# Master of Science Mathematics

## (M.Sc. Mathematics)

## MMT-202

## Complex Analysis

## Semester-II

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**SURESH GYAN VIHAR UNIVERSITY**

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# Suresh Gyanvihar University
## Department of Mathematics
## School of Science

**M.Sc., Mathematics - Syllabus – I year- II Semester (Distance Mode)**

| | | |
|---|---|---|
| **COURSE TITLE** | : | COMPLEX ANALYSIS |
| **COURSE CODE** | : | **MMT- 202** |
| **COURSE CREDIT** | : | **4** |

**COURSE OBJECTIVES**

While studying the **COMPLEX ANALYSIS**, the Learner shall be able to:

CO 1: Review the singular points.

CO 2: Discuss the concept of multiple connected region.

CO 3: Represent Weierstrass theorem on entire functions.

CO 4: Identify the applications of Harnack Principle.

CO 5: Describe the concept of canonical basis.

**COURSE LEARNING OUTCOMES**

After completion of the **COMPLEX ANALYSIS**, the Learner will be able to:

CLO 1: Interpret the Cauchy's integral formula, identify them to solve a problem by using Cauchy's integral formula.

CLO 2: Describe the concept of mean value property and properties of harmonic functions.

CLO 3: Enable to extend the Riemann zeta function to the whole complex plane.

CLO 4:Demonstrate and identify that the unit disk can be mapped conformally onto any simply connected region in the plane, other than the plane itself.

CLO 5: Demonstrate and identify the Weierstrass function, then able to prove the differential equation satisfied by Weierstrass function.

**BLOCK I: COMPLEX INTEGRATION**

Fundamental Theorems: Cauchy's Theorem for a Rectangle- Cauchy's Theorem in a Disk. Cauchy's Integral Formula: The Index of a point with respect to a closed curve – The Integral formula – Higher derivatives. Local Properties of analytical Functions:Removable Singularities-Taylors's Theorem – Zeros and poles – The local Mapping – The Maximum Principle.

**BLOCK II: COMPLEX INTEGRATION**

The Genral Form of Cauchy's Theorem: Chains and cycles- Simple Continuity - Homology - The General statement of Cauchy's Theorem - Proof of Cauchy's theorem - Locally exact differentials- Multilply connected regions - Residue theorem - The argument principle. Evaluation of Definite Integrals and Harmonic Functions: Evaluation of definite integrals - Definition of Harmonic function and basic properties - Mean value property - Poisson formula.

**BLOCK III: SERIES AND PRODUCT DEVELOPMENTS**

Partial Fractions and Entire Functions: Partial fractions - Infinite products – Canonical products – Gamma Function- Jensen's formula – Hadamard's Theorem

Riemann Theta Function and Normal Families: Product development – Extension of $\zeta(s)$ to the whole plane – The zeros of zeta function – Equicontinuity – Normality and compactness – Arzela's theorem – Families of analytic functions – The Classical Definition.

**BLOCK IV: CONFORMAL MAPPINGS**

Riemann mapping Theorem: Statement and Proof – Boundary Behaviour – Use of the Reflection Principle. Conformal mappings of polygons: Behaviour at an angle – Schwarz-Christoffel formula – Mapping on a rectangle. Harmonic Functions: Functions with mean value property – Harnack's principle.

**BLOCK V: ELLIPTIC FUNCTIONS**

Simply Periodic Functions : Representation by Exponentials-The Fourier Development - Functions of Finite Order. Doubly Periodic Functions:The Period Module-Unimodular Transformations - The Caninical Basis-General Properties of Elliptic Functions. Weierstrass Theory: The Weierstrass $\wp$-function – The functions $\zeta(s)$ and $\sigma(s)$ – The differential equation – The modular equation $\lambda(\tau)$ – The Conformal mapping by $\lambda(\tau)$.

