \vec{r} \times \vec{F} \tag{5.2.2}$$

where $\vec{F}$ is the force acting on the particle. i.e., the rate of change of angular momentum of a particle about a fixed point is equal to the moment of the applied force about that point.

If we take $O$ as the origin of rectangular axes $Oxyz$ and resolve vectors in the directions of these axes we get from (5.1.3) and (5.2.2), the following equations, where $X, Y, Z$ are the components of $F$.

$$\text{i.e., } \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ X & Y & Z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ m\ddot{x} & m\ddot{y} & m\ddot{z} \end{vmatrix}$$

$$\text{i.e., } \vec{i}(yZ - Yz) - \vec{j}(xZ - Xz) + \vec{k}(xY - Xy) = \vec{i}(my\ddot{z} - m\ddot{y}z) - \vec{j}(mx\ddot{z} - m\ddot{x}z) + \vec{k}(mx\ddot{y} - m\ddot{x}y)$$

i.e.,

$$\begin{aligned} yZ - zY &= m(y\ddot{z} - z\ddot{y}) \\ zX - xZ &= m(z\ddot{x} - x\ddot{z}) \\ xY - yX &= m(x\ddot{y} - y\ddot{x}) \end{aligned} \tag{5.2.3}$$

## 5.3 Principle of Energy

For a moving particle

$$\dot{T} = \dot{W} \tag{5.3.1}$$

where $\dot{T}$ is the rate of increase of the kinetic energy and $\dot{W}$ is the rate at which the applied forces do work. This form of principle of energy is useful in the case where the working forces are conservative. Then $\dot{W} = -\dot{V}$, where $V$ is the potential energy of the particle. Thus (1.3.1) becomes $\dot{T} = -\dot{V}$. On integrating

$$T + V = E \tag{5.3.2}$$

where $E$ is a constant. ( $E =$ total energy)

**Example 6.** Apply the principle of energy to the motion of a particle on a smooth sphere. Use cylindrical coordinates $(R, \psi, z)$ with origin at the center of the sphere, the axis of $z$ being directed vertically upward.

**Solution.** The equation of the required sphere is

$$R^2 = a^2 - z^2 \tag{5.3.3}$$

The force acting on the particle are its weight $mg$ and the normal reaction $N$ of the sphere. By resolving these forces along the parametric lines of $R, u, z$ we get from (5.3.3)

$$\begin{aligned} m(\ddot{R} - R\dot{\psi}^2) &= F_n \\ m\frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) &= F_\psi \\ m\ddot{z} &= F_z \end{aligned} \tag{5.3.4}$$

The above three equations along with (5.3.3) can be used to find $N, R, \psi, z$ as functions of time. Hence we notice that, since $N$ does no work, the principle of energy applies. The potential energy of the particle is $mgz$ and its kinetic energy is $\frac{1}{2}mv^2$, where $v$ is the velocity of the particle. Hence from $T + V = E$, we have

$$\frac{1}{2}mv^2 + mgz = mE, \tag{5.3.5}$$

, where $E$ is the energy constant per unit mass. From (1.2.2) the components of velocity are

$\dot{R},\ R\dot{\psi}$ and $\dot{z}$.

$$\therefore (5.3.5) \Rightarrow \frac{1}{2}m(\dot{R}^2 + R^2\dot{\psi} + \dot{z}^2) + mgz = mE \tag{5.3.6}$$

$N$ and the weight have no moment about $Oz$. Hence the angular momentum about $Oz$ is constant.

The components of linear momentum in the $R$ and $z$ directions have no moments about $Oz$. The $\varphi$ component is $mR\dot{\psi}$ and its moment is $mR^2\dot{\psi}$. Hence

$$R^2\dot{\psi} = h \tag{5.3.7}$$

where $h$ is a constant. This result also follows from the second equation of (5.3.4), since $F_\psi = 0$ in this case. When the initial position and velocity are known, the constants $E$ and $h$ can be obtained and the equations (5.3.3), (5.3.6) and (5.3.7) provide three equations to find $R,\ \psi$ and $Z$. By differentiating (5.3.3) we have

$$\dot{R} = \frac{1}{2\sqrt{a^2 - z^2}}(-2z\dot{z}) = -\frac{z\dot{z}}{\sqrt{a^2 - z^2}}.$$

From (5.3.7) $\dot{\psi} = \frac{h}{R^2}$. Substituting $\dot{R}$ and $\dot{\psi}$ in (5.3.6), we have

$$\frac{1}{2}\left(\frac{z^2\dot{z}^2}{a^2 - z^2} + R^2\frac{h^2}{R^4} + \dot{z}^2\right) + gz = E$$

$$\frac{z^2\dot{z}^2}{a^2 - z^2} + \frac{h^2}{R^2} + \dot{z}^2 + 2gz = 2E$$

$$\dot{z}^2\left(\frac{z^2}{a^2 - z^2} + 1\right) = 2E - 2gz - \frac{h^2}{R^2}$$

$$\dot{z}^2 = \frac{2g(a^2 - z^2)}{a^2}\left(\frac{E}{g} - z - \frac{h^2}{2gR^2}\right)$$

$$\dot{z}^2 = \frac{2g}{a^2}\left[(a^2 - z^2)\left(\frac{E}{g} - z\right) - \frac{h^2}{2g}\right], \quad since\ a^2 - z^2 = R^2 \tag{5.3.8}$$

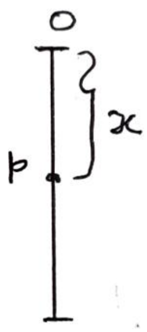
The above equation is a single equation for $z$ as a function of $t$. When this equation is solved, we can get $R$ in terms of $t$ from $R^2 = a^2 - z^2$ and $\psi$ by a quadrature from $R^2\dot{\psi} = h$.

## Check Your Progress

1. Verify the principle of conservation of energy for a particle falling freely under gravity.
2. Verify the principle of conservation of energy for a particle sliding down as inclined plane freely under gravity.

## Answer to Check Your Progress

*1.* Hint: Let the particle start from rest at $O$ and hits the surface of the earth at $A$ with $OA = h$ (say). At time $t$ let $p$ be the position of the body with $OP = x.$ $T = \frac{1}{2}mv^2 = \frac{1}{2}m(2gx) = mgx$

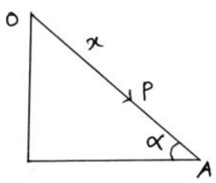



$[v^2 = u^2 + 2as]$

$V = mg(AP) = mg(h - x).$

$T + V = mgx + mg(h - x) = mgh = constant.$

*2.* Hint: Let $OP = x,$ where $P$ is the position of the particle at time $t.$



$$
\begin{aligned}
T + V &= \frac{1}{2}mv^2 + mg(OA - x)\sin\alpha \\
&= \frac{1}{2}m(2g\sin\alpha) + mg\sin\alpha(OA - x) \\
&= mgOA\sin\alpha \\
T + V &= \textit{constant.}
\end{aligned}
$$

# BLOCK-II

# UNIT 6

# Motion of a System

**Objectives**

**6.1 Principle of Linear Momentum**

**6.2 Motion of the Mass Centre**

**6.3 Principle of Angular Momentum**

**Objectives**

Upon completion of this Unit, the student is exposed to

- motion of the mass centre.
- principle of angular momentum.

## 6.1 Principle of Linear Momentum

If $m_i$ and $\vec{v}_i$ are the mass and velocity of the $i^{th}$ particle of a system, then the linear momentum is

$$\vec{P} = \sum_{i=1}^{n} m_i \vec{v}_i \tag{6.1.1}$$

where $n$ is the number of particles. From (6.1.1),

$$\dot{\vec{P}} = \sum_{i=1}^{n} m_i \dot{\vec{v}}_i = \sum_{i=1}^{n} m_i a_i = \vec{F}, \tag{6.1.2}$$

where $\vec{F}$ is the vector sum of the extremal forces. This is known as the *principle of linear momentum*. It is stated as follows: “The rate of increase of the linear momentum of a system is equal to the vector sum of the external forces ”.

## 6.2 Motion of the Mass Centre

If $\overrightarrow{v_m}$ is the velocity of the mass center and $m$ is the total mass, the linear momentum $\vec{P}$ is $m\overrightarrow{v_m}$ and (6.1.2) gives

$$m\dot{\overrightarrow{v_m}} = \vec{F} \tag{6.2.1}$$

This is the equation of motion for a single particle of mass $m$ under a force $\vec{F}$. We have the following result: "The mass center of a system moves equal to the mass of the system, acted on by a force equal to the vector sum of the external forces acting on the system".

The above statement is useful in the sense that it reduces the determination of motion of the mass center of any system under known external forces to a problem in particle dynamics. As examples, we may consider the motion of a high - explosive shell or of the earth in its orbit round the sun. To find the motion of the mass center of the shell, we need to know only the sum of the forces exerted by the air on the elements of its surface and the weight of the shell.

Similarly, in the case of the earth, its mass center moves like a particle subject to the gravitational fields of the sun, moon and other bodies in the solar system.

## 6.3 Principle of Angular Momentum

The angular momentum of a system of a particles about a point $O$ is

$$\vec{H} = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i), \tag{6.3.1}$$

where $m_i$ is the mass of the $i^th$ particle, $\vec{r}_i$ is the position vector of the $i^{th}$ particle relative to $O$, $\vec{v}_i$ is the velocity of the $i^{th}$ particle relative to $O$, $n$ is the number of particles in the system.

Let us consider $O$ to be either a fixed point in a Newtonian frame of reference or the mass center of the system. The rate of change of $\vec{H}$ is

$$\dot{\vec{H}} = \sum_{i=1}^{n} \left( \dot{\vec{r}}_i \times m_i \vec{v}_i + \vec{r}_i \times m_i \dot{\vec{v}}_i \right)$$

$$= \sum_{i=1}^{n} \left( \vec{v}_i \times m_i \vec{v}_i + \vec{r}_i \times m_i \vec{a}_i \right)$$

$$\dot{\vec{H}} = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i), \tag{6.3.2}$$

where $\vec{a}_i$ is the acceleration of the $i^{th}$ particle relative to $O.$

If $O$ is a fixed point then $\vec{a}_i$ is the acceleration relative to a Newtonian frame and hence

$$m_i \vec{a}_i = \vec{F}_i + \vec{F}_i^{\,J},$$

where $\vec{F}_i,$ $\vec{F}_i^{\,J}$ are the external and internal forces on the $i^{th}$ particle respectively. Hence (**??**) becomes,

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i + \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^{\,J}. \tag{6.3.3}$$

The internal forces have no moment about any point and so the second summation vanishes. Hence

$$\dot{\vec{H}} = \vec{G} \tag{6.3.4}$$

where $\vec{G}$ is the total moment of the external forces about the fixed $O.$ If $O$ is the mass center, the acceleration of the $i^th$ particle relative to a Newtonian frame $S$ is $\vec{a}_m + \vec{a}_i,$ where $\vec{a}_m$ is the acceleration of $O$ relative to $S.$

Therefore the equation of motion of the $i^th$ particle is

$$m_i(\vec{a}_m + \vec{a}_i) = \vec{F}_i + \vec{F}_i^{\,J} \tag{6.3.5}$$

Substituting for $m_i \vec{a}_i$ in (6.3.3), we have

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times (\vec{F}_i + \vec{F}_i^{\,J} - m_i \vec{a}_m) = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i +$$

$$\sum_{i=1}^{n} \overrightarrow{r_i} \times \overrightarrow{F_i}' - \sum_{i=1} m_i \overrightarrow{r_i} \times \overrightarrow{a_m}$$

As before, the second summation vanishes and

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times (\vec{F}_i + \vec{F}_i' - m_i \vec{a}_m) = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i - \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{a}_m$$

The last term vanishes, since $\sum_{i=1}^{n} m_i \vec{r}_i = 0$. Hence, we obtain

$$\dot{\vec{H}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = \vec{G}$$

as in (6.3.4), where $G$ is now the total moment of the external forces about the mass center.

Hence the principle of angular momentum may be stated as follows: "The rate of change of the angular momentum of a system about a fixed point, either fixed or moving with the moving center, is equal to the total moment of the external forces about that point. Symbolically, $\dot{\vec{H}} = \vec{G}$.

**Example 7.** Let us consider a cylinder rolling down an inclined plane. The mass center moves in a vertical plane and so the vectors $\dot{\vec{v}}_m$ and $\vec{F}$ lie in this plane. The angular velocity $\vec{\omega}$ is parallel to the axis of the cylinder and the angular momentum about the mass center is $\vec{H} = I\vec{\omega}$, where $I$ is the moment of inertia about the axis of the cylinder.

**Principle of Energy** The law of conservation of energy states that the total mechanical energy of a system is constant.

$$i.e., T + V = E,$$

where $T$ is the kinetic energy, $V$ is the potential energy and $E$ is the constant total energy.

# BLOCK-II

# UNIT 7

# Motion of a Rigid Body

**Objectives**

**7.1 Rigid Body with a Fixed Point**

**7.2 General Motion of a Rigid Body**

**7.3 worked Examples**

**Objectives**

Upon completion of this Unit, the students will be able to

x identify the rigid body with a fixed point.

x understand general motion of a rigid body.

## 7.1 Rigid Body with a Fixed Point

Consider a rigid body constrained to rotate about a fixed point $O$. The angular momentum about $O$ is

$$\vec{H} = A\omega_1 \vec{i} + B\omega_2 \vec{j} + C\omega_3 \vec{k} \tag{7.1.1}$$

where $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors in the directions of principal axes of inertia at $O$. $A, B, C$ are the principal moments of inertia at $O$, and $\omega_1, \omega_2, \omega_3 =$ components of the angular velocity $\vec{\omega}$ of the body in the directions of, $\vec{i}, \vec{j}, \vec{k}$ respectively.

In general, the principal axes at $O$ are fixed in the body, where the triad $\vec{i}, \vec{j}, \vec{k}$ has the angular velocity $\vec{\omega}$. In some cases, a principal triad which is neither fixed in the body nor in space may be used. Hence to check for all possibilities, let us denote the angular velocity of the triad by $\vec{\Omega}$, where $\vec{\Omega} = \vec{\omega}$ if the triad is fixed in the body.

$$Let\ \vec{\Omega} = \Omega_1 \vec{i} + \Omega_2 \vec{j} + \Omega_3 \vec{k} \tag{7.1.2}$$

If $\vec{P}$ is any vector, then using rate of change of vector,

$$\frac{d}{dt}\vec{P} = \frac{\delta \vec{P}}{\delta t} + \vec{\Omega} \times \vec{P} \tag{7.1.3}$$

Applying (7.1.3) to the angular momentum $\vec{H}$, we have

$$\dot{\vec{H}} = \frac{\delta H}{\delta t} + \vec{\Omega} \times \vec{H} \tag{7.1.4}$$

$$\dot{\vec{H}} = A\dot{\omega}_1 \vec{i} + B\dot{\omega}_2 \vec{j} + C\dot{\omega}_3 \vec{k} + (\Omega \vec{i} + \Omega_2 \vec{j} + \Omega_3 \vec{k}) \times (Aw_1 \vec{i} + Bw_2 \vec{j} + Cw_3 \vec{k})$$

$$\dot{\vec{H}} = A\dot{\omega}_1 \vec{i} + B\dot{\omega}_2 \vec{j} + C\dot{\omega}_3 \vec{k} + \vec{i}\,(C\Omega_2\omega_3 - Bw_2\Omega_3) - \vec{j}\,(Cw_3\Omega_1 - A\Omega_3\omega_1) + \vec{k}\,(Bw_2\Omega_1 - Aw_1\Omega_2)$$

$$\dot{\vec{H}} = (A\dot{\omega}_1 - Bw_2\Omega_3 + C\Omega_2\omega_3)\,\vec{i} + (B\dot{\omega}_2 - Cw_3\Omega_1 + Aw_1\Omega_3)\,\vec{j} + (C\dot{\omega}_3 - Aw_1\Omega_2 + Bw_2\Omega_1)\,\vec{k} \tag{7.1.5}$$

We know that $\dot{\vec{H}} = \vec{G}$, where $\vec{G}$ is the total moment of the external forces about $O$. Hence

(7.1.5) gives "the equations of motion of a rigid body with a fixed point"which are

$$A\dot{\omega}_1 - Bw_2\Omega_3 + Cw_3\Omega_2 = G_1$$

$$B\dot{\omega}_2 - Cw_3\Omega_1 + Aw_1\Omega_3 = G_2$$

$$C\dot{\omega}_3 - Aw_1\Omega_2 + Bw_2\Omega_1 = G_3$$

where $G_1, G_2, G_3$ are the components of $\vec{G}$ along $\vec{i}, \vec{j}, \vec{k}$.

If $\vec{i}, \vec{j}, \vec{k}$ are fixed in the body, so that $\vec{\Omega} = \vec{\omega}$, the equations becomes

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = G_1$$

$$B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = G_2$$

$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 =$$

(7.1.6)

(7.1.7)

The above equations (7.1.7) are called the "Euler's equation of motion of a rigid body with a fixed point. "If $T$ is the kinetic energy given by $\frac{1}{2}mv^2$ and $V$ is the potential energy, then $T + V = E$ by the principle of energy, where $E$ is the energy constant.

When the working forces are conservative, we can use in place of any of the three equations in (7.1.6) or (7.1.7), the following equation is deduced from the above principle of energy.

$$\frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2) + V = E. \tag{7.1.8}$$

## 7.2 General Motion of a Rigid Body

Let us consider a rigid body moving in a general motion. Let $\vec{F}$ be the total external force and $\vec{G}$ be the total moment of the external forces about the mass center.

If $m$ is the mass of the body and $\vec{a}$ is the acceleration of the mass center, then $\vec{F} = m\vec{a}$. If the angular momentum about the mass center is $G$, then $\dot{\vec{H}} = \vec{G}$, for any motion relative to the mass center.

Let us resolve the vectors $\vec{a}, \vec{F}, \vec{H}, \vec{G}$ along a principal triad $\vec{i}, \vec{j}, \vec{k}$ at the mass center. Its angular velocity will be denoted by $\vec{\Omega}$. If the triad is fixed in the body, then $\vec{\Omega} = \vec{\omega_1}$ the angular velocity of the body. Then by (7.1.3)

$$a = \frac{\delta \vec{v}}{\delta t} + \Omega \times \vec{v}, \tag{7.2.1}$$

where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is the velocity of the mass center. Substituting for $\vec{a}$ in $\vec{F} = m\vec{a}$ and noting that $\dot{\vec{H}} = \vec{G}$ leads to equations of the form (7.1.6), we have the following scalar equations of motion.

$$\begin{aligned} m(\dot{v}_1 - v_2\Omega_3 + v_3\Omega_2) &= F_1 \\ m(\dot{v}_2 - v_3\Omega_1 + v_1\Omega_3) &= F_2 \\ m(\dot{v}_3 - v_1\Omega_2 + v_2\Omega_1) &= F_3 \\ A\dot{\omega}_1 - Bw_2\Omega_3 + Cw_3\Omega_2) &= G_1 \\ B\dot{\omega}_2 - Cw_3\Omega_1 + Aw_1\Omega_3) &= G_2 \\ C\dot{\omega}_3 - Aw_1\Omega_2 + Bw_2\Omega_1) &= G_3 \end{aligned} \tag{7.2.2}$$

Here the constants $A$, $B$, $C$ are the principal moments of inertia at the mass center. The equations (7.2.2) are six equations for the components of velocity of the mass center and the components of angular velocity of the body. For any one of these six equations, we can substitute the law of conservation of energy

$$T + V = E \tag{7.2.3}$$

provided the external forces are conservative. Explicitly (7.2.3) yields

$$\frac{1}{2}m(v_1^2 + v_2^2 + v_3^2) + \frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2) + V = E \tag{7.2.4}$$

## 7.3 Worked Examples

**Example 8.** Show from Euler equations (7.1.7) that if $\vec{G} = \vec{0}$ and $A = B,$ then $\vec{\omega}$ is a constant.