**REFERENCE BOOKS:**

1. Lars F. Ahlfors, Complex Analysis, (3rd Edition) McGraw Hill Book Company, NewYork, 1979.

2.H.A. Presfly, Introduction to complex Analysis, Clarendon Press, oxford, 1990.

3.J.B. Corway, Functions of one complex variables, Springer - Verlag, International student Edition, Narosa Publishing Co.

4.E. Hille, Analytic function Thorey (2 vols.), Gonm& Co, 1959.

5.M.Heins, Complex function Theory, Academic Press, NewYork,1968.

**Web resources**

https://www.youtube.com/watch?v=AruIhF83CIY

https://www.youtube.com/watch?v=7JlBuU1CfUg

https://www.youtube.com/watch?v=rc8Y0rRU-Bg

https://www.youtube.com/watch?v=VBvuQbkvG7I

https://www.youtube.com/watch?v=o77UV7YrWvw

https://www.youtube.com/watch?v=lVvHHbkEkmw

https://www.youtube.com/watch?v=awHcEvjMTSI

https://www.youtube.com/watch?v=tgD1Bznt5wo

https://www.youtube.com/watch?v=sslQQHAchMY

https://www.youtube.com/watch?v=zqt7ubXbIe8

https://www.youtube.com/watch?v=ztOoZsRm_x0

https://www.youtube.com/watch?v=HgFAwLVkR00

https://www.youtube.com/watch?v=vFMs2IcqMxU

https://www.youtube.com/watch?v=Vmp4bWnvGJU

https://www.youtube.com/watch?v=4BldK5lKVjY

https://www.youtube.com/watch?v=4mpy3VxXRDc

https://www.youtube.com/watch?v=FMJUm6z_Nrg

https://www.youtube.com/watch?v=K3yMlbcxkP4

# BLOCK - I



# BLOCK - II



# BLOCK - III



# BLOCK - IV

# BLOCK - V

# Contents

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# BLOCK-I

# UNIT 1

# Fundamental Theorems

**Objectives**

Upon completion of this Unit, students will be able to

- identify Cauchy's theorem for a rectangle.

## 1.1 Introduction

In this Block we take up complex integral calculus. Many important properties of analytic functions are very difficult to prove without use of complex integration. For instance, it is only recently that it became possible to prove, without resorting to complex integrals or equivalent tools, that the derivative of an analytic function is continuous, or that the higher derivatives exist.

As in the real case we distinguish between definite and indefinite integrals. An indefinite integral is a function whose derivative equals a given analytic function in a region, in many elementary cases indefinite integrals can be found by inversion of known derivation formulas. The definite integrals are taken over differentiable or piecewise differentiable arcs and are not limited to analytic functions. The reader must be thoroughly familiar with the theory of definite integrals of real continuous functions.

## 1.2 Cauchy's Theorems for a Rectangle

There are several forms of Cauchy's theorem, but they differ in their topological rather than in their analytical content. It is natural to begin with a case in which the topological consideration are trivial.

We consider, specifically, a rectangle $R$ defined by inequalities $a \leq x \leq b$, $c \leq y \leq d$. Its perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that $R$ lies to the left of the directed segments. The order of the vertices is thus $(a, c)$, $(b, c)$, $(b, d)$, $(a, d)$. We refer to this closed curve as the boundary curve or contour of $R$, and we denote it by $\partial R$.

We emphasize that $R$ is chosen as a closed point set and hence, is not a region. In the following theorem, we consider a function which is analytic on the rectangle $R$.