**Solution.** $\vec{G} = \vec{0} \Rightarrow G_1\vec{i} + G_2\vec{j} + G_3\vec{k} = \vec{0} \Rightarrow G_1 = G_2 = G_3 = 0.$ Putting $A = B,$ the three Euler's equations of motion become

$$\begin{aligned} A\dot{\omega}_1 - (A - C)\omega_2\omega_3 &= G_1 = 0 \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= G_2 = 0 \\ C\dot{\omega}_3 - (A - A)\omega_1\omega_2 &= G_3 = 0 \end{aligned}$$

The third equation become $C\dot{\omega}_3 = 0 \Rightarrow \omega_3 = constant = k_1(say)$

Substituting $\omega_3 = k_1$ in the first two equations, we have

$$\begin{aligned} A\dot{\omega}_1 - (A - C)\omega_2 k_1 &= 0 \\ A\dot{\omega}_2 - (C - A)\omega_1 k_1 &= 0 \end{aligned} \tag{7.3.1}$$

$$\begin{aligned} \frac{(A - C)\omega_2 k_1}{\dot{\omega}_1} &= \frac{(C - A)\omega_1 k_1}{\omega_2} \\ \frac{\omega_2}{\dot{\omega}_1} &= -\frac{\omega_1}{\dot{\omega}_2} = k_2 \ (say) \\ \Rightarrow \dot{\omega}_2 &= k_3\omega_1 \quad (k_3 = -\frac{1}{k_2}) \\ \dot{\omega}_1 &= k_4\omega_2 \quad (k_4 = \frac{1}{k_2}) \end{aligned}$$

Hence differentiating the two equations (7.3.1) with respect to time, we have

$$\begin{aligned} A\ddot{\omega}_1 - (A - C)k_1\dot{\omega}_2 &= 0 \\ A\ddot{\omega}_2 - (C - A)k_1\dot{\omega}_1 &= 0 \\ i.e.,\ A\ddot{\omega}_1 - (A - C)k_1 k_3\omega_1 &= 0 \\ A\ddot{\omega}_2 - (C - A)k_1 k_4\omega_2 &= 0 \end{aligned}$$

The above two equations represent the equation of simple harmonic motion given by $\ddot{x} + n^2 x = 0$, where $n$ is the frequency about $z$ axis.

$$n = \sqrt{\frac{A - C}{A}}\, k, \tag{7.3.2}$$

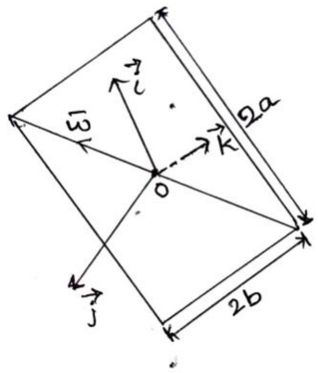
where $K$ is a constant. We know the simple harmonic oscillation can be written as $\omega_1 = C_1$ constant and $\omega_2 = C_2 \sin nt$ where $C_1$ and $C_2$ are constants and $n$ is given by (7.3.2). Thus

$$\begin{aligned} \vec{\omega} &= \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k} \\ \vec{\omega} &= C_1 \cos nt\, \vec{i} + C_2 \sin nt\, \vec{j} + k_1 \vec{k},\ (\omega_3 = k_1). \end{aligned}$$

The vector $\omega_1 \vec{i} + \omega_2 \vec{j} = C_1 \cos nt\, \vec{i} + C_2 \sin nt\, \vec{j}$ is a vector rotating round the fixed vector $\omega_3 \vec{k}$ with frequency $n$. Hence the angular velocity $\vec{\omega}$ is a constant.

**Example 9.** A rectangular plate spins with constant angular velocity $\omega$ about a diagonal. Find the couple which must out on the plate in order to maintain this motion.

**Solution.** Let $O$ be the mass center of the plate and let $\vec{i}$, $\vec{j}$ and $\vec{k}$ be the unit vectors along the perpendicular axes of inertia at $O$.
The vector $\vec{k}$ is normal to the plate and $\vec{i}$ and $\vec{j}$ lie on the plane of the plate, $\vec{i}$ being parallel




to the length. The principal moments of inertia at $O$ are

$$A = \frac{1}{3}mb^2, \quad B = \frac{1}{3}ma^2, \quad C = \frac{1}{3}m(a^2 + b^2) \tag{7.3.3}$$

where $m$ is the mass of the plate, $2a$ its length and $2b$ its breadth. Let $\alpha$ be the angle between $\vec{i}$ and the axis of rotation so that $\tan\alpha = \frac{b}{a}$. Then the angular velocity of the plate is

$$\vec{\omega} = \omega\cos\alpha\, \vec{i} + \omega\sin\alpha\, \vec{j}.$$

Therefore, the $\vec{i}$, $\vec{j}$ and $\vec{k}$ components of $\vec{\omega}$ are

$$\omega_1 = \omega\cos\alpha, \quad \omega_2 = \omega\sin\alpha \quad \omega_3 = 0 \tag{7.3.4}$$

respectively. Substituting (7.3.3) and (7.3.4) in the Euler's equations of motion, we have

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = G_1$$
$$A\dot{\omega}_2 - (C - A)\omega_3\omega_1 = G_2$$
$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = G_3$$

implies that $G_1 = 0,\ G_2 = 0,\ G_3 = 0 - \left(\frac{1}{3}ma^2 - \frac{1}{3}mb^2\right)\omega^2 \sin\alpha\cos\alpha$

$$i.e., G_3 = \frac{1}{3}m(a^2 - b^2)\omega^2 \sin\alpha\cos\alpha.$$

$$\tan\alpha = \frac{b}{a} \Rightarrow \sin\alpha = \frac{b}{\sqrt{a^2+b^2}},\ \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}.$$

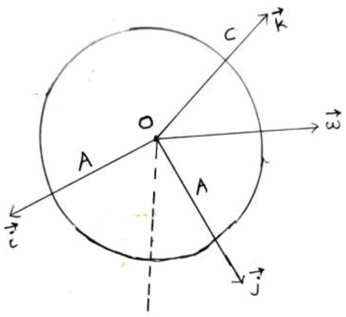
$$\therefore G_3 = \frac{1}{3}m\omega^2(a^2 - b^2)\frac{ab}{\sqrt{a^2+b^2}}$$

These are the components of the couple $\vec{G}$ acting on the plate.

**Example 10.** A circular disc of radius $r$ and mass $m$ is supported on a needle point at its center. It is set spinning with angular velocity $\omega_0$ about a line making an angle $\theta$ with the normal to the disk.
(i) Find the angular velocity of the disk at any subsequent time.
(ii) Find the motion in space of the disk.

**Solution.** In the following figure, $\vec{k}$ is a unit vector normal to the disk at the center $O$, and $\vec{j}$ are fixed in the plane of the disk. Let us choose $\vec{j}$ so that the initial velocity of the disk lies in the plane of $\vec{k}$ and $\vec{j}$.

The angular velocity at any time is $\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$. At $t = 0$, $\omega_1 = 0$, $\omega_2 = \omega_0 \sin\theta$,




$\omega_3 = \omega_0 \cos\theta$. The principal moments of inertia at $O$ are

$$A = \frac{1}{4}ma^2, \quad B = \frac{1}{4}ma^2, \quad C = \frac{1}{2}ma^2.$$

The external forces are the reaction and the weight of the disk. These forces have no moment about $O$, the Euler's equations become

$$\begin{aligned} A\dot{\omega}_1 - (A - C)\omega_2\omega_3 &= 0 \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 &= 0 \end{aligned}$$

From the last equation, $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = constant.$

$$\omega_3 = \omega_0 \cos\theta.$$

Multiplying the second equation by $i(=\sqrt{-1})$ and adding with the first equation.

$$\begin{aligned} A(\dot{\omega}_1 + i\dot{\omega}_2) + (C - A)\omega_3(\omega_2 - i\omega_1) &= 0 \\ A(\dot{\omega}_1 + i\dot{\omega}_2) - i(C - A)\omega_0(\omega_1 + i\omega_2) &= 0 \end{aligned}$$

Let $\psi = \omega_1 + i\omega_2$, then we have

$$A\dot{\psi} - i(C - A)\omega_0 \cos\theta(\omega_1 + i\omega_2) = 0.$$

$$A = \frac{1}{4}ma^2 \qquad C = \frac{1}{2}ma^2 \Rightarrow C = 2A$$

Hence the above equations becomes

$$\begin{aligned} A\dot{\psi} - i(A)\omega_0 \cos\theta(\psi) &= 0 \\ \Rightarrow \dot{\psi} - i\omega_0 \cos\theta(\psi) &= 0 \end{aligned}$$

Solving the above differential equation

$$\psi = ke^{i\omega_0 t\cos\theta},$$

where $k$ is a constant. Now $\psi = \omega_1 + i\omega_2$, At $t = 0$, $\psi = 0 + i\omega_0 \sin\theta = k.$ Hence

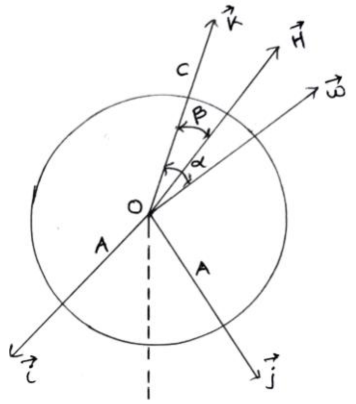
$$\psi = i\omega_0 \sin\theta \left[\cos(\omega_0 t\cos\theta) + i\sin(\omega_0 t\cos\theta)\right].$$

Equating the real and imaginary parts of $\psi$, we have

$$
\begin{aligned}
\omega_1 &= -\omega_0 \sin\theta \sin(\omega_0 t \cos\theta) \\
\omega_2 &= \omega_0 \sin\theta \cos(\omega_0 t \cos\theta)
\end{aligned}
$$

and $\omega_3 = \omega_0 \cos\theta.$ (ii) To find the motion of the disc in sphere:

In figure, $\vec{H}$ is the angular momentum, $\vec{\omega}$ is the angular velocity, $\vec{i}$ and $\vec{j}$ are unit vectors




in the plane of the disk, but not fixed in it, $\vec{k}$ is a unit vector normal to the plane. The vector $\vec{j}$ is taken in the plane determined by $\vec{H}$ and $\vec{k}$. The following facts are noted: (a) The external forces have no moment about $O$. Hence $\vec{H}$ is a constant vector. i.e., it has a fixed direction in space determined by the initial conditions.

(b) $\vec{H} = A\omega_1 \vec{i} + B\omega_2 \vec{j} + C\omega_3 \vec{k}$ and since $A = B$, $\vec{H} = A\omega_1 \vec{i} + A\omega_2 \vec{j} + C\omega_3 \vec{k}$ and $H_1 = A\omega_1 = 0$ implies $\omega_1 = 0$. So $\omega$ lies in the plane of $\vec{j}$ and $\vec{k}$.

(c) The orthogonal triad $\vec{i}, \vec{j}, \vec{k}$ is not fixed in the disk. So its angular velocity $\vec{\Omega}$ is not same as $\vec{\omega}$. The vector $\vec{k}$ is fixed both in the triad and in the disk. Here the extremity of $\vec{k}$ has a velocity $\vec{\Omega} \times \vec{k}$ (considering as a point of the triad) and a velocity $\vec{\omega} \times \vec{k}$ (considering as a point of the disk).

$$\therefore (\Omega_1 \vec{i} + \Omega_2 \vec{j} + \Omega_3 \vec{k}) \times \vec{k} = (\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}) \times \vec{k}$$

$$\Rightarrow \Omega_1 = \omega_1 = 0 \quad \textit{and} \quad \Omega_2 = \omega_2$$

Substituting these in the equations of motion of a rigid body with a fixed point which are given by

$$A\dot{\omega}_1 - B\omega_2\Omega_3 + C\omega_3\Omega_2 = G_1$$
$$B\dot{\omega}_2 - C\omega_3\Omega_1 + A\omega_1\Omega_3 = G_2$$
$$C\dot{\omega}_3 - A\omega_1\Omega_2 + B\omega_2\Omega_1 = G_3$$

We have

$$\begin{aligned} -B\omega_2\Omega_3 + C\omega_3\Omega_2 &= 0 \\ A\dot{\omega}_2 &= 0 \\ C\dot{\omega}_3 &= 0 \end{aligned}$$

$$\begin{aligned} i.e., -A\omega_2\Omega_3 + C\omega_3\Omega_2 &= 0 \ (since\ A = B) \\ A\dot{\omega}_2 &= 0 \\ C\dot{\omega}_3 &= 0 \end{aligned}$$

Thus $\omega_2, \omega_3, H_2, H_3$ are constants and

$$\frac{\Omega_3}{\Omega_2} = \frac{\Omega_3}{\omega_2} = \frac{C\omega_3}{A\omega_2} = \frac{H_3}{H_2}$$

From the first equation, $A\omega_2\Omega_3 = C\omega_3\Omega_2 \Rightarrow \dfrac{\Omega_3}{\Omega_2} = \dfrac{\Omega_3}{\omega_2} = \dfrac{C\omega_3}{A\omega_2}$ Therefore the angular velocity $\Omega$ of the triad has a constant magnitude and lies along the fixed direction $\vec{H}$.

Hence we have the following obvious facts.

(a) The disc spins about its normal $\vec{k}$ at a constant rate $\omega_1$.

(c) The angle $\beta$ between $\vec{k}$ and $\vec{H}$ given by $H\cos\beta = h_3$ is constant; the normal to the disc moves on a cone with axis $\vec{H}$, turning about $\vec{H}$ at the constant rate $\Omega$.

(c) The angle $\theta$ between $\vec{\omega}$ and $\vec{k}$ given by $\omega\cos\theta = \omega_3$ is constant; the angular velocity $\vec{\omega}$ describes a cone about the normal to the disk. This is the body cone. The angle $\theta - \beta$ between $\vec{\omega}$ and $\vec{H}$ is also a constant and so the space cone has constant semi vertical angle $\theta - \beta$ and axis $\vec{H}$. It lies inside the body cone.

# BLOCK-III

# UNIT 8

# Motion of a rigid body with a fixed point under no forces



**Objectives**

Upon completion of this Unit, the students will be able to

- identify the method of poinsot.
- find quantitative description of motion by using analytic method.

## 8.1 Introduction

If a rigid body is constrained to turn about a smooth fixed axis under no forces other than the reaction of the axis, the motion is extremely simple: the body spins with constant angular velocity. But if instead of fixing "a line "in the body, we fix "a point "only, the motion under no forces is very much complicated. The problem of finding the motion in the latter case is interesting than the former case. The mouting of a body so as to fix only one point may be done by an arrangement of light rings called as "Cardan's suspension ". The body is represented by the inner circle. The points $P$ and $Q$ are the fixed points. Rotation of the ring $r_1$ about $PQ$ gives one degree of freedom. Rotation of the ring $r_2$ about *RS* gives the second degree of freedom. Rotation of the body itself about *AB* gives the third. The point $C$ is the common intersection of *PQ*, *RS* and *AB*. The body can take up all positions in which the point $C$ of the body is fixed in space. In the

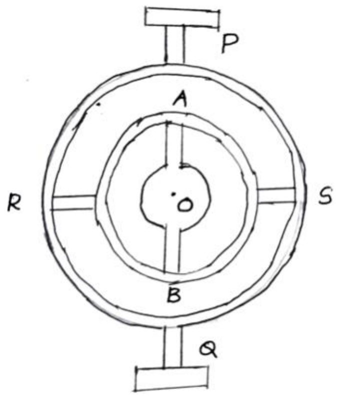



Figure 8.1.1: Cardan's suspension

mathematical theory, all the apparatus except the body itself is to be regarded as massless. But this is practically impossible. The masses of $r_1$ and $r_2$ are made as small as possible compared with the mass of the body.

The problem of motion of a body with a fixed point under no force can be treated in two ways viz the descriptive method and the analytic method. The descriptive method otherwise known as the method of Poinsot gives a good qualitative idea of the motion. The analytic method gives a quantitative description.

# The Method of Poinsot

Let $O$ be the fixed point in the body and $A, B, C$ the principal moments of inertia at $O$. Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors fixed in the body and directed along the principal axes at $O$. The angular velocity and angular momentum are given by

$$\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k} \tag{8.1.1}$$

and

$$\vec{H} = A\omega_1 \vec{i} + B\omega_2 \vec{j} + C\omega_3 \vec{k} \tag{8.1.2}$$

respectively. When we say that the body is under no forces, we mean that the forces acting on the body have no moment about $O$. i.e., the external forces do no work and have no moment about $O$. Then we have

(i) The kinetic energy $T$ is constant.

(ii) The angular momentum $\vec{H}$ is a constant vector.

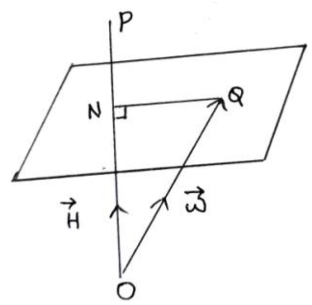
$$(i) \Rightarrow T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) = constant$$

$$i.e., 2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = constant \tag{8.1.3}$$

The angular momentum $\vec{H}$ has a direction fixed in space and a constant magnitude.

$$\text{(ii)} \Rightarrow A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 = constant \tag{8.1.4}$$

We draw through $O$ a line $OP$ in the fixed direction of $\vec{H}$ (see Fig.) This line is called the "invariable line. "Let $\overrightarrow{OQ}$ represent the angular velocity $\vec{\omega}$ at any instant. Drop the perpendicular $QN$ on $OP$. Then $ON = \vec{\omega}.\frac{\vec{H}}{H}$



From (8.1.1) and (8.1.2),

$$\begin{aligned}\vec{\omega}.\vec{H} &= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \\ &= 2T \quad from\ (8.1.3) \\ i.e., ON &= \frac{2T}{H} = constant.\end{aligned}$$

Then $N$ is a fixed point during the motion and so the plane through $N,$ perpendicular to the invariable line $OP,$ is a fixed plane; It is called the "invariable plane. "The point $Q$ of the angular velocity vector $\vec{\omega}$ moves on the invariable plane.

Let us consider the point of the view of an observer who moves with the body. To him the vectors $\vec{i}, \vec{j}, \vec{k}$ are fixed, but both the vectors $\vec{h}$ and $\vec{\omega}$ are changing. If $\vec{i}, \vec{j}, \vec{k}$ are taken as the co-ordinate axes and the extremity of the vector $\vec{\omega}$ is given co-ordinates $x, y, z.$ Then $x = \omega_1, y = \omega_2, z = \omega_2.$ Therefore (8.1.3) and (8.1.4) become

$$A^2x^2 + B^2y^2 + C^2z^2 = h^2 \tag{8.1.5}$$

which represents the equations of ellipsoid. i.e., "To an observer moving with the body, the extremity $Q$ of the angular velocity $\vec{\omega}$ describes a curve which is the intersection of the two ellipsoids (8.1.5) fixed in the body. "The first ellipsoid is similar to the momental ellipsoid and has the same axes. It is called "the poinsot ellipsoid."

The invariable plane is fixed in space, but it is moving plane to the observer moving with the body. It touches a sphere of radius $ON,$ but it has another remarkable property, which states, "the invariable plane touches the poinsot ellipsoid at the extremity of the angular velocity vector."

**Proof.** The tangent plane to the poinsot ellipsoid at the point $(\omega_1, \omega_2, \omega_3)$ is

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = 2T$$

The direction ratios of the normal to the ellipsoid at this point are $A\omega_1, B\omega_2, C\omega_3$. But these are the components of angular momentum. Hence the normal to the poinsot ellipsoid at the extremity of the angular velocity vector is parallel to the angular momentum vector, i.e., parallel to $OP$ which proves the result.