The following is a preliminary version of Cauchy's theorem:

**Theorem 1.2.1.** *If the function $f(z)$ is analytic on $R$, then*

$$\int_{\partial R} f(z)dz = 0, \tag{1.2.1}$$

*where $\partial R$ denotes the boundary of rectangle or contour.*

**Proof.** The proof is based on the method of bisection.

$$\eta(R) = \int_{\partial R} f(z)dz. \tag{1.2.2}$$

Divide the rectangle $R$ into four congruent rectangles $R^1$, $R^2$, $R^3$, $R^4$ by joining of the opposite sides of $R$ then from (12.2.2), we have

$$\eta(R) = \int_{\partial R^1} f(z)dz + \int_{\partial R^2} f(z)dz + \int_{\partial R^3} f(z)dz + \int_{\partial R^4} f(z)dz$$

$$\eta(R) = \eta(R^1) + \eta(R^2) + \eta(R^3) + \eta(R^4) \tag{1.2.3}$$

$$|\eta(R)| \leq |\eta(R^1)| + |\eta(R^2)| + |\eta(R^3)| + |\eta(R^4)|$$

It follows from (8.2.3) that at least one of the rectangles $R^k$, $k = 1, 2, 3, 4,$ must satisfy the condition

$$|\eta(R)| \leq 4|\eta(R^k)|$$

We denote this rectangle by $R_1$. Continuing this process indefinitely, we get a sequence of nested rectangles $R_1$, $R_2$, $R_3, R_n \cdot \cdot \cdot$, where $R_n$ is the $n^{th}$ bisection of the rectangles of $R$ such that

$R \supset R_1 \supset R_2 \supset \cdots \supset R_n \cdots$ and this rectangles satisfies the condition

$$|\eta R_n| \geq \frac{1}{4}|\eta(R_{n-1}) \geq \frac{1}{4}\eta(R_{n-2})$$

$$|\eta(R_n)| \geq \frac{1}{4^2}|\eta(R_{n-2})| \cdots\cdots$$

$$|\eta(R_n)| \geq \frac{1}{4^n}|\eta(R)|$$

$$\therefore\ |\eta(R)| \leq 4^n|\eta(R_n)| \tag{1.2.4}$$

which is sufficiently large the rectangle $R_n$ converges to a point say $z_0$ belongs to all the rectangles $R_1, R_2, R_3, \cdots R_n \cdots$

Choose a number $\delta > 0$ so small such that the rectangle $R_n$ is contained in $|z - z_0| < \delta$.

$$i.e.,\ R_n \subset |z - z_0| < \delta$$

Since the function $f(z)$ is analytic on a rectangle $R$, it is also analytic at the point $z_0$ and therefore differentiable at $z_0$.

Therefore, if given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon \qquad for \quad |z - z_0| < \delta$$

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \varepsilon\,|z - z_0| \tag{1.2.5}$$

Now,

$$\int_{\partial R_n}[f(z) - f(z_0) - (z - z_0)f'(z_0)]dz = \int_{\partial R_n} f(z)dz - f(z_0)\int_{\partial R_n} dz - f'(z_0)\int_{\partial R_n} zdz + z_0 f'(z_0)\int_{\partial R_n} dz$$

But

$$\int_{\partial R_n} dz = \int_{\partial R_n} zdz = 0.$$

$$\therefore \int_{\partial R_n} [f(z) - f(z_0) - (z - z_0) f'(z_0)]dz = \int_{\partial R_n} f(z)dz - 0 \quad 0 \pm 0 = \int_{\partial R_n} f(z)dz$$

$$\therefore \left| \int_{\partial R_n} [f(z) - f(z_0) - (z - z_0) f'(z_0)]dz \right| = |\eta(R_n)|$$

$$|\eta(R_n)| \leq \int_{\partial R_n} |f(z) - f(z_0) - (z - z_0) f'(z_0)|\,|dz|.$$

$$|\eta(R_n)| < \int_{\partial R_n} \varepsilon\, |z - z_0|\, |dz| \quad by \quad (1.2.5) \tag{1.2.6}$$

If $d_n$ denotes the diagonal of the rectangle $R_n$ then $|z - z_0| \leq d_n$.