**Remark.** There are two different points of view :
(a) the point of view of an observer $S$ fixed in space
(b) the point of view of an observer $S'$ fixed in the body.
It would be confusing to try to look simultaneously from two points of view. We have to see the situation separately.
(a) The observer $S$ fixed in space, cuts away (in his imagination) all the body except an ellipsoid pointsot ellipsoid. His attention is on the moving ellipsoid and on a fixed plane (the invariable plane). As the bodymoves, the ellipsoid always touches the plane. It actually rolls on the plane, since it has angular velocity vector which passes through the point of contact of the ellipsoid and the plane. This complicated motion becomes simple when the poitsot ellipsoid is a surface of revolution. If we think of the invariable plane as a sheet of paper and the poinsot ellipsoid as an inked surface, then a curve can be drawn in ink on the invariable plane in the course of motion. On joining the fixed point $O$ to the points on this curve, we get the "space cone."
(b) The observer $S'$ fixed in the body, turns his attention to the two ellipsoids fixed (according to him) and in particular to their curve of intersection. The angular velocity vector traces out a cone ("body cone ") formed by joining the fixed point $O$ to this curve.
In general, the body cone rolls on the space cone. The difference between the two cases is " $S$ regards the space cone as fixed, but $S'$ regards the body cone as fixed." ■

## 8.2 Analytic Method

The Poinsot method gives a qualitative description of the motion where as the analytic method gives a quantitative description.

Since the external forces have no moment about $O$, Euler's equation give

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = 0 \quad B\dot{\omega}_2 - (C - A)\omega_2\omega_1 = 0 \quad C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = 0 \tag{8.2.1}$$

From (8.1.3) and (8.1.4)

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T$$
$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 \tag{8.2.2}$$

where $H$ and $T$ are constants, which can be obtained by inserting the values of $\omega_1, \omega_2, \omega_3$ at $t = 0$.

Let us assume that $A, B, C$ are all distinct. We may suppose $\vec{i}, \vec{j}, \vec{k}$ be choosen, so that $A > B > C$.

From (8.2.2) $2AT - H^2 = (A - B)B\omega_2^2 + (A - C)C\omega_3^2$

$\implies 2AT - H^2 > 0$ (By our assumption $A > B > C$).

Again from (8.2.2) $2GT - H^2 = (G - A)A\omega_1^2 + (G - B)B\omega_2^2 + (G - C)C\omega_3^2$

$\implies 2GT - H^2 < 0$

Equation (8.2.1) has three simple particular solutions. They are

$\omega_1 =$ constant, $\omega_2 = 0$, $\omega_3 = 0$

$\omega_2 =$ constant, $\omega_3 = 0$, $\omega_1 = 0$

$\omega_3 =$ constant, $\omega_1 = 0$, $\omega_2 = 0$

Thus three solutions correspond to steady rotations about the three principal axes of inertia. These are the only axes about which the body will spin steadily under no forces. To find the most general solution of (8.2.1) we have to eliminate two of the unknowns to get a differential equation involving one unknown.

Let us solve (8.2.2) for $\omega_1^2$ and $\omega_3^2$ to obtain

$$\omega_1^2 = P - Q\omega_2^2$$
$$\omega_3^2 = R - S\omega_2^2 \tag{8.2.3}$$

where $p, Q, R, S$ are positive expressions involving $A, B, C, T, H$.

Substitution of the above in $B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = 0$

We get $\dot{\omega}_2 = \frac{(C - A)}{B}\omega_3\omega_1$

squaring

$$\dot{\omega}_2^2 = \frac{(C - A)^2}{B^2}\omega_3^2\omega_1^2$$

$$= \frac{(C - A)^2}{B^2}(P - Q\omega_2^2)(R - S w_2^2) \tag{8.2.4}$$

Let $\zeta = \frac{\omega_2}{\beta}, \quad \tau = pt$

where $\beta, p$ are constants which are positive functions of $A, B, C, T, H$.

Then (8.2.4) becomes,

$$\left(\frac{d\zeta}{d\tau}\right)^2 = (1 - \zeta^2)(1 - k^2\zeta^2) \tag{8.2.5}$$

where

$$\zeta = \frac{\omega_2}{\beta}, \quad \tau = pt \tag{8.2.6}$$

the constants $\beta, p, k$ being positive functions $A, B, C, T, H$ with $k < 1$.

Hence

$$\omega_2 = \beta\, sn\,[p(t - t_0)] \tag{8.2.7}$$

where $sn$ is the elliptic function and $t_0$ is a constant of integration.

Substituting in (8.2.3) we have either

$$\omega_1 = \alpha\, dn\,[p(t - t_0)], \quad \omega_3 = \gamma\, cn\,[p(t - t_0)] \tag{8.2.8}$$

$$or \quad \omega_1 = \alpha\, cn\,[p(t - t_0)], \quad \omega_3 = \gamma\, dn\,[p(t - t_0)] \tag{8.2.9}$$

where $\alpha$ and $\gamma$ are functions of $A, B, C, T, H$ determined except for sign.

Substituting in (8.2.1) we have $\alpha\beta\gamma$ to be negative.

For definiteness we may take '$\alpha$' positive by suitable choice of the sense of the vector $\vec{i}$ , then $\gamma$ is negative.

The constants $\beta, p, k$ are chosen in such a way that $k$ is less than unity.

For the case $H^2 > 2BT$ , we get

$\omega_1 = \alpha\, dn\,[p(t - t_0)]$

$\omega_3 = \gamma\, cn\,[p(t - t_0)]$

For $H^2 < 2BT$ , we get

$\omega_1 = \alpha\, cn\,[p(t - t_0)]$

$\omega_3 = \gamma\, dn\,[p(t - t_0)]$

But this does not complete the solution of the problem.

From the given initial conditions, we should be able to tell the position of the body at any time.

For this, let the directions of the trial $\vec{i}, \vec{j}, \vec{k}$ relative to a triad $\vec{I}, \vec{J}, \vec{K}$ fixed in space be expressed by means of the Eulerian angles $\theta, \varphi$ and $\psi$ . Then

$$\begin{aligned} \omega_1 &= \sin\psi\dot{\theta} - \sin\theta\cos\psi\dot{\varphi} \\ \omega_2 &= \cos\psi\dot{\theta} + \sin\theta\sin\psi\dot{\varphi} \\ \omega_3 &= \cos\theta\dot{\varphi} + \phi \end{aligned} \tag{8.2.10}$$

If we substitute for $\omega_1, \omega_2, \omega_3$ for (8.2.7), (8.2.8), (8.2.9), we get three differential equations for $\theta, \varphi, \psi$ .

We choose the vector $\vec{k}$ in the direction of the invariable line defined by the constant vector $\vec{H}$ .

Then the components of $\vec{H}$ along $\vec{i}\ \vec{j}\ \vec{k}$ are found by multiplying $H$ by the direction cosines of $\vec{k}$ relative to $\vec{i}$, $\vec{j}$, $\vec{k}$, which are given by $-sin\theta cos\psi$, $sin\theta sin\psi$, $cos\theta$.

Hence

$$Aw_1 = -Hsin\theta cos\psi bw_2 = Hsin\theta sin\psi Cw_3 = Hcos\theta \tag{8.2.11}$$

From the equation (8.2.11), we have

$$cos\theta = \frac{Cw_3}{H} tan\psi = -\frac{Bw_2}{a\omega_1} \tag{8.2.12}$$

$\therefore$ We get $\theta$ and $\psi$ as functions of '$t$' without any integration.

From the first two equations of (8.2.10)

$$\omega_2 sin\psi = sin\psi cos\psi\dot{\theta} + sin\theta sin^2\psi\dot{\varphi}$$

$$\omega_1 cos\psi = sin\psi cos\psi\dot{\theta} - sin\theta cos^2\psi\dot{\varphi}$$

$$\omega_2 sin\psi - \omega_1 cos\psi = sin\theta\dot{\psi} \tag{8.2.13}$$

From the above equation $\varphi$ is obtained by a quadrature (Since $\theta, \Psi, \omega_1, \omega_2$ are already known functions of $t$).

From the periodic property of elliptic funcxtions, we see that $\theta$, $sin\psi$, $cos\psi$, $\dot{\varphi}$ are periodic functions of $t$.

In general $\varphi$ does not increase by a multiple of $2\pi$ in a period, and the motion is not periodic as a whole.

# Check Your Progress

*1.* Find the kinetic energy of rotation of a rigid body with respect to the principal axes terms of Eulerian angles and interpret the result when $A = B$.

2. If $T$ is the kinetic energy, $\vec{G}$ is the external torque about the instantaneous axis of rotation and $\omega$ is the angular velocity, then prove that $\frac{dT}{dt} = \vec{G} \cdot \vec{\omega}$ 3. If $A, B, C, D, E, F, G, H$ are the moments and products of inertia of a rigid body about three mutually perpendicular and concurrent axes. Prove that the moment of inertia of the rigid body about an axis making angles $\alpha, \beta, \gamma$ with the original axes is given by $I = A\cos^{\alpha} + B\cos^2\beta + C\cos^2\gamma - 2F\cos\beta\cos\gamma - 2G\cos\gamma\cos\alpha - 2H\cos\alpha\cos\beta$.

# Answer to Check Your Progress

1. Hint: $T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$

$$T = \frac{1}{2}A(\sin\phi\,\dot{\theta} - \sin\theta\cos\phi\,\dot{\varphi})^2 + \frac{1}{2}B(\cos\phi\,\dot{\theta} + \sin\theta\sin\phi\dot{\varphi})^2 + \frac{1}{2}C(\cos\theta\,\dot{\varphi} + \dot{\phi})^2$$

$$when \quad A = B,\ T = \frac{1}{2}A(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{C}{2}(\dot{\varphi}\cos\theta + \dot{\phi})^2$$

2. Hint: $\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$, $\vec{G} = G_1\vec{i} + G_2\vec{j} + G_3\vec{k}$, $T = \frac{1}{2}A\omega_1^2 + \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2$

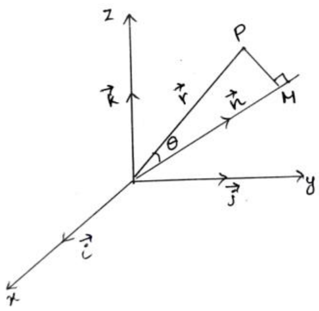
$$\begin{aligned}
\frac{dT}{dt} &= A\omega_1\dot{\omega}_1 + B\omega_2\dot{\omega} + C\omega_3\dot{\omega}_3 \\
&= \omega_1[G_1 + (B-C)\omega_2\omega_3] + \omega_2[G_2 + (C-A)\omega_3\omega_1] + \omega_3[G_3 + (A-B)\omega_1\omega_2]\ from\ Euler's\ equation \\
&= G_1\omega_1 + G_2\omega_2 + G_3\omega_3 \\
\frac{dT}{dt} &= \vec{G} \cdot \vec{\omega}
\end{aligned}$$

3. Hint:

$$PM = r\sin\theta = |\vec{r}\times\vec{n}|,\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$PM = |\vec{r}\times\vec{n}|^2 = (y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2$

$I = {}^{\cdot}\; m\,PM^2$ gives the required expression.



# BLOCK-III

# UNIT 9

# The Spinning Top



**Objectives**

Upon completion of this Unit, the student is exposed to

$x$ general motion of a top.

## 9.1 Introduction

The spinning top is one of the most familiar example of a gyroscope system. The word "gyroscope"is used for any system in which a rapidly rotating body is a mainted that it may change the direction of its angular velocity vector.

A top means a rigid body with an axis of symmetry acted on by the force of gravity. A point on the axis of symmetry is fixed. Thus we idealise the ordinary top by supposing it to terminate in s a sharp point (vertex) and to spin on a floor rough enough to prevent slipping.

## 9.2 Steady Precession of a Top

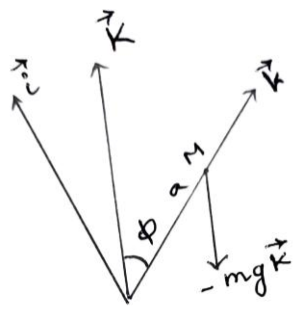
The motion of any rigid body with fixed point $O$ satisfies the equation

$$\dot{\vec{H}} = \vec{G} \tag{9.2.1}$$

where $\vec{H}$ is the angular momentum about $O$ and $\vec{G}$ is the moment of the external forces about $O$. In most of dynamical problems, we usually consider the forces as "given "and the motions as "unknown. "In these cases $\vec{G}$ is given and $\vec{H}$ is to be found. But we look at (9.2.1) in another way. We may regard the motion as "prescribed "so that $\vec{H}$ is known as a vector function of the time. Then (9.2.1) shows directly the moment $\vec{G}$ which must be applied to the body in order to give this motion.

Let us now describe a simple motion of a top, called "steady precession "and find what forces must act on the top in order that this motion may take place. In steady precession, the axis of symmetry of the top describes with constant angular velocity a right circular cone with the vertical for axis. At the same instant, the top spins about its axis of symmetry with constant angular velocity.

We use the following notations:




$m =$ mass of the top.

$a =$ distance of mass center $M$ from fixed vertex $O.$

$A =$ transverse moment of inertia at $O.$

$C =$ axial moment of inertia at $O.$

$\vec{K} =$ unit vector directed vertically upward

$\varphi =$ inclination of $OM$ to the vertical.
$\vec{i}, \vec{j}, \vec{k}$ = unit orthogonal triad with $\vec{k}$ along $OM$ and $\vec{i}$ in the plane of $\vec{k}$ and $\vec{K}$.
We see that $\vec{K} = \sin\varphi\,\vec{i} + \cos\varphi\,\vec{k}$. The angular velocity vector $\vec{\omega}$ of the top lies in the plane $(\vec{k}, \vec{K})$. We resolve $\vec{\omega}$ along $\vec{i}$ and $\vec{k}$,

$$i.e., \vec{\omega} = \omega_1\vec{i} + s\vec{k} \tag{9.2.2}$$

where $s$ denotes the "spin "of the top. The velocity of the point $M$ is

$$\begin{aligned}\vec{\omega} \times a\vec{k} &= (\omega_1\vec{i} + s\vec{k}) \times a\vec{k}\\ &= -\omega_1 a\vec{j}, \; (\vec{i} \times \vec{k} = -\vec{j}).\end{aligned}$$

We under by the "precession $p$ "the angular velocity with which $OM$ rotates about $\vec{K}$. The velocity of $M$ is

$$\begin{aligned}\varphi\vec{K} \times a\vec{k} &= pa\sin\varphi(-\vec{j})\\ &= -pa\sin\varphi\,\vec{j}\end{aligned}$$

Equating the two expressions for the velocity of $M$, we have

$$-\omega_1 a\vec{j} = -pa\sin\varphi\,\vec{j}$$

$$\omega_1 = p\sin\varphi \tag{9.2.3}$$

In the study precession, $\varphi$, $s$ and $p$ are constants. The angular momentum is

$$\vec{H} = A\omega_1\vec{i} + Cs\vec{k}$$

$$\vec{H} = Ap\sin\varphi\,\vec{i} + Cs\vec{k} \tag{9.2.4}$$

The above vector lies in the plane $(\vec{k}, \vec{K})$ and rotates rigidly with it. Thus $\dot{\vec{H}}$ is the velocity of a point with position vector $\vec{H}$ in a rigid body whose angular velocity is $p\vec{K}$. Hence

$$\begin{aligned}
\dot{\vec{H}} &= p\vec{K} \times \vec{H} \\
&= p(\sin\varphi\,\vec{i} + \cos\varphi\,\vec{k}) \times (Ap\sin\varphi\,\vec{i} + Cs\,\vec{k}) \\
&= p\sin\varphi Cs(-\vec{j}) + Ap\sin\varphi p\cos\varphi\vec{j} \quad \vec{k} \times \vec{i} = \vec{j} \\
&= p\sin\varphi(Ap\cos\varphi - Cs)\vec{j}
\end{aligned}$$

$$\dot{\vec{H}} = p\sin\varphi(Ap\cos\varphi - Cs)\vec{j} \tag{9.2.5}$$

The steady precession takes place with assigned values of $\varphi$, $p$ and $s$ provided that the moment of about $O$ of all forces (including gravitational force) is

$$\vec{G} = p\sin\varphi(Ap\cos\varphi - Cs)\vec{j}, \textit{ since } (\dot{\vec{H}} = \vec{G}) \tag{9.2.6}$$

Now the weight of the top is a force given by $-mg\vec{k}$ at $M$ and hence has a moment

$$a\vec{k} \times (-mg\vec{k}) = -mga\sin\varphi\vec{j} \tag{9.2.7}$$

about $O$. If this moment is equal to $\vec{G}$, then from (9.2.6) and (9.2.7) no force other than the weight of the top is required to maintain the motion. Thus the steady precession takes place under gravitational force alone if

$$p\sin\varphi(Ap\cos\varphi - Cs) = -mga\sin\varphi$$

$$\textit{i.e.}, \; p(Cs - Ap\cos\varphi) = mga \tag{9.2.8}$$

This is the single equation connecting the three constants $\varphi$, $p$ and $s$. Therefore, there is a doubly infinite set of steady precessions corresponding to arbitrary values of two out of the three constants. However, it is not possible to assign completely arbitrary values of two of the constants;

these values must be such that (9.2.8) yields a real value for the third constant.

If we see a top spinning, $\varphi$ and $p$ are easy to observe. Therefore $s$ in terms of $\varphi$ and $p$ is given by

$$s = \frac{mga}{pC} + \frac{Ap\cos\varphi}{C} \tag{9.2.9}$$

We see that if the precession is small, the spin is great and is given approximately by

$$s = \frac{mga}{pC} \tag{9.2.10}$$

## 9.3 General Motion of a Top

Let $\vec{I}$, $\vec{J}$, $\vec{K}$ be a fixed orthogonal triad, $\vec{K}$ being directed vertically upward. Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be an orthogonal triad with $\vec{k}$ pointing along *OM,* the axis of symmetry of the top, and $\vec{i}$ coplanar with $\vec{k}$ and $\vec{K}$; thus $\vec{j}$ is horizontal. The triad $\vec{i}$, $\vec{j}$, $\vec{k}$ is fixed neither in space nor in the top, but $\vec{k}$ is fixed on the top.

Let $\varphi$, $\phi$ represent the polar angles of $\vec{k}$ relative to the fixed triad. Variation in $\varphi$ are referred to as “nutation ”and variations in $\phi$ are referred to as “precession. ”Let

$$\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k} \tag{9.3.1}$$

be the angular velocity of the top, and

$$\Omega = \Omega_1\vec{i} + \Omega_2\vec{j} + \Omega_3\vec{k} \tag{9.3.2}$$

be the angular velocity of the triad $\vec{i}$, $\vec{j}$, $\vec{k}$. Now

$$\Omega_1 = \dot{\phi}\sin\varphi, \Omega_2 = -\dot{\varphi} \quad \dot{\phi}\cos\varphi \tag{9.3.3}$$

The relative motion of the top and the triad $\vec{i}, \vec{j}, \vec{k}$ consists only of a rotation about $\vec{k}$.

$$\begin{aligned} \omega_1 &= \Omega_1 = \dot{\phi} \sin\varphi \\ \omega_2 &= \Omega_2 = -\dot{\varphi} \end{aligned} \tag{9.3.4}$$

The angular momentum is

$$\vec{H} = A\omega_1 \vec{i} + A\omega_2 \vec{j} + C\omega_3 \vec{k} \tag{9.3.5}$$

and its rate of change is

$$\dot{\vec{H}} = A\dot{\omega}_1 \vec{i} + A\dot{\omega}_2 \vec{j} + C\dot{\omega}_3 \vec{k} + \vec{\Omega} \times \vec{H} \tag{9.3.6}$$

(since $\frac{d\vec{p}}{dt} = \frac{\delta \vec{p}}{\delta t} + \vec{\Omega} \times \vec{p}$, $\vec{p}$ any vector, $\Omega$ angular velocity relative to a frame $s$)

The moment about $O$ of the weight of the top is

$$\vec{G} = a\vec{k} \times (-mg\vec{K})$$

$$\vec{G} = -mga \sin\varphi \vec{j} \tag{9.3.7}$$

The motion of the top satisfies $\dot{\vec{H}} = \vec{G}$. Whenever substitute the expressions from (9.3.6) and (9.3.7), the vector equation $\dot{\vec{H}} = \vec{G}$ gives three scalar equations for $\varphi$, $\phi$ and $\omega_3$. The third component in the direction of $\vec{k}$ gives $C\dot{\omega}_3 = 0$, since from (9.3.4) and (9.3.5) $\vec{\Omega}\ \vec{H}$ has no component in the direction of $\vec{k}$.

Hence $\omega_3 = s$, a constant. That is the spin of the top is a constant. Resolving $\vec{K}$ along $\vec{i}$ and $\vec{k}$, we have

$$\vec{K} = \sin\varphi \vec{i} + \cos\varphi \vec{k}. \tag{9.3.8}$$

Since the weight of the top has no moment about $\vec{K}$, the component of angular momentum in this fixed direction is constant and hence

$$\vec{H} \cdot \vec{K} = \alpha, \text{ a constant} \tag{9.3.9}$$

From (9.3.8), (9.3.4) and (9.3.5), we have

$$
\begin{aligned}
(A\omega_1 \vec{i} + A\omega_2 \vec{j} + C\omega_3 \vec{k}) \cdot (\sin\varphi \vec{i} + \cos\varphi \vec{k}) &= \alpha \\
A\omega_1 \sin\varphi + C\omega_3 \cos\varphi &= \alpha \\
i.e.,\ A(\dot{\phi} \sin\varphi) \sin\varphi + Cs \cos\varphi &= \alpha
\end{aligned}
$$

$$
i.e.,\ A\dot{\phi} \sin^2\varphi + Cs\cos\varphi = \alpha \tag{9.3.10}
$$

The equation of energy is

$$
T + V = E \tag{9.3.11}
$$

where

$$
T = \frac{1}{2}A(\omega_1^2 + \omega_2^2) + \frac{1}{2}C\omega_3^2
$$

is the kinetic energy, $E$ being a constant. From $\omega_3 = s$ and (9.3.4), (9.3.11) becomes

$$
\frac{1}{2}A(\dot{\varphi}^2 + \dot{\phi}^2 \sin^2\varphi) + \frac{Cs^2}{2} = E - mga\cos\varphi
$$

$$
\Rightarrow a(\dot{\varphi}^2 + \dot{\phi}^2 \sin^2\varphi) + Cs^2 = 2(E - mga\cos\varphi) \tag{9.3.12}
$$

Let us take $Cs = \beta$. Then (9.3.10) and (9.3.12) become

$$
\left.
\begin{aligned}
A\dot{\phi}\sin^2\varphi &= \alpha - \beta\cos\varphi \\
A(\dot{\varphi}^2 + \dot{\phi}^2 \sin^2\varphi) + \frac{\beta^2}{C} &= 2(E - mga\cos\varphi)
\end{aligned}
\right\} \tag{9.3.13}
$$

Let $x = \cos\varphi$. Then $1 - x^2 = \sin^2\varphi$ and $\dot{x} = \sin\varphi(\dot{\varphi})$. Multiplying the second equation in (9.3.13) by $\sin^2\varphi$ and substituting for $\dot{\phi}$ from the first equation, we get

$$
A(\dot{\varphi}^2 \sin^2\varphi + \dot{\phi}^2 \sin^4\varphi) + \frac{\beta^2}{C}\sin^2\varphi = 2(E - mga\cos\varphi)(\sin^2\varphi)
$$

$$
A\dot{x}^2 + \frac{(\alpha - \beta x)^2}{A^2} + \frac{\beta^2}{C}(1 - x^2) = 2(E - mgax)(1 - x^2) \tag{9.3.14}
$$

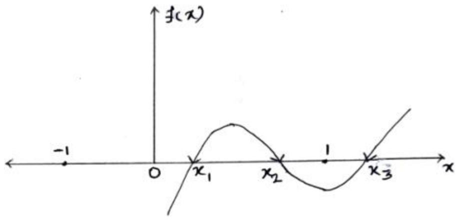
The above equation can be written as

$$
\dot{x}^2 = f(x)
$$

where

$$
\begin{aligned}
f(x) &= \frac{1}{A}\left[2(E - mgax)(1-x^2) - \frac{\beta^2}{C}(1-x^2) - \frac{(\alpha-\beta x)^2}{A^2}\right] \\
\Rightarrow\ f(x) &= \frac{1}{A}\left[2E - 2mga - \frac{\beta^2}{C}\right](1-x^2) - \frac{(\alpha-\beta x)^2}{A^2} \qquad (9.3.15)
\end{aligned}
$$

This is a cubic in $x$. Its graph is shown in figure. The function $f(x)$ has three real zeros $x_1, x_2, x_3$ such that $-1 < x_1 < x_2 < 1 < x_3$. Thus $f(x)$ may be written as




$$
f(x) = \frac{2mgx}{A}(x-x_1)(x-x_2)(x-x_3) \qquad (9.3.16)
$$

Let $u = x - x_1$. Then $u^2 = x - x_1$ and $x = u^2 + x_1$. $2u\dot{u} = \dot{x}$. From $\dot{x}^2 = f(x)$, we have

$$
\begin{aligned}
4u^2\dot{u} &= \frac{2mga}{A}u^2(x-x_2)(x-x_3) \\
\dot{u}^2 &= \frac{mga}{2A}(x_2-x)(x_3-x)
\end{aligned}
$$

$$
\dot{u}^2 = \frac{mga}{2A}(x_2 - x_1 - u^2)(x_3 - x_1 - u^2) \qquad (9.3.17)
$$

This suggests the differential equation $\dot{x}^2 = f(x)$ for the elliptic function *sn.*

$$
\text{Let} \quad \omega = \frac{u}{\sqrt{x_2-x_1}} = \sqrt{\frac{x-x_1}{x_2-x_1}}
$$

$$
k = \sqrt{\frac{x_2-x_1}{x_3-x_1}}, \quad p = \sqrt{\frac{mga(x_3-x_1)}{2A}}
$$

(9.3.16) becomes then

$$\begin{aligned}
\dot{\omega}^2 &= \frac{mga}{2A}(x_2 - x_1)\left(1 - \frac{u^2}{- x_1}\right)(x_3 - x_1)\left(1 - \frac{u^2}{x_3 - x_1}\right)\\
&= \frac{mga}{2A}(x_3 - x_1)(1 - \omega^2)\left(1 - \frac{(x_2 - x_1)\omega^2}{x_3 - x_1}\right)\\
&= p^2(1 - \omega^2)(1 - k^2\omega^2)
\end{aligned}$$

$$i.e., \dot{\omega}^2 = p^2(1 - \omega^2)(1 - k^2\omega^2) \tag{9.3.18}$$

$\therefore\ \omega = sn(p(t - t_0))$ where $t_0$ is a constant of integration.

$$\begin{aligned}
x - x_1 &= (x_2 - x_1)sn^2(p(t - t_0))\\
x_2 - x &= (x_2 - x_1)cn^2(p(t - t_0))\\
x_3 - x &= (x_3 - x_1)dn^2(p(t - t_0))
\end{aligned} \tag{9.3.19}$$

Any one of these three equations gives $x$ as a function of $t$. From the first equation of (9.3.19)

$$x = x_1 + (x_2 - x_1)sn^2(p(t - t_0))$$

$$\cos\theta = x_1 + (x_2 - x_1)sn^2(p(t - t_0)) \tag{9.3.20}$$

where $p$ and the modulus $k$ of the elliptic function

$$p^2 = \frac{mga(x_3 - x_1)}{2A}, \quad k^2 = \frac{x_2 - x_1}{x_3 - x_1}$$

The constants $x_1$, $x_2$, $x_3$ are functions of the constants occuring in (9.3.15). i.e., the constants of the top and $\alpha$, $\beta$, $E$. The constants $\alpha$, $\beta$ and $E$ are known when the initial position and regular velocity of the top are known. The complete solution for the motion of the axis of the top is given by (9.3.20) and

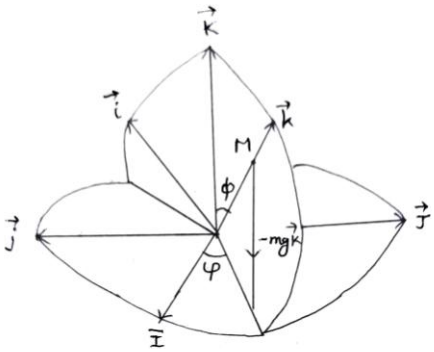
$$\dot{\phi} = \frac{\alpha - \beta x}{A(1 - x^2)} \tag{9.3.21}$$

Since $x$ is a known function as a function of $t$, this last equation gives $\phi$ by a quadrature. This analytic solution does not give a clear idea of the way in which the top behaves. However, we can construct the essential features of the motion, by fixing our concentration on the intersection of the axis of the top with a unit sphere having its center at $O$.

From (9.3.20) we see that the representative point on the unit sphere oscillates between two levels $\varphi = \varphi_1$ and $\varphi = \varphi_2$ given by

$$\cos\varphi_1 = x_1 \quad \cos\varphi_2 = x_2.$$

In the case of the top, we may have loops on the curve. The absence of loops are the presence




of loops(see figure) depends on the way in which the motion is started i.e., on the values of the constants $\alpha$, $\beta$, $E$. The criterion for the existence of a loop is that $\phi$ should sometimes increase and sometimes discrease, and the condition for this is that $\phi$ should vanish during the motion. By (9.3.21), $\phi = 0$ implies

$$\frac{\alpha - \beta x}{A(1 - x^2)} = 0 \Rightarrow x = \frac{\alpha}{\beta}$$

Since $x$ oscillates between $x_1$ and $x_2$, the presence of loops depends on whether $\frac{\alpha}{\beta}$ lies in this range of oscillation. If it lies in the range, there are loops; if not, there are no loops.

# BLOCK-IV

# UNIT 10

# Introduction to Lagrange's Equations

| |
|---|
| **Objectives** |
| **10.1 Lagrange's Equations for a particle in a plane** |

| |
|---|
| **Objectives** |
| Upon completion of this Unit, the students will be able to |
| $x$ identify the Lagrange's equations for a particle in a plane. |

The methods of Lagrange and Hamilton are useful in helping us to carry out the primary task of dynamics namely "how systems move".

## 10.1 Lagrange's Equations for a particle in a plane

Let us consider a particle of mass '$m$' moving in a plane. If $oxy$ are the rectangular axes and $X, Y$ are the components of the force acting on the particle, then by equation of motion (force = mass $\times$ acceleration), we have

$$m\ddot{x} = X, \quad m\ddot{y} = Y \tag{10.1.1}$$

Let $q_1 q_2$ be any curvilinear co-ordinates. Then $(x, y)$ are functions of $(q_1, q_2)$ and hence

$$x = x(q_1, q_2), y = y(q_1, q_2) \tag{10.1.2}$$

The four partial derivatives

$$\frac{\partial x}{\partial q_1}, \frac{\partial x}{\partial q_2}, \frac{\partial y}{\partial q_1}, \frac{\partial y}{\partial q_2} \tag{10.1.3}$$

are the functions of $q_1$ and $q_2$.

If the particle moves in any manner, then $(x, y, q_1, q_2)$ are all functions of time- $t$. Differentiating (10.1.2), we have

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2, \qquad \dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 \tag{10.1.4}$$

Let

$$\begin{aligned} \dot{x} &= f(q_1, q_2, \dot{q}_1, \dot{q}_2) \\ \dot{y} &= g(q_1, q_2, \dot{q}_1, \dot{q}_2) \end{aligned} \tag{10.1.5}$$

Equations (10.1.4) and (10.1.5) are one and the same.

Differentiating (10.1.4) w.r. to $\dot{q}_1$ and $\dot{q}_2$, we get

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \ \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}, \ \frac{\partial \dot{y}}{\partial \dot{q}_1} = \frac{\partial y}{\partial q_1}, \ \frac{\partial \dot{y}}{\partial \dot{q}_2} = \frac{\partial y}{\partial q_2} \tag{10.1.6}$$

The above result is called "The cancellation of the dots".

Again Differentiating (10.1.4)

$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2$$

$$\frac{\partial \dot{x}}{\partial q_2} = \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2^2}\dot{q}_2 \tag{10.1.7}$$

But $\frac{\partial x}{q_1}$ and $\frac{\partial x}{q_2}$ are functions of $q_1$ and $q_2$ and these are in term functions of $t$.

Hence $$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_2}\right) = \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2^2}\dot{q}_2 \tag{10.1.8}$$

Comparing (10.1.7) and (10.1.8),

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1}, \quad \frac{d}{dt}\left(\frac{\partial x}{\partial q_2}\right) = \frac{\partial \dot{x}}{\partial q_2}$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial q_1}\right) = \frac{\partial \dot{y}}{\partial q_1}, \quad \frac{d}{dt}\left(\frac{\partial y}{\partial q_2}\right) = \frac{\partial \dot{y}}{\partial q_2} \tag{10.1.9}$$

The above result is very important (i.e) " the interchange of $d$ and $\partial$ " in the above equation is an important result.

Let the motion of the particle be arbitrary. Its kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{10.1.10}$$

From (10.1.4)

$$T = \frac{1}{2}m\left[\left(\frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2\right)^2 + \left(\frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2\right)^2\right] \tag{10.1.11}$$

Therefore $T$ can be expressed as a function of $q_1, q_2, \dot{q}_1, \dot{q}_2$. (i.e),

$$T = T(q_1, q_2, \dot{q}_1, \dot{q}_2) \tag{10.1.12}$$

From (10.1.4), we see that $T$ is a quadratic in $\dot{q}_1$ and $\dot{q}_2$ so that

$$T = \frac{1}{2}(a\dot{q}_1^2 + 2h\dot{q}_1\dot{q}_2 + b\dot{q}_2^2) \tag{10.1.13}$$

where $a, h, b$ are functions of $q_1, q_2$. From (10.1.11), we get

$$\frac{\partial T}{\partial q_1} = \frac{1}{2} m \left( 2\dot{x}\frac{\partial \dot{x}}{\partial q_1} + 2\dot{y}\frac{\partial \dot{y}}{\partial q_1} \right)$$

$$= m\dot{x}\frac{\partial \dot{x}}{\partial q_1} + m\dot{y}\frac{\partial \dot{y}}{\partial q_1} \tag{10.1.14}$$

From equation (10.1.6), the above equation becomes

$$\frac{\partial T}{\partial \dot{q}_1} = m\dot{x}\frac{\partial x}{\partial q_1} + m\dot{y}\frac{d}{dt}\left( \frac{\partial x}{\partial q_1} \right) \tag{10.1.15}$$

Differentiating the above equation w.r. to '$t$'

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = m\ddot{x}\frac{\partial x}{\partial q_1} + m\dot{x}\frac{d}{dt}\left( \frac{\partial x}{\partial q_1} \right) + m\ddot{y}\frac{\partial y}{\partial q_1} + m\dot{y}\frac{d}{dt}\left( \frac{\partial y}{\partial q_1} \right)$$

Using (10.1.9), the above equation becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = m\ddot{x}\frac{\partial x}{\partial q_1} + m\ddot{y}\frac{\partial y}{\partial q_1} + m\dot{x}\frac{\partial \dot{x}}{\partial q_1} + m\dot{y}\frac{\partial \dot{y}}{\partial q_1} \tag{10.1.16}$$

(10.1.16)-(10.1.14) gives,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = m\ddot{x}\frac{\partial x}{\partial q_1} + m\ddot{y}\frac{\partial y}{\partial q_1}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = m\ddot{x}\frac{\partial x}{\partial q_2} + m\ddot{y}\frac{\partial y}{\partial q_2} \tag{10.1.17}$$

Consider small displacement of the particle corresponding to increments $\partial q_1, \partial q_2$ in theco-ordinates $q_1.q_2$. Then,

$$\partial x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2, \; \partial y = \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2 \tag{10.1.18}$$

The work done by the forces in this displacement is

$$\delta W = X\delta x + Y\delta y \tag{10.1.19}$$

substituting from (10.1.18), we have

$$\delta W = X\left(\frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2\right) + Y\left(\frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2\right)$$

(i.e),

$$\delta W = Q_1\delta q_1 + Q_2\delta q_2 \tag{10.1.20}$$

where

$$Q_1 = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1}; \quad Q_2 = X\frac{\partial x}{\partial q_2} + Y\frac{\partial y}{\partial q_2} \tag{10.1.21}$$

From (10.1.1) and from the first equation (10.1.17), we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} \tag{10.1.22}$$

and from (10.1.21), the above equation becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1 \tag{10.1.23}$$

Similarly the second equation of (10.1.17) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2 \tag{10.1.24}$$

where $q_1, q_2$ are any curvilinear co-ordinates, $T$ is the kinetic energy (expressed classificaation of dynamical systems as a function of $q_1, q_2, \dot{q}_1, \dot{q}_2$ ) and $Q_1, Q_2$ can be obtained from (10.1.20) for the work done in an arbitrary small displacement. (10.1.23) and (10.1.24) are the Lagrange's equations of motion, $q_1 q_2$ are the generalised co-ordinates $Q_1, Q_2$ are the generalised forces.

If the system is conservative, with potential energy $V$, then

$$\delta W = -\delta V, \quad Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2} \tag{10.1.25}$$

Here $V$ is a function of $q_1, q_2$.

Define the Lagrangian function $L$ as

$$L = T - V \tag{10.1.26}$$

where

$$L = L(q_1, q_2, \dot{q}_1, \dot{q}_2) \tag{10.1.27}$$

Hence we have

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1}, \quad \frac{\partial L}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{q}_2}$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial T}{\partial q_1} - \frac{\partial V}{\partial q_1}, \quad \frac{\partial L}{\partial q_2} = \frac{\partial T}{\partial q_2} - \frac{\partial V}{\partial q_2} \tag{10.1.28}$$

The Lagrange's equations (10.1.23) and (10.1.24) may be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0 \tag{10.1.29}$$

# BLOCK-IV

# UNIT 11

# Classification of Dynamical Systems



**Objectives**

Upon completion of this Unit, the students will be able to

- find the Lagrange's equation for simple dynamical system.
- understand Lagrange's for non - holonomic systems with moving constraints

## 11.1 Introduction

A system may be classified as **scleronomic** or **Rheonomic** according as which has only **fixed constraints** or **moving constraints** respectively. As an example, a pendulum with a fixed support is Scleronomic whereas a pendulum for which the point of support is given an assigned motion is rheonomic. The next classification is based on the generalised forces. If the generalised forces are derivable from a potential energy $V$, then the system is said to be **conservative**. Otherwise it is said to be **non-conservative**.

We classify the system as **holonomic** accordingly as the arbitrary independent variations can be given to the generalised co-ordinates without violating the constraints. In the case of **non-holonomic** systems this cannot be done.

We refer the system as simple if it is scleronomic, conservative or holonomic.

## 11.2 Lagrange's equations for simple Dynamical System

Let us consider a simple system with $n-$ degrees of freedom and generalised co-ordinates $q_1, q_2, \cdots, q_n$. The system has a potential energy $V$ which is a function of the $q$'s given by

$$V = V(q) \tag{11.2.1}$$

Let $N$ be the number of particles, $m_i$ be the mass of the $i^{th}$ particle and $\vec{r}_i$ be its position vector, which is a function of $q$'s.