If $l_n$ denotes the perimeter of the rectangle $R_n$ then

$$\int_{\partial R_n} |dz| = l_n.$$

$\therefore$ from (1.2.6), we get

$$|\eta(R_n)| < \varepsilon\, d_n\, l_n \tag{1.2.7}$$

If $L$, $D$ denote the perimeter and diagonal of the rectangle $R$ and $l$ and $d$ denotes the perimeter and diagonal of the rectangle $R_n$, we have

$$d_1 = \frac{D}{2}$$

$$l_1 = \frac{L}{2}.$$

Similarly $l_n$, $d_n$ perimeter and diagonal of the rectangle $R_n$, then

$$
\begin{aligned}
l_2 &= \frac{l_1}{2}\\
l_2 &= \frac{L}{2^2}\\
d_2 &= \frac{d_1}{2}\\
d_2 &= \frac{D}{2^2}\\
&.\\
&.\\
&.\\
l_n &= \frac{L}{2^n}\\
d_n &= \frac{D}{2^n}
\end{aligned}
$$

$$\therefore (1.2.7) \Rightarrow |\eta(R_n)| < \epsilon \frac{D}{2^n}\frac{L}{2^n}$$

$$|\eta(R_n)| < \epsilon\frac{DL}{4^n} \tag{1.2.8}$$

From (1.2.4), we get

$$
\begin{aligned}
|\eta(R)| &\leq 4^n\,|\eta(R_n)|\\
&< 4^n\,\epsilon\frac{DL}{4^n}\\
|\eta(R)| &< \epsilon DL
\end{aligned}
$$

Since $\epsilon$ is arbitrary, we can only have

$$\eta(R) = 0.$$

$$i.e., \int_{\partial R} f(z)dz = 0.$$

Hence the theorem is proved. ■

This beautiful proof, which could hardly be simpler, is due to E. Goursat who discovered that the classical hypothesis of a continuous $f'(z)$ is redundant. At the same time the proof is simpler than the earlier proofs in as much as it leans neither on double integration nor on differentiation under the integral sign.

The hypothesis in Theorem.1.2.1 can be weakened considerably. We shall prove at once the following stronger theorem which will find very important use.

**Theorem 1.2.2.** *Let $f(z)$ be analytic on the set $R'$ obtained from a rectangle $R$ by omitting a finite number of interior points $\zeta_j$. If it is true that*

$$\lim_{z\to\zeta_j}(z-\zeta_j)f(z)=0$$

*for all $j$, then*

$$\int_{\partial R} f(z)dz = 0.$$

**Proof.** It is enough to consider the case of a single exceptional point $\zeta$ for evidently rectangle $R$ can be divided into smaller rectangles containing at most one $\zeta_j$.
We divide the rectangle $R$ into nine rectangles and let $R_0$ be the center of the rectangle containing the point $\zeta_j$ then by applying Theorem.1.2.1, we have

$$\int_{\partial R} f(z)dz = \int_{\partial R_0} f(z)dz + 0,$$

$$\left|\int_{\partial R} f(z)dz\right| = \left|\int_{\partial R_0} f(z)dz\right|$$

$$\left|\int_{\partial R} f(z)dz\right| \leq \int_{\partial R_0} |f(z)|\,|dz|. \tag{1.2.9}$$

But we are given

$$\lim_{z\to\zeta_j}(z-\zeta_j)f(z)=0$$

Corresponding to $\varepsilon > 0$ we can write

$$\left|(z-\zeta_j)f(z)\right| < \varepsilon$$
$$\left|z-\zeta_j\right| |f(z)| < \varepsilon$$
$$|f(z)| < \frac{\varepsilon}{|z-\zeta_j|}$$

$$\left|\int_{\partial R} f(z)dz\right| < \int_{\partial R_0} \frac{\varepsilon}{|z-\zeta_j|}|dz|$$