Hence

$$\vec{r}_i = \vec{r}_i(q) \tag{11.2.2}$$

which gives

$$\dot{\vec{r}}_i = \sum_{p=1}^{n} \frac{\partial \vec{r}_i}{\partial q_p} \dot{q}_p \tag{11.2.3}$$

The quantities $\dot{q}_p$ are the generalised co-ordinates, $p = 1, 2, \cdots, n$.

We may write $\dot{\vec{r}}_i = \vec{v}_i(q, \dot{q})$ which is a function of $2n$ co-ordinates $q_p, \dot{q}_p$ $(p = 1, 2, \cdots, n)$.

thus (11.2.2) gives

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\sigma} = \frac{\partial \vec{r}_i}{\partial q_\sigma} \tag{11.2.4}$$

Again from (11.2.3)

$$\frac{\partial \dot{\vec{r}}_i}{\partial q_\sigma} = \sum_{p=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_\sigma \partial q_p} \dot{q}_p \tag{11.2.5}$$

But

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_\sigma} = \sum_{p=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_p \partial q_\sigma}\dot{q}_p \tag{11.2.6}$$

comparing (11.2.5) and (11.2.6), we have

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_\sigma} = \frac{\partial \dot{\vec{r}}_i}{\partial q_\sigma} \tag{11.2.7}$$

(11.2.7) shows that $d$ and $\partial$ can be interchanged.

The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(q, \dot{q}) \tag{11.2.8}$$

$$\frac{\partial T}{\partial q_p} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_p} \tag{11.2.9}$$

and

$$\frac{\partial T}{\partial \dot{q}_p} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_p} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_p} \tag{11.2.10}$$

which follows from (11.2.4).

Differentiating (11.2.10) w.r. to '$t$' and subtracting (11.2.9), we have from (11.2.7)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_p}\right) - \frac{\partial T}{\partial q_p} = \sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{r}_i}{\partial q_p} \tag{11.2.11}$$

where

$$\vec{a}_i = \dot{\vec{v}}_i = \ddot{\vec{r}}_i \tag{11.2.12}$$

is the acceleration of the $i^{th}$ particle.

Let $\vec{F}_i$ be the total force acting on the $i^{th}$ particle including both applied forces and constraint forces. By Newton's law,

$$\vec{F}_i = m_i \vec{a}_i \tag{11.2.13}$$

Thus (11.2.11) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} = \sum_{1=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_p} \tag{11.2.14}$$

Let us consider a new set of arbitrary infintinetsimal increments $\delta q_p$. From (11.2.2)

$$\delta \vec{r}_i = \sum_{p=1}^{n} \frac{\partial \vec{r}_i}{\partial q_p} . \delta q_p \tag{11.2.15}$$

The work done in these displacements is

$$\delta W = \sum_{i=1}^{N} \vec{F} \cdot \delta \vec{r}_i = \sum_{p=1}^{n} \left[ \sum_{i=1}^{N} F_i \frac{\partial \vec{r}_i}{\partial q_p} \right] \delta q_p \tag{11.2.16}$$

But this work can also be expressed as

$$\delta W = -\delta V = -\sum_{p=1}^{n} \frac{\partial V}{\partial q_p} . \delta q_p \tag{11.2.17}$$

Comparing (11.2.16) and (11.2.17) as $\delta q_p$ are arbitrary, we have

$$\sum_{i=1}^{N} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_p} = -\frac{\partial V}{\partial q_p} \tag{11.2.18}$$

Using (11.2.18), the equations (11.2.14) may be written as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} = -\frac{\partial V}{\partial q_p} \tag{11.2.19}$$

Let us introduce the Lagrangean $L$ as

$$L = T - V = L(q, \dot{q}) \tag{11.2.20}$$

Thus (11.2.19) transforms into

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} - \frac{\partial L}{\partial q_p} = 0 \tag{11.2.21}$$

Thus any simple dynamical system moves in accordance with Lagrange's equations of the form given by (11.2.21). We note that if the conservative condition is removed, Lagrange's equations take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} = Q_p \tag{11.2.22}$$

where $Q_p$ are the generalised forces obtained from

$$\delta W = \sum_{p=1}^{n} Q_p.\delta q_p \tag{11.2.23}$$

where $\delta W$ is the work done in an arbitrary displacement.

The equations (11.2.21) form a set of $n-$ oridnary differential equations of second order. The solution of these equations will contain $2n$ arbitrary constants.

## 11.3 Lagrange's equations for Non-holonomic systems with moving constraints

Consider a system of $N-$ particles with masses $m_i$, position vectors $\vec{r}_i$ and accelerations $\vec{a}_i (i = 1, 2, \cdots, N)$. Let $\vec{F}_i$ be the total force acting on the $i^{th}$ particle. Then by Newton's law

$$\vec{F}_i = m_i \vec{a}_i, \, i = 1, 2, \cdots, N \tag{11.3.1}$$

We note that these $N$ vector equations are equivalent to the single scalar equation

$$\sum_{i=1}^{N} (m_i \vec{a}_i - \vec{F}_i) \vec{P}_i = 0 \tag{11.3.2}$$

where $\vec{P}_i$ indicates a set of $N-$ arbitrary vectors. If $\vec{P}_i = \delta \vec{r}_i$ and consider $\delta \vec{r}_i$ as arbitrary virtual displacements, we have

$$\sum_{i=1}^{N} m_i \vec{a}_i \delta \vec{r}_i = \delta W \tag{11.3.3}$$

where

$$\delta W = \sum_{i=1}^{N} \vec{F}_i . \delta \vec{r}_i \tag{11.3.4}$$

which is the virtual work done by the forces in the virtual displacements.

Let us suppose that the system is subject to non-holonomic constraints. the generalised co-ordinates $q_p$ $(p = 1, 2, \cdots, n)$ together with time $t$ determine the positions of the particles.

Let us consider the functions

$$r_i = r_i(q, t) \tag{11.3.5}$$

as known functions. The time $t$ is included to allow for moving constraints (Rheonomic system). Theses equations of non-holonomic constraints are of the form

$$\sum_{p=1}^{n} A_{\alpha p}\dot{q}_p + A_\alpha = 0 \quad (\alpha = 1, 2, \cdots, m < n). \tag{11.3.6}$$

where the $A$ 's are functions of $q$ 's and $t$.

Equivalently, we may write

$$\sum_{p=1}^{n} A_{\alpha p} dq_p + A_\alpha dt = 0 \quad (\alpha = 1, 2, \cdots, m) \tag{11.3.7}$$

The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i . \dot{\vec{r}}_i \tag{11.3.8}$$

By (11.3.5), we have

$$\dot{\vec{r}}_i = \sum_{p=1}^{n} \frac{\partial \vec{r}_i}{\partial q_p}\dot{q}_p + \frac{\partial \vec{r}_i}{\partial t} \tag{11.3.9}$$

and so we can write

$$T = T(q, \dot{q}, t) \tag{11.3.10}$$

We now define $S_p$ by

$$S_p = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} \tag{11.3.11}$$

Again from the previous section

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_p} = \frac{\partial \vec{r}_i}{\partial q_p}, \quad \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_p} = \frac{\partial \dot{\vec{r}}_i}{\partial q_p} \tag{11.3.12}$$

From (11.2.13) and (11.3.11), we have

$$S_p = \sum_{i=1}^{N} m_i \vec{a}_i . \frac{\partial \vec{r}_i}{\partial q_p} \tag{11.3.13}$$

Let $\delta q_p$ satisfy the equations

$$\sum_{p=1}^{n} A_{\alpha p}\delta q_p = 0 \quad (\alpha = 1, 2, \cdots, m) \tag{11.3.14}$$

Otherwise let there be arbitrary.

By (11.3.5) these variations generate virtual displacements

$$\delta \vec{r}_i = \sum_{p=1}^{n} \frac{\partial \vec{r}_i}{\partial q_p} .\delta q_p \tag{11.3.15}$$

We note that (11.3.14) agrees with (11.3.7) except for the omission of the terms in $dt$, we say that $\delta \vec{r}_i$ are the virtual displacements satisfying the instantaneous constraints. When we substitute from (11.3.15) in the D-Alembert's equation (11.3.3) and apply (11.3.11), we have

$$\sum_{p=1}^{n} S_p.\delta q_p = \delta W \tag{11.3.16}$$

Here $\delta W$ represents the work done by all forces in the virtual displacements (11.3.15). These forces are split into (i) applied forces (such as gravity) and (ii) constraint forces. Let us neglect the effect of sliding friction. Then the constraint forces do no work in virtual diaplacements satisfying the instantaneous constraints and so the work $\delta W$ arises only from the applied forces. We can write

$$\delta W = \sum_{p=1}^{n} Q_p.\delta q_p \tag{11.3.17}$$

where $Q_p$ are the generalised forces expressible in terms of the applied forces and hence (11.3.17) becomes

$$\sum_{p=1}^{n} (S_p - Q_p).\delta q_p = 0 \tag{11.3.18}$$

If $\delta q_p$ were arbitrary, we could conclude that $S_p = Q_p$, which would be again in the form of Lagrange's equations $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} = Q_p$. But $\delta q_p$ are not arbitrary. It is necessary to subject the constraint forces with the following conditions, to eliminate them

$$\sum_{p=1}^{n} A_{\alpha p}\delta q_p = 0 \quad (\alpha = 1, 2, \cdots, m) \tag{11.3.19}$$

and hence we have to find the consequences of (11.3.18) when it holds for all $\delta q_p$ which satisfy (11.3.19).

Let $S_p - Q_p = B_p$ and define $F$ by

$$\begin{aligned} F &= (B_1 - \lambda_1 A_{11} - \lambda_2 A_{21} - \cdots - \lambda_m A_{m1})\delta q_1 \\ &+(B_2 - \lambda_1 A_{12} - \lambda_2 A_{22} - \cdots - \lambda_m A_{m2})\delta q_2 + \cdots \\ &+(B_n - \lambda_1 A_{1n} - \lambda_2 A_{2n} - \cdots - \lambda_m A_{mn})\delta q_n. \end{aligned} \tag{11.3.20}$$

where the $\lambda$'s are arbitrary at this instant. We note that $F = 0$ for all $\delta q_p$ satisfying (11.3.19), since (11.3.19) implies (11.3.18). Let us choose $\lambda$'s to satisfying the '$m$' equations

$$\begin{aligned} B_1 &= \lambda_1 A_{11} + \lambda_2 A_{21} + \cdots + \lambda_m A_{m1} \\ B_2 &= \lambda_1 A_{12} - \lambda_2 A_{22} - \cdots - \lambda_m A_{m2} \\ &\cdots\cdots\cdots \\ B_m &= \lambda_1 A_{1m} - \lambda_2 A_{2m} - \cdots - \lambda_m A_{mm} \end{aligned} \tag{11.3.21}$$

So that $F$ reduces to

$$\begin{aligned} F = &(B_{m+1} - \lambda_1 A_{1,m+1} - \cdots - \lambda_m A_{m,m+1})\delta q_{m+1} \\ &+(B_{m+2} - \lambda_1 A_{1,m+2} - \cdots - \lambda_m A_{m,m+2})\delta q_{m+2} \\ &+\cdots\cdots\cdots \\ &+(B_n - \lambda_1 A_{1n} - \cdots - \lambda_m A_{mn})\delta q_n. \end{aligned} \tag{11.3.22}$$

this must vanish for arbitrary values of $\delta q_{m+1}, \cdots, \delta q_n$, since when these values are given, we can always choose $\delta q_1, \cdots, \delta q_m$ to satisfy the $m-$ equations (11.3.19) and we have seen that these imply $F = 0$.

Hence we have,

$$\begin{aligned} B_{m+1} &= \lambda_1 A_{1,m+1} + \lambda_2 A_{2,m+1} + \cdots + \lambda_m A_{m,m+1} \\ B_{m+2} &= \lambda_1 A_{1,m+2} - \lambda_2 A_{2,m+2} - \cdots - \lambda_m A_{m,m+2} \\ &\cdots\cdots\cdots \\ B_n &= \lambda_1 A_{1n} - \lambda_2 A_{2n} - \cdots - \lambda_m A_{mn} \end{aligned} \tag{11.3.23}$$

combining the equations (11.3.23) and (11.3.21), we see that there exist $\lambda_1 \lambda_2 \cdots \lambda_m$ ( called Lagrange multipliers) such that

$$S_p - Q_p = B_p = \sum_{\alpha=1}^{m} \lambda_\alpha A_{\alpha p} \tag{11.3.24}$$

Hence the extension of Lagrange's equations for systems which are rheonomic, non-conservative and non-holonomic with '$n$' generalised co-ordinates and constraints is the form

$$\sum_{p=1}^{n} A_{\alpha p} \dot{q}_p + A_\alpha = 0 \quad (\alpha = 1, 2, \cdots, m < n) \tag{11.3.25}$$

The equations of motion consist of the above $m$ - equations and the following $n$ - equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_p} - \frac{\partial T}{\partial q_p} = Q_p + \sum_{\alpha=1}^{m} \lambda_\alpha A_{\alpha p} \quad (p = 1, 2, \cdots, n) \tag{11.3.26}$$

where $T$ is the kinetic energy and $Q_p$ are the generalised forces calculated from the applied forces by (11.3.17). These $(m + n)$ equations are to be solved for the $(m + n)$ quantities $q_1 q_2 \cdots q_n, \lambda_1 \lambda_2 \cdots \lambda_m$.

## 11.4 Worked Examples

**Example 11.** A particle of mass $m$ is moving in a plane under an attractive force $\frac{\mu m}{r^2}$ directed to the origin of polar co-ordinates $r, \theta$. Find the Lagrange's equations of motion.

**Solution.** The kinetic energy of the system is

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

The potential energy of the system is $V = -\frac{\mu m}{r}$

Lagrangean $L = T - V$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}.$$

The Lagrangean equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (11.4.1)$$

where $r$ and $\theta$ are the generalised co-ordinates. $\therefore$ (11.4.1) becomes

$$\frac{d}{dt}\left(\frac{1}{2} m.2\dot{r}\right) - \left(\frac{1}{2} m(2r)\dot{\theta}^2 - \frac{\mu m}{r^2}\right) = 0$$

and $\frac{d}{dt}\left(\frac{1}{2} m.2\dot{\theta}\right) - (0) = 0$

(i.e) $m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0$

and $m\frac{d}{dt}(r^2\dot{\theta}) = 0$ are the equations of motion.

**Example 12.** A particle of mass $m$ is moves under gravity on a smooth sphere of radius - $b$. Find the Lagrangean equations of motion, taking $x, y, z$ as the generalised co-ordinates. The generalised co-ordinates are the rectangular cartesian co-ordinates with the origin at the center of the sphere and $z$ measured vertically upward.

**Solution.** The equation of constraint is

$$x^2 + y^2 + z^2 = b^2 \tag{11.4.2}$$

[Equation of sphere with centre at origin and radius - $b$ units].

The constraint equation is of the form

$$\sum_{p=1}^{n} A_{\alpha p}\dot{q}_p + A_\alpha = 0 \quad (\alpha = 1, 2, \cdot\cdot\cdot, m < n)$$

Expressing (11.4.2) in the above form, we have

$$x\dot{x} + y\dot{y} + z\dot{z} = 0 \tag{11.4.3}$$

The kinetic energy of the system is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The generalised forces are $X = 0, Y = 0$ and $Z = -mg$ (gravitational force).

Since there is only one constraint (11.4.3), there is only one Lagrange multiplier $\lambda$. The $(m + n)$ equations are given by (11.3.15) and (11.3.16).

From (11.3.15) we have $x\dot{x} + y\dot{y} + z\dot{z} = 0$, (11.3.16) yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = 0 + \lambda x \implies \frac{1}{2}(2\ddot{x}).m - 0 = \lambda x$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = 0 + \lambda y \implies m\ddot{y} = \lambda y$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = -mg + \lambda z \implies m\ddot{z} = -mg + \lambda z$$

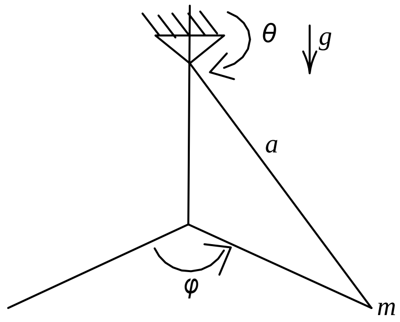
Hence the four equations are

$x\dot{x} + y\dot{y} + z\dot{z} = 0$, $m\ddot{x} = \lambda x$, $m\ddot{y} = \lambda y$, and $m\ddot{z} = -mg + \lambda z$.

**Example 13.** Find the Lagrange's equations of motion of a spherical pendulum consisting of a particle of mass - $m$ , which moves under gravity on a smooth sphere of radius - $a$ . The generalised co-ordinates are the spherical polar angles $\theta$ and $\varphi$.

**Solution.** The kinetic energy of the system is

$T = \frac{1}{2}ma^2(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2)$ (under spherical polar co-ordinates).




The potential energy $V = -mgacos\theta$.

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}ma^2(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2) + mgacos\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2}ma^2(2\dot{\theta}) = ma^2\dot{\theta} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ma^2\ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= \frac{1}{2}ma^2(2sin\theta cos\theta\dot{\varphi}^2) - mgasin\theta. \end{aligned}$$

$\therefore$ The $\theta$ - equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\implies \quad ma^2\ddot{\theta} - ma^2 sin\theta cos\theta\dot{\varphi}^2 + mgasin\theta = 0 \tag{11.4.4}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2}ma^2 sin^2\theta(2\dot{\varphi})$$

$$\frac{\partial L}{\partial \varphi} = 0.$$

$\therefore$ The $\varphi$ - equation is

$$ma^2\frac{d}{dt}(sin^2\theta\dot{\varphi}) = 0. \tag{11.4.5}$$

(11.4.4) and (11.4.5) are the Lagrange's equations of motion.

**Example 14.** Consider a particle of unit mass moving in space, whose position is described by the spherical polar co-ordinates $r, \theta, \varphi$. Find the components of acceleration along the parametric lines.

**Solution.** The kinetic energy $T = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2 + r^2 sin^2\theta\dot\varphi^2)$.

Let $R$, $\theta$ and $\psi$ be the generalised forces acting on the particle.

Then the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = R$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = \psi$$

which takes the form,

$$\frac{d}{dt}\left(\frac{1}{2}2\dot r\right) - \frac{1}{2}(2r\dot\theta^2 + 2r sin^2\theta\dot\varphi^2) = R$$

(i.e), $\ddot r - r\dot\theta^2 - r sin^2\theta\dot\varphi^2 = R$

The other two equations are

$$\frac{d}{dt}(r^2\dot\theta) - r^2 sin\theta cos\theta\varphi^2 = \theta$$

$$\frac{d}{dt}(r^2 sin^2\theta\dot\varphi) = \psi .$$

Let $a_r$, $a_\theta$, $a_\varphi$ be the components of acceleration along the parametric lines. Since the particle is of unit mass, these are equal to the components of force in these directions. Let us equate the two expressions for the work done in an arbitrary displacement.

$$\delta W = a_r\delta_r + a_\theta r\delta\theta + a_\varphi r sin\theta\delta\varphi = R\delta r + \theta\delta\theta + \psi\delta\varphi$$

Equating the co-efficients of $\delta r$, $\delta\theta$ and $\delta\varphi$, we get

$$a_r = R = \ddot r - r\dot\theta^2 - r sin^2\theta\dot\varphi^2$$

$$a_\theta = \frac{1}{r}\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) - r sin\theta cos\theta\dot\varphi^2$$

$$a_\varphi = \frac{\psi}{r sin\theta} = \frac{1}{r sin\theta}\frac{d}{dt}(r^2 sin^2\theta\dot\varphi) .$$

which are the components of acceleration.