Let $a$ be a side of the rectangle which we consider $R_0$ is a square of the centre, then

$$|z-\zeta_j| \geq \frac{a}{2}$$

$$\therefore \frac{1}{|z-\zeta_j|} \leq \frac{2}{a}$$

$$\therefore \left|\int_{\partial R} f(z)dz\right| < \int_{\partial R} \varepsilon \frac{2}{a}|dz|$$

$$< \frac{2\varepsilon}{a}\int |dz|$$

$$< \frac{2\varepsilon}{a} 4a$$

$$\left|\int_{\partial R} f(z)dz\right| < 8\varepsilon.$$

Since $\varepsilon$ is arbitrary, we get

$$\int_{\partial R} f(z)dz = 0.$$

Hence the theorem is proved. ■

# BLOCK-I

# UNIT 2

# Cauchy's Theorem in a Disk

**Objectives**

Upon completion of this Unit, students will be able to

- prove Cauchy's theorem in a disk.

## 2.1 Introduction

It is not true that the integral of an analytic function over a closed curve is always zero. Indeed, we have found that

$$\int_C \frac{dz}{z-a} = 2\pi i$$

when $C$ is a circle about $a$. In order to make sure that the integral vanishes, it is necessary to make sure that the integral vanishes, it is necessary to make a special assumption concerning the region $\Omega$ in which $f(z)$ is known to be analytic and to which the curve $\gamma$ is restricted. We are not yet in a position to formulate this condition, and for this reason we must restrict attention to a very special case. In what follows we assume that $\Omega$ is an open disk $|z - z_0| < \rho$ to be denoted by $\Delta$.

**Theorem 2.1.1.** *If $f(z)$ is analytic in an open disk $\Delta$, then*

$$\int_\gamma f(z)dz = 0$$

*for every closed curve $\gamma$ in $\Delta$.*

**Proof.** Let us define a function

$$F(z) = \int_{\sigma} f(z)dz$$

where $\sigma$ consists of the horizontal line segment $CD$ vertical line segment $DP$. $C$ is the point $(x_0, y_0)$, $D$ is the point $(x, y_0)$, $P$ is a point $(x, y)$ and $B$ is a point $(x_0, y)$ and all the points lie inside the $\Omega$.

$$\therefore F(z) = \int_{CD+DP} f(z)dz$$

$$F(z) = \int_{CD} f(z)dz + \int_{DP} f(z)dz.$$

Along $CD$, $y$ is a constant. Therefore $dy = 0$ This implies that $dz = dx$. Along $DP$, $x$ is constant. Therefore $dx = 0 \Rightarrow dz = idy$.

$$F(z) = \int_{x_0}^{x} f(z)dz + \int_{y_0}^{y} f(z)idy$$

Differentiating with respect to $y$ partially, we get

$$\frac{\partial}{\partial y}[F(z)] = 0 + i f(z)$$

$$\frac{\partial}{\partial y}[F(z)] = i f(z) \tag{2.1.1}$$

Now complete the rectangle $CDPBC$ by dot lines. Therefore $B$ is a point $(x_0, y)$. Let $\sigma'$ consists of the line segments $CD$ and $BP$ and $-\sigma'$ is $PB$ and $BC$.
Therefore $\sigma + (-\sigma')$ is the closed rectangle $CDPBC$.
By Cauchy's theorem for rectangle,

$$\int_{\sigma+(-\sigma')} f(z)dz = 0$$

$$\int_{\sigma} f(z)dz + \int_{-\sigma'} f(z)dz = 0$$

$$\int_{\sigma} f(z)dz - \int_{\sigma'} f(z)dz = 0$$

$$F(z) = \int_{\sigma'} f(z)dz$$

$$F(z) = \int_{CB} f(z)dz + \int_{BP} f(z)dz$$

Along $CD$ $x$ is constant. Therefore $dz = idy$ Along $BP$ $y$ is constant. Therefore $dz = dx$

$$\therefore F(z) = \int_{y_0}^{y} f(z)idy + \int_{x_0}^{x} f(z)dx$$

Differentiating with respect to $x$ partially, we get

$$\frac{\partial}{\partial x}[F(z)] = 0 + f(z)$$

$$\frac{\partial}{\partial x}[F(z)] = f(z) \tag{2.1.2}$$

From (9.2.1) and (2.1.2) we get

$$\frac{\partial}{\partial x}[F(z)] = -i\frac{\partial}{\partial y}[F(z)] = f(z).$$