**Example 15.** A particle moves in space with the Lagrangean $L = \frac{1}{2}m(\dot x^2+\dot y^2+\dot z^2) - V + \dot xA + \dot yB + \dot zC$ where $V, A, B, C$ are given functions of $x, y, z$. Show that the equations of motion are $m\ddot x = -\frac{\partial V}{\partial x} + \dot y\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) - \dot z\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)$ and two similar equations.

**Solution.** $\dfrac{\partial L}{\partial \dot{x}} = \dfrac{1}{2}m(2\dot{x}) + A = m\dot{x} + A$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x} + A) = m\ddot{x} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z}$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} + \dot{x}\frac{\partial A}{\partial x} + \dot{y}\frac{\partial B}{\partial x} + \dot{z}\frac{\partial C}{\partial x}$$

$\therefore$ The Lagrange's equation is $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} = 0$

(i.e), $m\ddot{x} + \dfrac{\partial A}{\partial x}\dot{x} + \dfrac{\partial A}{\partial y}\dot{y} + \dfrac{\partial A}{\partial z}\dot{z} + \dfrac{\partial V}{\partial x} - \dot{x}\dfrac{\partial A}{\partial x} - \dot{y}\dfrac{\partial B}{\partial x} - \dot{z}\dfrac{\partial C}{\partial x} = 0$

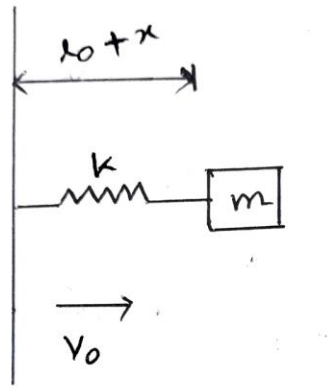
$$m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) - \dot{z}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)$$

which is the first equation of motion. In a similar manner, the other two equations can be found.

## Check Your Progress

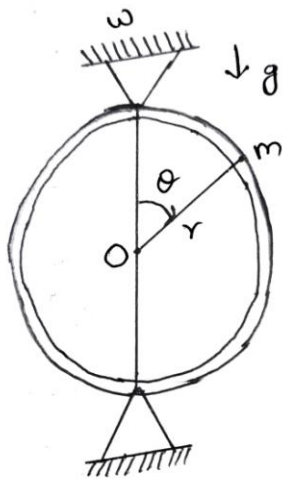
1. A particle of mass $m$ can slide without friction on the inside of a small tube which is bent in the form of a circle of radius $r$. The tube rotates about a vertical diameter with a constant angular velocity $\omega$ as shown in the figure. Find the differential equation of motion.

2. Suppose a mass spring system is attached to a frame which is translating with a uniform velocity $v_0$ as shown in the figure. Let $l_0$ be the unstressed spring length and use the elongation $x$ as the generalised co-ordinate. Find the differentia equation of motion.



# Answer to Check Your Progress

1. Hint: $T = \frac{1}{2}mr^2(\dot{\theta}^2 + \omega^2 \sin^2\theta)$, $v = mgr \cos\theta$.




$mr^2\ddot{\theta} - mr^2\omega^2 \sin\theta \cos\theta - mgr \sin\theta = 0$.

2. Hint: $T = \frac{1}{2}m(v_0 + \dot{x})^2$, $v = \dfrac{Kx^2}{2}$.

$m\ddot{x} + Kx = 0$.

# BLOCK-IV

# UNIT 12

# Hamilton's Equations



**Objectives**

Upon completion of this Unit, the students will be able to

- obtain Lagrangean from Hamiltonian.
- identify conservative systems

## 12.1 Introduction

The Lagrangean $L$ is a function of $n$ - quantities $q_p$, their derivatives $\dot{q}_p$ with respect to $t$ and itself and hence we write

$$L = L(q, \dot{q}, t) \tag{12.1.1}$$

write,

$$p_p = \frac{\partial L}{\partial \dot{q}_p} \tag{12.1.2}$$

These $n$ - quantities are known as the generalised momenta. In these equations, we see $p_p$ expressed as functions of $q$'s, the $\dot{q}$'s and $t$. (i.e), expressed as functions of $q$'s, $p$'s and $t$. Then $L$ itself can be regarded as a function of the $q$'s, the $p$'s and $t$.

Define the quantity $H$ as

$$H = \sum_{p=1}^{n} \dot{q}_p \frac{\partial L}{\partial \dot{q}_p} - L \tag{12.1.3}$$

The quantity $H$ is the Hamiltonian function. We write

$$H = H(q, p, t) \tag{12.1.4}$$

in the case of a simple dynamical system, We have

$$L = T(q, \dot{q}) - V(q) \tag{12.1.5}$$

in the above equation, $T$ is homogenous and quadratic in the generalised velocities.
By Euler's theorem for homogenous functions,

$$\sum_{p=1}^{n} \dot{q}_p \frac{\partial T}{\partial \dot{q}_p} = 2T \tag{12.1.6}$$

$\therefore$ (12.1.1) becomes

$$H = 2T - (T - V) = T + V \tag{12.1.7}$$

which is the total energy consisting of the kinetic and potential energy.
Let us write (12.1.1) in the form

$$H = \sum_{p=1}^{n} \dot{q}_p P_p - L \tag{12.1.8}$$

The connection between $L$ and $H$ is given by (12.1.9) where we regard $n$ - equations connecting the $3n + 1$ quantities.

$$q, \dot{q}_p, p_p, t \tag{12.1.9}$$

thus giving variations to the quantities in (12.1.9), we have from (12.1.8).

$$\delta + 1 = \sum_{p=1}^{n} p_p \delta \dot{q}_p + \sum_{p=1}^{n} \dot{q}_p \delta p_p - \sum_{p=1}^{n} \frac{\partial L}{\partial \dot{q}_p} \delta \dot{q}_p - \frac{\partial L}{\partial q_p} \delta q_p - \frac{\partial L}{\partial t} \delta t \tag{12.1.10}$$

the first and third terms of the R.H.S of the above equation cancel each other by (12.1.7).

The remaining differentials $\delta q_p$, $\delta p_p$, $\delta t$ being $2n + 1$ in number, may be regarded as independent and arbitrary, and since $H = H(q, p, t)$, $\delta H = \frac{\partial H}{\partial q_p} \delta q_p + \frac{\partial H}{\partial p_p} \delta p_p + \frac{\partial H}{\partial t} \delta t$. We have from (12.1.10)

$$\frac{\partial H}{\partial p_p} = \dot{q}_p, \quad \frac{\partial H}{\partial q_p} = -\frac{\partial L}{\partial q_p}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{12.1.11}$$

The $2n$ - equations of motion from (12.1.7)

$$\dot{q}_p = \frac{\partial H}{\partial p_p}, \quad \dot{p}_p = \frac{\partial H}{\partial q_p} \tag{12.1.12}$$

The Hamilton's equations of motion or Hamilton's canonical equations of motion.

## 12.2 Lagrangean from Hamiltonian

Suppose that we are given a function $H(q, p, t)$ and the motion of the system satisfies the canonical equations

$$\dot{q}_p = \frac{\partial H}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H}{\partial q_p} \tag{12.2.1}$$

We solve the first set of equations in the above equation for the $p$'s in terms of the $q$'s, $\dot{q}$'s and $t$.

Then we write

$$L = \sum_{p=1}^{n} \dot{q}_p p_p - H \tag{12.2.2}$$

and express $L$ as a function of the $q$'s, $\dot{q}$'s and $t$. This is the required Lagrangean and the motion which satisfies the Hamilton's canonical equations, also satifies the Lagrange's equations of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right) - \frac{\partial L}{\partial q_p} = 0$ and vice versa.

## 12.3 Ignorable Co-ordinates

Suppose that one co-ordinate say $q_1$ is absent from $H$, so that, $\frac{\partial H}{\partial q_1} = 0$. Then by (12.2.1), $\dot{p}_1 = 0 \implies p_1$ is a constant of the motion.

Let $p - 1 = q_1$.
substituting $p_1 = a_1$ in (12.2.1), we have

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial a_1} \quad and \quad \dot{p}_1 = \frac{\partial H}{\partial q_1} \;(i.e)\; 0 = -\frac{\partial H}{\partial q_1}$$

Dropping the above equations from the set of $2n$ equations, we have a set of canonical equations for $2(n-1)$ quantities $q_2, q_3, \cdots, q_n, p_2, p_3, \cdots, p_n$.

The co-ordinate $q_1$ is called an ignorable co-ordinate. We see that if there is an ignorable co-ordinate, the number of degrees of freedom is reduced by unity, without loss of the canonical form of the equations. The ignorable co-ordinate $q_1$ is to be found from the equation $\dot{q}_1 = \frac{\partial H}{\partial p_1}$.

If there are $m$ - ignorable co-ordinates then the number of degrees of freedom is reduced by $m$.
From $\frac{\partial H}{\partial q_p} = -\frac{\partial L}{\partial q_p}$, we see that $\frac{\partial H}{\partial q_1} = 0$ is equivalent to $\frac{\partial L}{\partial q_1} = 0$.
So, if we start from a Lagrangean instead of a Hamilton, an ignorable co-ordinate can be detected through its absence from $L$.

## 12.4 Conservative Systems

By (12.1.11), the equations

$$\frac{\partial L}{\partial t} = 0, \quad \frac{\partial H}{\partial t} = 0 \tag{12.4.1}$$

are equivalent; We shall say that a system is **conservative** if they are satisfied or equivalently if $L$ or $H$ does not depend explicitly on $t$.

On applying the canonical equations of motion (12.2.1), we have

$$\dot{H} = \sum_{p=1}^{n} \frac{\partial H}{\partial q_p}\dot{q}_p + \sum_{p=1}^{n} \frac{\partial H}{\partial p_p}\dot{p}_p + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{12.4.2}$$

This vanishes if (12.4.1) is satisfied; Hence for a conservative system, the Hamiltonian $H$ is a constant of the motion, its value being determined by the initial conditions.

## 12.5 Worked Examples

**Example 16.** Write the Hamilton's canonical equations of motion for a particle of mass $m$ moving in a plane with potential energy $V(x, y)$.

**Solution.** The Lagrangean is $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$ ······················ (∗)

where we know that the kinetic energy $T = \frac{1}{2}mv^2$.

The generalised momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}}, \quad p_y = \frac{\partial L}{\partial \dot{y}}$$

With the aid of (*), the above equations become, $p_x = m\dot{x}$ and $p_y = m\dot{y}$

$\implies \dot{x} = \frac{p_x}{m}$ and $\dot{y} = \frac{p_y}{m}$.

The Hamiltonian $H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(x, y)$

(i.e) $H = \frac{1}{2}m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2}\right) + V(x, y)$

(i.e) $H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x, y)$.

The canonical equations of motion are

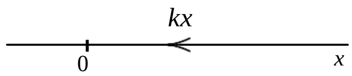
$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{2m}(2p_x) = \frac{p_x}{m}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\dot{p}_y = \frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y}$$

**Example 17.** The Harmonic oscilator consistes of a particle which can move on a straight line (say) along $x$ - axis. It is attracted towards the origin by a controlling force $-kx\vec{i}$ where $\vec{i}$ is the unit vector along the positive direction of $x$ - axis. Show that it is a simple system with $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}k\dot{x}^2$, $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\dot{x}^2$, $p = m\dot{x}$, $H = \frac{1}{2m}p^2 + \frac{kx^2}{2}$. Hence obtain the Lagrangian equation of motion $m\ddot{x} + kx = 0$ and the canonical equations of motion $\dot{x} = \frac{p}{m}$, $\dot{p} = -kx$.




**Solution.** The motion is along $x$ - axis only, hence the system is simple with kinetic energy.

$$T = \frac{1}{2}m\dot{x}^2,$$

The potential energy $V = \frac{1}{2}k\dot{x}^2$ $[\frac{k}{2}\times$ square of the distance $]$

$$\therefore \text{ Lagrangean } L = T - V = \frac{1}{2}k\dot{x}^2 - \frac{1}{2}k\dot{x}^2.$$

The Lagrangean equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \text{ (i.e), } \frac{d}{dt}\left(\frac{1}{2}m2\dot{x}\right) - \frac{1}{2}(-2kx) = 0$$

$$\text{(i.e) } \frac{d}{dt}(m\dot{x}) + kx = 0$$

$$\Longrightarrow m\ddot{x} + kx = 0$$

The Hamiltonian $H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k\dot{x}^2$

The generalised momentum $p = \frac{\partial L}{\partial \dot{x}}$

$$\text{(i.e), } p = \frac{m}{2}(2\dot{x}) = m\dot{x}$$

$$\Longrightarrow \dot{x} = \frac{p}{m}$$

$$\therefore \quad H = \frac{1}{2}m\left(\frac{p^2}{m^2}\right) + \frac{kx^2}{2} = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$$

The Hamilton canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial x}$$

$$\therefore\ \dot{x} = \frac{1}{2m}(2p) \text{ and } \dot{p} = -\frac{k(2x)}{2}$$

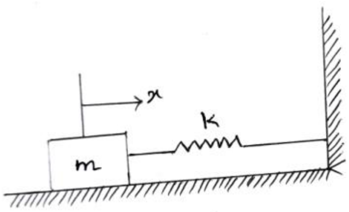
are the canonical equations.

(i.e) $\dot{x} = \frac{p}{m}$ and $\dot{p} = -kx$.

# Check Your Progress

*1.* A particle of mass $m$ is attracted to a fixed point $O$ by an inverse square force $-\frac{\mu m}{r^2}$, where $\mu$ is the gravitational coefficient. Using the polar coordinates $(r, \theta)$, find the Hamilton's canonical equations of motion.

2. Given a mass spring system consisting of a mass $m$ and a linear spring of stiffness $K$, find the Hamilton's canonical equations of motion.




# Answer to Check Your Progress

1. Hint: $H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 - \frac{\mu m}{r})$

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r},\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$, where $L = T - V$.

Answer: $\dot{r} = \frac{p_r}{m},\ \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\mu m}{r^2}$

$\dot{\theta} = \frac{p_\theta}{mr^2},\ \dot{p}_\theta = 0.$

2. Hint: $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{kx^2}{2},\ \ p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$H = T + V = \frac{p^2}{2m} + \frac{kx^2}{2}$$

Answer: $\dot{x} = \frac{p}{m},\ \dot{p} = -kx$ (or) $m\ddot{x} + kx = 0.$

# BLOCK-V

# UNIT 13

# Natural Motions

**Objectives**



**Objectives**

Upon completion of this Unit, the students will be able to

- identify the space of events.
- find the Jacobi's principle of least action

## 13.1 Introduction

Let us consider a dynamical system with $n$ - degrees of freedom and a Hamiltonian $H(q, p, t)$ where $q, p$ refers to the $2n$ quantities $q_p, p_p$ $(p = 1, 2, \cdot\cdot\cdot, n)$. The canonical equations are

$$\dot{q}_p = \frac{\partial H}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H}{\partial q_p} \tag{13.1.1}$$

We call a motion to be "natural" if it satisfies the above equations (13.1.1)

## 13.2 The space of Events

Let us consider the set of numbers $(q_1, q_2, \cdot \cdot \cdot, q_n, t)$ as a point in a **representative space** of $(n + 1)$ dimensions. Since a point corresponds to a configuration or position of the system at a certain time, we may refer to a **point** of the representative space as an **event** and call the space as **space of events** $E_{n+1}$ to distinguish it from other representative spaces.

Any motion of the system (not necessarily a natural motion) may be described by considering the q's as functions of $t$.

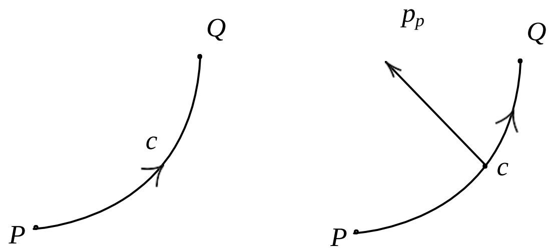



Fig 13.2 (a) Fig 13.2 (b)

The geometrical image of this motion in $E_{n+1}$ is a curve $c$, analogous to the curve in ordinary space given by the equations $x = x(t),\ y = y(t),\ z = z(t)$ as shown in figure 13.2 (a).

For $n = 2$, we can make a model of the curve $c$ in wire and obtain our picture by projection on the plane of the paper. We cannot do this for $n > 2$, but we can still think of $c$ as a projection of an $(n + 1)$ dimensional model on the plane of the paper.

## Action for an Arbitrary motion

We can describe a motion from an event $P$ to an event $Q$ by writing

$$q_p = q_p(u), \quad p_p = p_p(u), \quad t = t(u) \tag{13.2.1}$$

These being $(2n + 1)$ functions of a parameter '$u$' which runs from $u = u_1$ at $P$ to $u = u_2$ at $Q$. We may think of the $p$'s as defining a momentum vector $p_p$ at each point of $c$. Fig 5.1(b). We

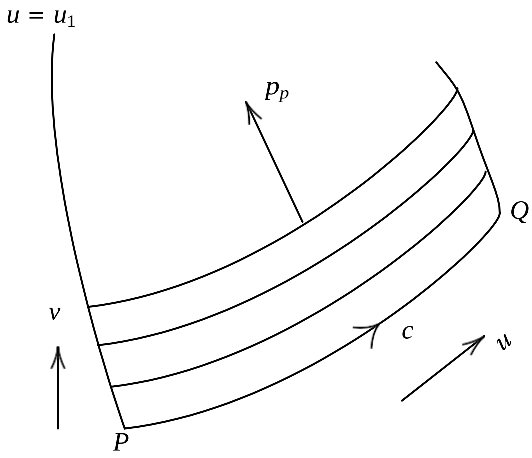



Fig 13.2 (c)

say that (13.2.1) define a "curve with momentum".

We define the **action** along $c$ to be the integral

$$S = \int_{u_1}^{u_2} \left( \sum_{p=1}^{n} p_p \frac{dq_p}{du} - H \frac{dt}{du} \right) du \tag{13.2.2}$$

Equivalently, we have

$$S = \int_{P}^{Q} \left( \sum_{p=1}^{n} p_p dq_p - H dt \right) \tag{13.2.3}$$

where the integration is taken along $c$.

## The Variation of Action

Consider an infinity of motions each with attached momentum. We describe them as

$$q_p = q_p(u, v), \quad p_p = p_p(u, v), \quad t = t(u, v) \tag{13.2.4}$$

where $u$ is a parameter which is constant for each motion this parameter '$v$' serves as a label to pick on any particular one of the motions. Let us take $u_1, u_2$ be constants independent of $v$.

The set of motions appear in $E_{n+1}$ as a set of curves (fig 13.2(c)) and for each there is an action which is a function of $V$. Fig 13.2(c) represents at singly infinite set of motions with attached moments.

We write

$$S(v) = \int_{u_1}^{u_2} \left( \sum_{p=1}^{n} p_p \frac{\partial q_p}{\partial u} - H \frac{\partial t}{\partial u} \right) du \tag{13.2.5}$$

with partial derivatives, since $v$ does not change as we follow a motion.