Therefore $F(z)$ is an analytic function and also $f(z)$ is the derivative of the analytic function $F(z)$ in $\Omega$.

$$i.e., f(z) = F'(z)$$

$$f(z) = \frac{d}{dz}[F(z)].$$

Therefore $f(z)dz$ is an exact differentiable. Thus

$$\int_{\gamma} f(z)dz = 0,$$

$\gamma$ is a closed curve in $\Omega$. ■

For the applications it is very important that the conclusion of Theorem 2.1.1 remains valid under the weaker condition of Theorem 1.2.2. We state this as the following theorem.

**Theorem 2.1.2.** *Let $f(z)$ be analytic in the region $\Delta'$ obtained by omitting a finite number of points $\zeta_j$ from an open disk $\Delta$. If $f(z)$ satisfies the condition*

$$\lim_{z\to\zeta_j}(z - \zeta_j)f(z) = 0$$

*for all* $j$*, then*

$$\int_{\gamma} f(z)dz = 0.$$

*holds for any closed curve* $\gamma$ *in* $\Delta'$.

**Proof.** The theorem is proved using the above theorem. ■

# BLOCK-I

# UNIT 3

# Cauchy's Integral Formula

**Objectives**

Upon completion of this Unit, students will be able to

- identify Cauchy's integral formula.
- solve problems by using Cauchy's integral formula.
- prove Liouville's theorem.

## 3.1 Introduction

Integral representation formulae are powerful tools for studying analytic functions. Through a very simple application of Cauchy's theorem it becomes possible to represent an analytic function $f(z)$ as a line integral in which the variable $z$ enters as a parameter. This representation, known as Cauchy's integral formula, has numerous important applications. One application of an integral representation is to estimate the size of the function being represented. The integral representation will allow us to show that all the derivatives of an analytic function are analytic. It will also allow us to obtain power series expansions for analytic functions.

## 3.2 The Index of a Point with Respect to a Closed Curve

As a preliminary to the derivation of Cauchy's formula we must define a notion which in a precise way indicates how many times a closed curve winds around a fixed point not on the curve. If the curve is piecewise differentiable, as we shall assume without serious loss of generality, the definition can be based on the following lemma:

**Lemma 3.2.1.** *If the piecewise differentiable closed curve* $\gamma$ *does not pass through the point* $a$, *then the value of the integral*

$$\int_\gamma \frac{dz}{z-a}$$

*is a multiple of* $2\pi i$.

**Proof. Case.i** We can write

$$\int_\gamma \frac{dz}{z-a} = \int_\gamma d(\log(z-a))$$

$$= \int_\gamma d(\log|z-a|) + i\int_\gamma d(\arg(z-a)) \quad \log z = \log|z| + i\arg z$$

When $z$ describes a closed curve, $\log|z-a|$ returns to its initial value and $\arg(z-a)$ increases and decreases by a multiple of $2\pi$.

$$\int_\gamma \frac{dz}{z-a} = o + in(2\pi)$$

$$\int_\gamma \frac{dz}{z-a} = n.2\pi i.$$

**Case.ii** The simplest proof is computational. If the equation of $\gamma$ is $z = z(t)$, $\alpha \leq t \leq \beta$, let us consider the function

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}dt, \tag{3.2.1}$$

where $h(t)$ is defined and continuous on the closed interval $[\alpha, \beta]$. The derivative of $h(t)$ is

$$h'(t) = \frac{z'(t)}{z(t)-a},$$

whenever $z'(t)$ is continuous.

$$\frac{d}{dt}[e^{-h(t)}(z(t) - a)] = 0.$$

Hence

$$e^{-h(t)}(z(t)-a) = a \ constant = k\,(say)$$

Put $t = \alpha$.

Therefore $e^{-h(\alpha)}[z(\alpha) - a] = k$. But by (14.3.1), $h(\alpha) = 0$.

$$\therefore e^{-0}[z(\alpha) - a] = k \Rightarrow z(\alpha) - a = k.$$

$$\begin{aligned} e^{-h(t)}[z(t) - a] &= z(\alpha) - a \\ e^{-h(t)} &= \frac{z(\alpha) - a}{z(t) - a} \\ e^{h(t)} &= \frac{z(t) - a}{z(\alpha) - a}. \end{aligned}$$

Put $t = \beta$,

$$e^{h(\beta)} = \frac{z(\beta) - a}{z(\alpha) - a}$$

Since $\gamma$ is a closed curve and $\alpha \le t \le \beta$.

$$\begin{aligned} \therefore z(\alpha) &= z(\beta) \\ e^{h(\beta)} &= \frac{z(\alpha) - a}{z(\alpha) - a} \\ e^{h(\beta)} &= 1 \\ \therefore h(\beta) &= i2\pi n, \end{aligned}$$

where $n$ is any integer.

$$\therefore \int_{\alpha}^{\beta} \frac{z'(t)}{z(t) - a} dt = i2\pi n.$$

$$\int_{\alpha}^{\beta} \frac{dz}{z - a} = i2\pi n,$$

where $z = z(t)$ and $dz = z'(t)dt$. Hence the Lemma proved. ■

**Definition.** The index of the point $a$ with respect to a curve $\gamma$ is defined by the equation

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}.$$

It is also called as winding number of $\gamma$ with respect to $a$.

## Properties of the index of the point

1. $n(-\gamma, a) = -n(\gamma, a)$.

**Proof.**

$$
\begin{aligned}
n(-\gamma, a) &= \frac{1}{2\pi i}\int_{-\gamma}\frac{dz}{z-a} \\
&= -\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} \\
n(-\gamma, a) &= -n(\gamma, a).
\end{aligned}
$$

∎

2. If $\gamma$ lies inside of a circle, then $n(\gamma, a) = 0$ for all points $a$ outside of the same circle.

**Proof.** Let $a$ lies outside the circle $\gamma$ then the function $\frac{1}{z-a}$ is analytic of $\gamma$ and on $\gamma$. Therefore by Cauchy's theorem,

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} &= 0 \\
\therefore n(\gamma, a) &= 0.
\end{aligned}
$$

∎

3. As $f(a)$, the index $n(\gamma, a)$ is constant in each of the regions determined by $\gamma$, and zero in the unbounded region.

**Proof.** Let $a$, $b$ be two different points in the bounded region determined by $\gamma$. The unbounded region contains point at infinity.

Let us join the points $a$ and $b$ by polygon which does not meet $\gamma$.
Now,

$$
\begin{aligned}
n(\gamma, a) - n(\gamma, b) &= \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} - \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b} \\
&= \frac{1}{2\pi i}\int_{\gamma}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)dz \\
n(\gamma, a) - n(\gamma, b) &= \frac{1}{2\pi i}\left[\log\left(\frac{z-a}{z-b}\right)\right]_{\gamma}
\end{aligned}
$$

Outside of this line segment, the function $\frac{z-a}{z-b}$ is never real and less than or equal to zero. Therefore the principal branch of $\log\left(\frac{z-a}{z-b}\right)$ is analytic in the complement of straight line segment.

$$\frac{1}{2\pi i}\cdot\log\left(\frac{z-a}{z-b}\right) = 0$$
$$n(\gamma, a) = n(\gamma, b)$$

If $a$ lies in the unbounded region then $|a|$ is very large. Choose $\rho$ such that $|z| < \rho < |a|$ then by the property (ii), $n(\gamma, a) = 0$. ■