We want to see how the acttion changes as we pass from motion to motion. For this we differentiate (13.2.5) w.r. to '$v$'.

$$\frac{ds}{dv} = \int_{u_1}^{u_2} \left( \sum_{p=1}^{n} \frac{\partial p_p}{\partial v} . \frac{\partial q_p}{\partial u} + \sum_{p=1}^{n} p_p \frac{\partial^2 q_p}{\partial v \partial u} - \frac{\partial H}{\partial v} \frac{\partial t}{\partial u} - H \frac{\partial^2 t}{\partial v \partial u} \right) du \tag{13.2.6}$$

As the second order partial derivatives are continuous. We have

$$\frac{\partial^2 q_p}{\partial v \partial u} = \frac{\partial^2 q_p}{\partial u \partial v} = \frac{\partial}{\partial u} \frac{\partial q_p}{\partial v} \tag{13.2.7}$$

Now consider $\int_{u_1}^{u_2} p_p \frac{\partial^2 q_p}{\partial v \partial u} du = \int_{u_1}^{u_2} p_p \frac{\partial}{\partial u} \frac{\partial q_p}{\partial v} du$ [from (13.2.7)]

$$= \left[ p_p \frac{\partial q_p}{\partial v} \right]_{u_1}^{u_2} - \int_{u_1}^{u_2} \frac{\partial p_p}{\partial u} \frac{\partial q_p}{\partial v} du \quad (\mathit{usingintegrationbyparts}) \tag{13.2.8}$$

Similarly $\int_{u_1}^{u_2} H \frac{\partial^2 t}{\partial v \partial u} du = \int_{u_1}^{u_2} H \frac{\partial}{\partial u} \frac{\partial t}{\partial v} du$ (From (13.2.7))

$$= \left[ H \frac{\partial t}{\partial v} \right]_{u_1}^{u_2} - \int_{u_1}^{u_2} \frac{\partial H}{\partial u} \frac{\partial t}{\partial v} du \tag{13.2.9}$$

Using (13.2.8) & (13.2.9) in (13.2.6), we have

$$\frac{ds}{dv} = \left[ \sum_{p=1}^{n} p_p \frac{\partial q_p}{\partial v} - H \frac{\partial t}{\partial v} \right]_{u_1}^{u_2} + \int_{u_1}^{u_2} \left( \sum_{p=1}^{n} \frac{\partial p_p}{\partial v} \frac{\partial q_p}{\partial u} - \sum_{p=1}^{n} \frac{\partial p_p}{\partial u} \frac{\partial q_p}{\partial v} - \frac{\partial H}{\partial v} \frac{\partial t}{\partial u} + \frac{\partial H}{\partial u} \frac{\partial t}{\partial v} \right) du \tag{13.2.10}$$

Let us consider the infinitesimal change in $S$ resulting from an infitesimal change $\partial v$ in $v$ (i.e), in passing from a curve $c$ to a neighbouring curve.

We write

$$\frac{\partial q_p}{\partial u} du = dq_p, \quad \frac{\partial p_p}{\partial u} du = dp_p, \quad \frac{\partial t}{\partial u} = dt, \quad \frac{\partial H}{\partial u} = dH \tag{13.2.11}$$

these being the increments in passing along $c$.

Now we write

$$\frac{ds}{dv}\delta v = \delta s, \quad \frac{\partial q_p}{\partial v}\delta v = \delta q_p, \quad \frac{\partial p_p}{\partial v}\delta v = \delta p_p, \quad \frac{\partial t}{\partial v}\delta v = \delta t, \quad \frac{\partial H}{\partial v}\delta v = \delta H \tag{13.2.12}$$

these being the variations resulting from the change in $v$.

Multiplying (13.2.10) by $\delta v$ we get,

$$\delta s = \left[\sum_{p=1}^{n} p_p \delta q_p - H\delta t\right]_{u_1}^{u_2} + \int_P \left(\sum_{p=1}^{n} \delta p_p dq_p - \sum_{p=1}^{n} \delta q_p dp_p - \delta H dt + \delta t dt\right) \tag{13.2.13}$$

[Using (13.2.11) and (13.2.12) in (13.2.10)]

where the integration is to be carried out w.r. to '$d$' and not $\delta$.

# Remark

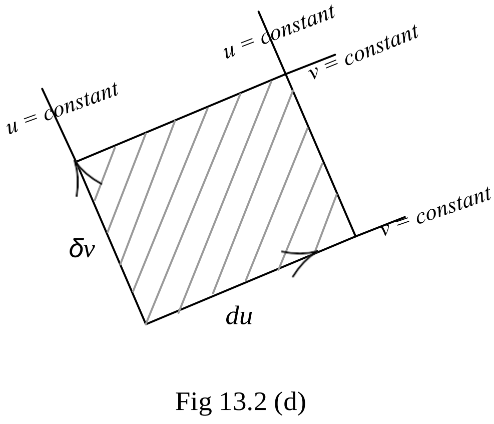



Fig 13.2 (d)

To avoid the use of partial derivatives, we proceed as follows: The curves in (13.2)(c) form a surface of two dimensions. If we draw on that surface the curves $u$ = constant and $v$ = constant, we get a network, a typical cell of which is shown in figure 13.2(d). The figure 13.2(d) shows a cell on the surface formed by a set of motions. The sides correspond to an increment $du$ and variation $\delta v$.

Since $u$ has the same bounds $(u_1, u_2)$ for all the curves, we can make $du$ and $\delta v$ constant infinitisimals over the whole of the surface. Now if $f$ is any function of $u$ and $v$, we have

$$\frac{\partial^2 f}{\partial u \partial v} = \frac{\partial^2 f}{\partial v \partial u} \tag{13.2.14}$$

Multiplying (13.2.14) by $du \ \delta v$ and using the fact that these infinitesimals are constants, we get

$$du \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\delta v\right) = \delta v \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u} du\right) \tag{13.2.15}$$

The above equation may be written with the aid of (13.2.11) and (13.2.12) as

$$d\delta f = \delta d f \tag{13.2.16}$$

(i.e),

$$d\delta = \delta d \tag{13.2.17}$$

We use to denote the set of $n$ - quantities $q_p$ and $p$ to denote the set of $n$ - quantities $p_p$.

For example, we write

$$\sum_{p=1}^{n} p_p dq_p = pdq \tag{13.2.18}$$

thus (13.2.18) in (13.2.5), we have

$$S = \int_P^Q (pdq - Hdt) \tag{13.2.19}$$

$$\delta S = \int_P^Q \delta pdq + p\delta dq - \delta Hdt - H\delta dt \tag{13.2.20}$$

Using (13.2.17) in (13.2.20) we have

$$\delta S = \int_P^Q (\delta p dq + p d\delta q - \delta H dt - H d\delta t)$$

$$\delta S = \int_P^Q (\delta p d\delta q - H d\delta t) + \int_P^Q \delta p dq - \delta H dt \tag{13.2.21}$$

$$= [p\delta q]_P^Q - \int_P^Q \delta q dp - [H\delta t]_P^Q - \int_P^Q \delta t dH + \int_P^Q \delta p dq - \delta H dt$$

(Applying integration by parts for the first integral in (13.2.21))

$$\therefore \quad \delta s = [p\delta q - H\delta t]_P^Q + \int_P^Q (\delta p dq - \delta q dp - \delta H dt + \delta t dH) \tag{13.2.22}$$

which is same as (13.2.13).

Hence the use of partial derivatives is eliminated.

## 13.3 Hamilton's Principle

We now discuss the variations of action when the end events $P$ and $Q$ are held fixed.

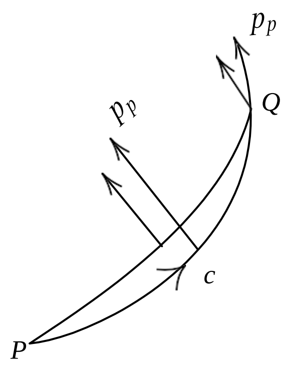



Fig 13.3 (e)

Figure 13.3(e) shows two cutves in $E_{n+1}$ with the same end points and with attached moments need not be the same at the end pointa for the two curves.

Since $\delta q = 0, \delta t = 0$ at $p$ and $Q$ (13.2.22) gives

$$\delta s = \int_p^Q (\delta p dq - \delta q dp - \delta H dt + \delta t dH) \tag{13.3.1}$$

As $H = H(q, p, t)$ we have

$$\delta H(q, p, t) = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial t}\delta t \tag{13.3.2}$$

Substituting (13.3.2) in (13.3.1), we get

$$\begin{aligned}\delta s &= \int_p^Q \left\{ \delta p dq - \delta q dp - \left[\frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial t}\delta t\right] dt + \delta t dH \right\} \\ &= \int_p^Q \left\{ \delta p \left( dq - \frac{\partial H}{\partial p} dt\right) - \delta q \left( dp + \frac{\partial H}{\partial q} dt \right) + \delta t \left( dH - \frac{\partial H}{\partial t} dt \right) \right\} \\ &= \int_p^Q \left\{ \delta p \left( \dot{q} - \frac{\partial H}{\partial p} \right) - \delta q \left( \dot{p} - \frac{\partial H}{\partial q} \right) + \delta t \left( \dot{H} - \frac{\partial H}{\partial t} \right) \right\} dt \end{aligned} \tag{13.3.3}$$

So far the motion represented by $c$ has been completely arbitrary.

Suppose now that it is a natural motion satisfying

$$\dot{q}_p = \frac{\partial H}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H}{\partial q_p}, \quad \dot{H} = \frac{\partial H}{\partial t}$$

Then by (13.3.3), we have $\delta s = 0$ , no matter what the variations $\delta q, \delta p, \delta t$ may be. Thus we say that " $s$ has a stationary value for the natural motion when compared with arbitrary adjacement motions with the same end events.

Let us now prove the converse ($v$) " If $s$ has a stationary value for variations $\delta q, \delta p, \delta t$ which are arbitrary for end conditions, then $c$ represents a natural motion".

To prove this, we choose the variation to be

$$
\begin{aligned}
\delta q &= -\left(\dot{p} + \frac{\partial H}{\partial q}\right) F\delta v \\
\delta p &= -\left(\dot{q} - \frac{\partial H}{\partial p}\right) F\delta v \\
\delta t &= -\left(\dot{H} - \frac{\partial H}{\partial t}\right) F\delta v
\end{aligned}
\tag{13.3.4}
$$

where $\delta v > 0$ and $F$ is any function along $c$ such that $F \geq 0$ and $F = 0$ at the ends. When we substitute (13.3.4) in (13.3.3) we get a non-negative integrand; then the assured stationary character of $s$ ($\delta s = 0$) implies the vanishing of the integrand along $c$, which implies immediately that the canonical equations

$$\dot{q}_p = \frac{\partial H}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H}{\partial q_p}$$

are satisfied.

The result what we have established is known as **Hamilton's Principle**. It states that "The integral of action

$$S = \int \left(\sum_{p=1}^{n} p_p dq_p - Hdt\right) \tag{13.3.5}$$

$$S = \int (pdq - Hdt) \tag{13.3.6}$$

has a stationary value for the natural motion when compared with adjacent motions having the same end events".

We can express Hamilton's principle by the variational equation

$$\delta \int (pdq - Hdt) = 0 \tag{13.3.7}$$

We have the connection between the Hamiltonian '$H$' and the Lagrangean '$L$' as $H = p\dot{q} - L$. (i.e),

$$L = p\dot{q} - H \tag{13.3.8}$$

Substituting (13.3.8) in (13.3.7), we get

$$\delta L dt = 0 \tag{13.3.9}$$

This is the usual form in which Hamilton's principle is quoted and equivalent to Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{13.3.10}$$

The equations (13.3.10) are called the Euler-Lagrange equations associated with the variational equation (13.3.9).

Consider the formula

$$\begin{aligned} \delta \int pdq &= \int (\delta p dq + p \delta dq) \\ &= \int (\delta p dq + p d \delta q) \end{aligned}$$

Applying integration by parts to second integral, we have

$$= [p\delta q] + \int (\delta p dq - \delta q dp) \tag{13.3.11}$$

Suppose that the motion $c$, from which the variation is made, is natural.

Then $dq = \frac{\partial H}{\partial p} dt$

$$dp = -\frac{\partial H}{\partial q} dt$$

(13.3.11) becomes

$$\begin{aligned} \delta \int pdq &= [p\delta q] + \int \left(\delta p \frac{\partial H}{\partial p} + \delta q \frac{\partial H}{\partial q}\right) dt \\ &= [p\delta q] + \int \delta H dt \end{aligned} \tag{13.3.12}$$

This vanishes if $\delta q = 0$ at the ends and $\delta H = 0$ along $c$.

We know that $H$ is a constant for any natural motion of a conservative system. $\left(\because \dot{H} = \frac{\partial H}{\partial t} = 0\right)$.

Hence we have the following variational principle:

$$\delta \int pdq = 0 \tag{13.3.13}$$

for variations from a natural motion of a conservative system provided the end configurations are fixed and $H$ has, in the varied motion, the same constant value which it has in the natural motion.

# Application

Let us apply this principle to a simple dynamical system with $n$ - degrees of freedom and generalised co-ordinates $q_p$ $(p = 1, 2, \cdots, n)$. Let $N$ be the number of particles in the system. Let $m_i$ be the mass of a typical particle and $\vec{r}_i$ its position vector. Let the system have the potential energy $v = v(q)$.

For such a system,

$$H = T + V, \quad p = \frac{\partial T}{\partial \dot{q}}, \quad pdq = \frac{\partial T}{\partial \dot{q}}\dot{q}dt = 2Tdt \tag{13.3.14}$$

Since $T$ is homogenous of the second degree in the velocities.

"In the natural motion $H = E$, a constant. (13.3.13) can be written as

$$\delta \int Tdt = 0 \quad (Using(13.3.14)) \tag{13.3.15}$$

With the understanding that the end configurations are fixed and that in it, $T + V = E$".

This principle is often called as the principle of least (or stationary) action.

For a simple system, the kinetic energy is of the form

$$T = \frac{1}{2}\sum_{p=1}^{n}\sum_{\sigma=1}^{n} a_{p\sigma}\dot{q}_p\dot{q}_\sigma \tag{13.3.16}$$

Now we write $T = T^{\frac{1}{2}}T^{\frac{1}{2}} = (E - V)^{\frac{1}{2}}T^{\frac{1}{2}}$

$$\therefore Tdt = (E - V)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{p=1}^{n}\sum_{\sigma=1}^{n} a_{p\sigma}dq_p dq_\sigma\right)^{\frac{1}{2}} \tag{13.3.17}$$

We note that a constant factor is of no significance in a variational principle. Hence we may express $\delta\int Tdt = 0$ in the form

$$\delta \int ds = 0 \tag{13.3.18}$$

where

$$ds^2 = (E - V)\sum_{p=1}^{n}\sum_{\sigma=1}^{n} a_{p\sigma} dq_p dq_\sigma \tag{13.3.19}$$

If we consider the $n$ - dimensional configuration space, as a Riemannian space with a distance $ds$ between adjacent points then (13.3.18) may be stated as "The natural curves of motion are geodesics or curves of stationary length".

The formula (13.3.18) is called Jacobi's principle of least (or stationary) action.

# BLOCK-V

# UNIT 14

# Phase Space



**Objectives**

Upon completion of this Unit, the student is exposed to

- the space of events.
- construction of generating function.

## 14.1 Introduction

Let us consider a space of $2n$ - dimensions in which the co-ordinates of a point are the $n$ - generalised co-ordinates $(q)$ and the $n$ - generalised momenta $(p)$ to represent the natural motions of a system with $n$ - degrees of function. This $2n$ - dimensional space is called **phase space**.

We discuss only conservative systems in this section so that $H = H(q, p)$, $\frac{\partial H}{\partial t} = 0$ and $H$ is a constant of the motion.

# The stream lines in phase space

At each point of phase space, the canonical equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{14.1.1}$$

Define the ratios $dq_1 : dq_2 : \cdot\cdot\cdot\cdot : dq_n : dp_1 : dp_2 : \cdot\cdot\cdot\cdot : dp_n$.
Thus they define a directionat the point and so the totality of natural motions give us a **congruence of curves**, filling phase space, with one curve through each point. Those curves are called as **stream lines**. We have to consider the representative points moving along the stream lines.

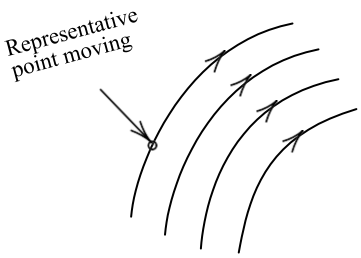



Figure 14.1.1: The congruence of streamlines in phase space

The increments in $(q, p)$ are related to the increment in the time by

$$dq = \frac{\partial H}{\partial p}dt, \quad dp = -\frac{\partial H}{\partial q}dt \tag{14.1.2}$$

**Example 18.** For a particle of unit mass moving on a straight line under no force, the Hamiltonian

$$H = \frac{p^2}{2} \quad (q \;\; and \;\; t \;\; happen \;\; to \;\; be \;\; absent) \tag{14.1.3}$$

(14.1.3) gives $dq = \frac{1}{2}(2p)dt \;\; and \;\; dp = 0$

(i.e), $dq = pdt, \; dp = 0$

To show diagramatically, let $q$ and $p$ be taken as a rectangular co-ordinates. The streamlines are parallel to the $q$ - axis [ $dp = 0 \implies p = constant$ ]. The representative point stays at rest on

$p = 0$ (equilibrium) and on the other streamlines it moves with a velocity proportional to $p$.

**Example 19.** The Hamiltonian for a simple harmonic oscillator may be written as

$$H = \frac{1}{2m} p^2 + \frac{1}{2} kq^2 \tag{14.1.4}$$

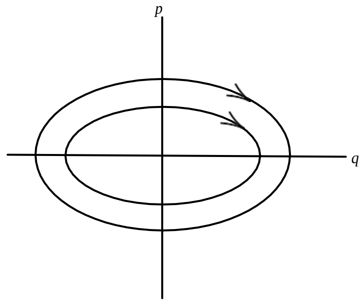
The streamlines are given by

$$dq = \frac{\partial H}{\partial p} dt, \quad dp = -\frac{\partial H}{\partial q} dt$$

(i.e) $dq = \frac{1}{2m} 2pdt \quad \Longrightarrow \quad dq = \frac{p}{m} dt$

$$dp = -\frac{k}{2}(2q)dt \quad \Longrightarrow \quad dp = -kqdt$$

Since $H$ is a constant during the motion, from (14.1.4) we see that the streamlines are ellipses.




# Canonical Transformations

For a given dynamical system defined physically. We are free to choose the generalised co-ordinates ‘ $q$ ’. Thus general dynamical theory is clearly invariant under transformations $q \to q^J$ (i.e), a set of $n$ - equations expressing one set of $n$ - generalised co-ordinates ‘ $q$ ’ interms of another set $q^J$. The term “invariant” here means that any general statement in dynamical theory is equally true no matter which system of co-ordinates is used. Hence the Hamilton’s canonical equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{14.1.5}$$

transform into

$$\dot{q}^J = \frac{\partial H}{\partial p^J}, \quad \dot{p}^J = -\frac{\partial H}{\partial q^J} \tag{14.1.6}$$

Such transformations are called **Canonical** or **contact** transformation.

## 14.2 The Bilinear Invariant

Let $(\partial q, \partial p)$ and $(\Delta q, \Delta p)$ be two arbitrary infinitesimal displacements in phase space. The expression

$$\delta q \Delta p - \delta p \Delta q \tag{14.2.1}$$

is called a bilinear form.

For $n$ - variables the above expression reads

$$\delta q_1 \Delta p_1 + \delta q_2 \Delta p_2 + \cdot\cdot\cdot + \delta q_n \Delta p_n - \delta p_1 \Delta q_1 - \delta p_2 \Delta q_2 - \cdot\cdot\cdot - \delta p_n \Delta q_n \tag{14.2.2}$$

For the transformation

$$q = q(q^J, p^J), \quad p = p(q^J, p^J) \tag{14.2.3}$$

We can express the differntials occuring in (14.2.2) in terms of $\delta q^J, \delta p^J, \Delta q^J, \Delta p^J$ and then substitute in (14.2.2).

**Theorem 14.2.1.** *If the transformation is canonical, then*

$$\delta q \Delta p - \delta p \Delta q = \delta q^J \Delta p^J - \delta p^J \Delta q^J \tag{14.2.4}$$

*Conversely if* (14.2.4) *holds, for all infinitesimal displacements, then the transformation*

$$q = q(q^J, p^J), \quad p = p(q^J, p^J) \tag{14.2.5}$$

*is canonical.*

**Proof.** To prove the if part, we note that if (14.2.5) is canonical then the following are satisfied

$$\dot{q} = \frac{\partial H}{\partial p}dt, \quad \dot{p} = -\frac{\partial H}{\partial q}dt \tag{14.2.6}$$

$$\dot{q}^J = \frac{\partial H}{\partial p^J}dt, \quad \dot{p}^J = -\frac{\partial H}{\partial q^J}dt \tag{14.2.7}$$

$\therefore$ for arbitrary $\delta q, \delta p$, we have

$$\delta q \left( dp + \frac{\partial H}{\partial q}dt \right) - \delta p \left( dq - \frac{\partial H}{\partial p}dt \right) = 0 \tag{14.2.8}$$

$$\delta q dp + \delta q \frac{\partial H}{\partial q} dt - \delta p dq + \delta p \frac{\partial H}{\partial p} dt = 0 \text{ (i.e),}$$

$$\delta q dp - \delta p dq + \delta H dt = 0 \tag{14.2.9}$$

Similarly

$$\delta q^J dp^J - \delta p^J dq^J + \delta H dt = 0 \tag{14.2.10}$$

(14.2.9)-(14.2.10) yields

$$\delta q dp - \delta p dq = \delta q^J dp^J - \delta p^J dq^J \tag{14.2.11}$$

Here $(\delta q, \delta p)$ are arbitrary, but $(dq, dp)$ are not since they correspond to a displacement along a streamline. A transformation is considered to be canonical if it must preserve the canonical form of equations of motion for every $H$. But by changing $H$, we can make $(dq, dp)$ as we like and so let us change $d$ to $\Delta$ in (14.2.11).

Hence $\delta q \Delta p - \delta p \Delta q = \delta q^J \Delta p^J - \delta p^J \Delta q^J$ which proves (14.2.4).
To prove the converse part, we have to prove (14.2.4) and (14.2.6) together reply (14.2.7). We see that (14.2.6) imply (14.2.9) and that with (14.2.4) [used instead of $\Delta$] implies (14.2.10).

This can be written as

$$\delta q^J \left( dp^J + \frac{\partial H}{\partial q^J}dt \right) - \delta p^J \left( dq^J + \frac{\partial H}{\partial p^J}dt \right) = 0 \tag{14.2.12}$$

From the above equation, we have

$$dq^J = \frac{\partial H}{\partial p^J}dt, \quad dp^J = -\frac{\partial H}{\partial q^J}dt$$

which gives (14.2.7), since $(\delta q^J, \delta p^J)$ are arbitrary which proves the converse part. ■

**Example 20.** Prove the transformation $p = \frac{1}{q'}$ and $q = p'q'^2$ is canonical.

**Solution.** $\delta p = -\frac{1}{q'^2}\delta q'$, $\quad \delta q = p'(2q')\delta q' + q'^2\delta p'$

$$\Delta p = -\frac{1}{q'^2}\Delta q', \quad \Delta q = p'(2q')\Delta q' + q'^2\Delta p'$$

Consider $\delta q\Delta p - \delta p\Delta q$

$$\begin{aligned}
&= (2p'q'\delta q' + q'^2\delta p')\left(-\frac{1}{q'^2}\Delta q'\right) - \left(-\frac{1}{q'^2}\right)\delta q'(2p'q'\Delta q' + q'^2\Delta p') \\
&= -\frac{2p'}{q'}\delta q'\Delta q' - \delta p'\Delta q' + \frac{2}{q'}p'\delta q'\Delta q' + \Delta p'\delta q' \\
&= \delta q'\Delta p' - \delta p'\Delta q'
\end{aligned}$$

Hence the transformation is canonical.

# Generating Functions

The Bilinear unvariant is used to test whether a given transformation is canonical or not. But it does not tell us how to construct canonical transformation.

# Construction of generating functions

Let $G(q, q')$ be a function of $2n$ quantities $(q, q')$. Let

$$p = \frac{\partial G}{\partial q}, \quad p' = -\frac{\partial G}{\partial q'} \tag{14.2.13}$$

which is equivalent to

$$p\delta q - p'\delta q' = \delta G = \frac{\partial G}{\partial q}\delta q + \frac{\partial G}{\partial q'}\delta q' \tag{14.2.14}$$

where $(\delta q, \delta q')$ is arbitrary.

In (14.2.14), we have $2n$ equations connecting $4n$ - quantities $(q, p, q', p')$. Solving for $(q, p)$ we have a transformation.

$$q = q(q', p'), \quad p = p(q', p') \tag{14.2.15}$$

We show that the transformation (14.2.15) is canonical.

Consider a surface with two dimensions in phase space with equations.

$$q = q(u, v), \quad p = p(u, v) \tag{14.2.16}$$

where $u$ and $v$ are parameters.

Let $\delta$, $\Delta$ denote the changes arising from constant infinitesimal variations $\delta u$, $\Delta v$ respectively. Then for any function $f$ on $u$ and $v$, we have

$$\delta f = \frac{\partial f}{\partial u}\delta u, \quad \Delta f = \frac{\partial f}{\partial v}\delta v \tag{14.2.17}$$

$$\Delta\delta f = \frac{\partial^2 f}{\partial v \partial u}\partial u \Delta v = \frac{\partial^2 f}{\partial u \partial v}\partial u \Delta v = v\delta\Delta f \tag{14.2.18}$$

(i.e),

$$\Delta\delta = \delta\Delta \tag{14.2.19}$$

Applying the operator $\Delta$ to (14.2.14)

$$\Delta p\delta q + p\Delta\delta q - \Delta p'\delta q' - \Delta p'\delta q' = \Delta\delta G \tag{14.2.20}$$

Applying (14.2.19) in (14.2.20) we have

$$\delta p\Delta q + p\delta\Delta q - \delta p'\Delta q' - \delta p'\Delta q' = \delta\Delta G \tag{14.2.21}$$

Subtracting (14.2.21) from (14.2.20) and using (14.2.19), we get

$$\delta q\Delta p - \delta p\Delta q = \delta q'\Delta p' - \delta p'\Delta q'$$

thus the invariance of the bilinear form is established. Hence the transformation (14.2.15) is canonical.

The function $G(q, q')$ is called the **generating function** of the canonical transformation.

**Example 21.** Find the generating function for the transformation $p = \frac{1}{q'}, \quad q = p'q'^2$

**Solution.** $p\delta q - p'\delta q' = \delta G$

(i.e), $\frac{1}{q}(\delta q) - \left(\frac{q}{q'^2}\right)\delta q' = \delta G$

(i.e), $\delta G = q\left(-\frac{1}{q'^2}\right)\delta q' + \left(\frac{1}{q'}\right)\delta q$

$= \delta\left(q\frac{1}{q'}\right)$

$= \delta\left(\frac{q}{q'}\right)$

Hence $G(q, q') = \frac{q}{q'}$ is the generating function.

**Example 22.** Find the generating function for the transformation $q = -p'$, $p = q'$

**Solution.** $p\delta q - p'\delta q' = \delta G$

$\Longrightarrow q'\delta q + q\delta q' = \delta G$

(i.e), $\delta G = q(\delta q') + q'(\delta q)$

$= \delta(qq')$

Hence $G(q, q') = qq'$

# Check Your Progress

1. Show that the following transformations are canonical.

(a) $q = \frac{1}{2}\left(q'^2 + p'^2\right)$ ; $p = -\tan^{-1}\left(\frac{q'}{p'}\right)$

(b) $q = \sqrt{2q'e^t}\cos p'$ ; $p = \sqrt{2q'e^{-t}}\sin p'$

(c) $q = \log\left(\frac{\sin p'}{q'}\right)$ ; $p = q'\cot(p')$

# Answer to Check Your Progress

1. Hint : Prove $\delta q\Delta p - \delta p\Delta q = \delta q'\Delta p' - \delta p'\Delta q'$

# BLOCK-V

# UNIT 15

# Poisson Brackets



**Objectives**

Upon completion of this Unit, the students will be able to

- identify canonical transformation generated by the motion.
- identify Liouville's theorem.

## 15.1 Introduction

Let $f(q, p)$ be any function of $(q, p)$. Then as the system moves in accordance with the canonical equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{15.1.1}$$

the rate of change of $f$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = [f, H] \tag{15.1.2}$$

where

$$[f, H] = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} \tag{15.1.3}$$

The last expression is a particular case of the **Poisson bracket**

The Poisson bracket for any two functions $f(q, p)$, $g(q, p)$ is defined as

$$[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{15.1.4}$$

In full form

$$[f, g] = \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} + \cdots + \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} \tag{15.1.5}$$

**Note**

1. It is clear that

   $$[f, g] = -[g, f] \tag{15.1.6}$$

   so that the Poisson brackets are skew symmetric.

2. $[f, f] = 0$

If $f = q - 1$ is substituted in (15.1.2) then

$$\dot{q}_1 = [q, H] \tag{15.1.7}$$

The above is an alternative way of meeting the first of the canonical equations (15.1.1). The full set of canonical equations may be expressed in terms of Poisson brackets as follows:

$\dot{q} = [q, H]$, $\dot{p} = [p, H]$

If $f = H$ is substituted in (15.1.2) then $\dot{H} = [H, H] = 0$ by (15.1.7).

If the Poisson brackets are considered with the co-ordinates themselves, then from (15.1.5) we have

(i)

$$[q_p, q_\sigma] = 0 \quad [p_p, p_\sigma] = 0 \tag{15.1.8}$$

(ii)

$$[q_p,\ p_\sigma] = -[p_\sigma, q_p] = \delta_{p\sigma} \tag{15.1.9}$$

where $p, \sigma = 1, 2, \cdots, n$ and $\delta_{p\sigma}$ is the kronecker delta defined by

$$\begin{aligned} \delta_{p\sigma} &= 1 \quad if \quad p = \sigma \\ and\ \ \delta_{p\sigma} &= 0 \quad if \quad p \neq \sigma \end{aligned} \tag{15.1.10}$$

the Poisson brackets plays an important role in the generalisation of quantum mechanics out of classical dynamics.

## 15.2 Canonical Transformations generated by the Motion

Consider the dynamical fluid in motion. Figure(5.2) shows a portion of the fluid at time ‘ $t_0$ ’ and that same portion as a later time $t$ . By “the same portion” we mean it is composed of the same “particles” of the dynamical fluid, each such particle moving in accordance with the canonical equations.

Let $A$ be the position of a particle at time ‘ $t$ ’. Now $A$ at $t_0$ and $B$ at $t$ are two events and between those two events there is an action $S\ (q_0,\ t_0,\ q,\ t)$ where $q_0$ and $q$ refer to $A$ and $B$ respectively.

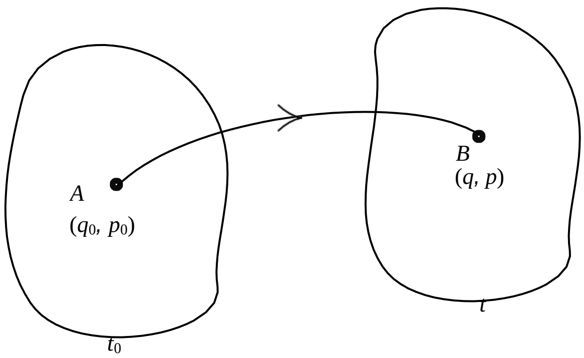



Figure 15.2.1: Canonical transformations generated by the motion of the dynamical fluid

Action is given by

$$S = \int_{A,t_0}^{B,t} (pdq - Hdt) \tag{15.2.1}$$

The moments at $A$ and $B$ are give by

$$p = \frac{\partial S}{\partial q}, \quad p_0 = -\frac{\partial S}{\partial q_0} \tag{15.2.2}$$

Remembering that $t_0$ and $t$ are merely two constants, we recognise in (15.2.2), the canonical transformations, $p = \frac{\partial G}{\partial q}, \quad p' = -\frac{\partial G}{\partial q'}$.
The characteristic function $S$ playing the part of generating function.

thus we have proved that "if $(q_0, p_0)$ is the position in phase space at time $t_0$ of a particle of the dynamical fluid and $(q, p)$ the position of the same particle at time '$t$', then the transformation

$$(q_0, p_0) \to (q, p)$$

is a canonical transformation and Hamilton's characteristic function is a generating function of that transformation".

**Note**

1. If we vary the parameters $t_0$ and $t$ of the transformation, we get a continuous group of canonical transformations.
2. The Poisson brackets are invariant under a canonical transformation . i.e., $[f, g]_{q,p} = [f, g]_{q', p'}$. Hence if the transformation is canonical then $[f, f] = [g, g] = 0$ and $[g, f] = 1$.

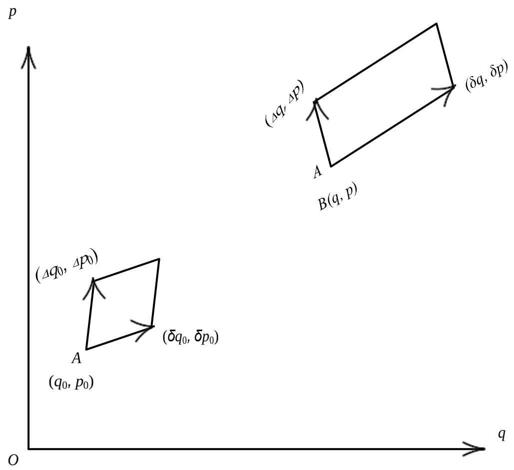
## 15.3 Liouville's Theorem

## For one degree of freedom

The area of any finite domain remains constant as that domain is carried along with the dynamical fluid.

(i.e) $\frac{d}{dt}\int\int \delta q\delta p = 0$, where $p$ and $q$ are the rectangular co-ordinate axes.

**Proof.** Let $A$ be the position of a particle at time $t_0$ and $B$ be the position of the same particle at time $t$.
At $A$, let us draw an infinitesimal parallelogram with edges $(\partial q_0, \partial p_0)$ and $(\Delta q_0, \Delta p_0)$




Its area is $\delta q_0\Delta p_0 - \delta p_0\Delta q_0$. This parallelogram is carried along by the motion of the dynamical fluid and at time '$t$' it has edges $(\delta q, \delta p)$ and $(\Delta q, \Delta p)$.

It's area at $B$ is $\delta q\Delta p - \delta p\Delta q$. But the transformation produced by the motion is canonical and therefore by the invariance of the bilinear form under canonical transformations, we have

$$\delta q\Delta p - \delta p\Delta q = \delta q_0\Delta p_0 - \delta p_0\Delta q_0 \tag{15.3.1}$$

= Thus the area of the infinitesimal parallelogram is unchanged by the motion. Since a finite area may be split up into infinitesimal parallelograms, it follows that "the area of any finite domain remains constant as that domain is carried along with the dynamical fluid" which proves the Liouville's theorem for a system with one degree of freedom.
(i.e),

$$\frac{d}{dt}\int\int \delta q\delta p = 0 \tag{15.3.2}$$

∎

**For $n$ - degrees of Freedom**

The volume

$$\int \cdots \int \delta q_1 \cdots \delta q_n, \delta p_1 \cdots \delta p_n$$

of any domain in phase space remains constant as that volume is carried along with the dynamical fluid, moving in accordance with the canonical equations.

**Proof.** Consider a system with $n$ - degrees of freedom phase space of $2n$ - dimensions and we regard the $2n$ - variables $(q, p)$ as rectangular cartesian co-ordinates in it.

The volume of any domain in phase-space is

$$\int \cdots \int \delta q_1 \cdots \delta q_n, \delta p_1 \cdots \delta p_n \tag{15.3.3}$$

At each point in phase space there is a "velocity vector with - $2n$ components,

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \ \dot{q}_2 = \frac{\partial H}{\partial p_2} \cdots \dot{q}_n = \frac{\partial H}{\partial p_n} \tag{15.3.4}$$

$$\dot{p}_1 = \frac{\partial H}{\partial q_1}, \ \dot{p}_2 = -\frac{\partial H}{\partial q_2} \cdots \dot{p}_n = \frac{\partial H}{\partial q_n} \tag{15.3.5}$$

Using the fact that "the divergence of the velocity vanishes", we have

$$\frac{\partial \dot{q}_1}{\partial q_1} + \cdots + \frac{\partial \dot{q}_n}{\partial q_n} + \frac{\partial \dot{p}_1}{\partial p_1} + \cdots + \frac{\partial \dot{p}_n}{\partial p_n} = 0 \tag{15.3.6}$$

For a moment let us consider an ordinary fluid moving in arbitrary space. Let $(u, v, w)$ be the components of velocity of the fluid at the point with co-ordinates $(x, y, z)$. Then for any volume bounded by a surface and we have by Green's theorem

$$\int\int\int \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) dxdydz = \int\int (lu + mv + nw) d\zeta \tag{15.3.7}$$

The integral on the LHS is taken throughout the volume and the integral on the RHS is taken over $\zeta$. The quantities $l, m, n$ are the direction cosines of the outward drawn normal to $\zeta$.

But

$$lu + mv + nw = V \tag{15.3.8}$$

the component of velocity normal to $\zeta$ and so as $\zeta$ is carried along with the fluid, in a time interval $dt$, the volume contained inside $\zeta$ increases by an amount

$$dt\int\int V d\zeta = dt\int\int\int\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)dxdydz \tag{15.3.9}$$

Thus

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{15.3.10}$$

is a necessary and sufficient condition that the volume of any domain remains constant when that domain is carried along with the fluid. The constancy of the volume is implied by the vanishing of the divergence of the velocity vector.

The equation is mathematically same as the one in (15.3.10) in $2n$ dimensions instead of in 3 dimensions. Green's theorem can be applied for a space of any dimensionality and so if we apply it to a $2n$ - dimensional volume in phase space bounded by a $(2n-1)$ dimensional surface, carried along with the dynamical fluid, then the above hydro-dynamical argument sends to the general form of Liouville's theorem.

**Note** The Liouville's theorem can be expressed as $\frac{d}{dt}\int \delta q\delta p = 0$, where the single integration here represents a $2n$ - fold integration and $\delta q\ \delta p$ represents the $2n$ - dimensional element in $\int \cdots \int \delta q_1 \cdots \delta q_n, \delta p_1 \cdots \delta p_n$. ■

## Check Your Progress

1. Using Poisson bracket, show that the transformation is canonical $q = \sqrt{2p'}\sin q'$; $p = \sqrt{2p'}\cos q'$

2. For what values of $m$ and $n$ do the transformation equations $q' = q^m \cos np$; $p' = q^m \sin np$ present a canonical transformation.

3. For what values of $\alpha$, the following transformation $q' = q\cos\alpha - p\sin\alpha$; $p' = q\sin\alpha + p\cos\alpha$ is canonical.

4. Show that $(ad - bc) = 1$, in order that the transformation $q' = aq + bp$; $p' = cq + dp$ is canonical.

# Answer to Check Your Progress

1. The transformation can be written as $\tan q' = \frac{q}{p}$; $p' = \frac{q^2 + p^2}{2}$; Prove $[q', p'] = [p', p'] = 0$ and $[q', p'] = 1$ for thr transformation to be canonical.

2. For the transformation to be canonical, we show that $[q', p'] = 1$. Using this condition we get $mnq^{2m-1} = 1$. Equating coefficient of $q^0$ on both sides, $2m - 1 = 0 \Rightarrow m = \frac{1}{2}$. Using $mnq^0 = 1$, and $m = \frac{1}{2}$, we get $n = 2$.

3. $[q', p'] = 1 \Rightarrow \cos^2\alpha + \sin^2\alpha = 1$, for all $\alpha$.

4. $[q', p'] = 1 \Rightarrow \frac{\partial q'}{\partial q}\frac{\partial p'}{\partial p} - \frac{\partial q'}{\partial p}\frac{\partial p'}{\partial q} = 1 \Rightarrow ad - bc = 1.$