**Lemma 3.2.2.** *Let $z_1$, $z_2$ be two points on a closed curve $\gamma$ which does not pass through the origin. Denote the subarc from $z_1$ to $z_2$ in the direction of the curve by $\gamma_1$ and the subarc from $z_2$ to $z_1$ by $\gamma_2$. Suppose that $z_1$ lies in the lower half plane and $z_2$ in the upper half plane. If $\gamma_1$ does not meet the negative real axis and $\gamma_2$ does not meet the positive real axis, then $n(\gamma, 0) = 1$.*

## 3.3 The Integral Formula

**Theorem 3.3.1.** *Suppose that $f(z)$ is analytic in an open disk $\Omega$, and let $\gamma$ be a closed curve in $\Omega$. For any point $a$ not on $\gamma$*

$$n(\gamma, a).f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz, \tag{3.3.1}$$

*where $n(\gamma, a)$ is the index of $a$ with respect to $\gamma$.*

**Proof.** Let $f(z)$ be analytic in an open disk $\Omega$. Let $\gamma$ be a closed curve in $\omega$ and let $a \in D$, where $a$ does not lie on $\gamma$.
Consider the function

$$F(z) = \frac{f(z) - f(a)}{z-a} \tag{3.3.2}$$

where $F(z)$ is analytic throughout $\gamma$ except at $z = a$.

$$\begin{aligned}\lim_{z\to a}(z-a).F(z) &= \lim_{z\to a}[f(z) - f(a)]\\ &= f(a) - f(a)\\ &= 0.\end{aligned}$$

The hypothesis of the Theorem 2.1.2. Therefore we conclude that

$$\begin{aligned}
\int_\gamma F(z)dz &= 0\\
\int_\gamma \frac{f(z)-f(a)}{z-a}dz &= 0\\
\int_\gamma \frac{f(z)}{z-a}dz - f(a)\int_\gamma \frac{dz}{z-a} &= 0\\
\int_\gamma \frac{f(z)}{z-a}dz &= f(a)\int_\gamma \frac{dz}{z-a}\\
\int_\gamma \frac{f(z)}{z-a}dz &= f(a).2\pi i.n(\gamma,a)\\
f(a).n(\gamma,a) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz
\end{aligned}$$

■

**Note.** When $n(\gamma, a) = 1$, we get,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

Let $a$ be interior point in $\gamma$, replace $a$ by $z$, then we get

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}d\zeta.$$

This formula is called the Cauchy's Integral Formula and this is valid only when $n(\gamma, a) = 1$.

**Example 3.3.1.** Compute $\int_{|z|=1} \frac{e^z}{z}dz$.

**Solution.** Let $\gamma$ be a unit circle.

*i.e.*, $\gamma : |z| = 1$.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

Here $f(z) = e^z$ and $a = 0$. Clearly the function $f(z)$ is analytic and the point $a = 0$ lies inside $\gamma$.

$$\therefore\ f(0) = \frac{1}{2\pi i}\int_{\gamma} \frac{e^z}{z-0}\,dz$$
$$\int_{\gamma} \frac{e^z}{z}\,dz = f(0)2\pi i$$
$$\int_{\gamma} \frac{e^z}{z}\,dz = 2\pi i \quad since\ f(0) = 1.$$

**Example 3.3.2.** Compute $\displaystyle\int_{|z|=2} \frac{dz}{z^2+1}$.

**Solution.** Let $\gamma$ be $|z| = 2$.
By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a}\,dz$$

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$$
$$\frac{1}{z^2+1} = -\frac{1}{2i}\frac{1}{z+i} + \frac{1}{2i}\frac{1}{z-i}$$
$$\int_{\gamma} \frac{dz}{z^2+1} = -\frac{1}{2i}\int_{\gamma} \frac{dz}{z+i} + \frac{1}{2i}\int_{\gamma} \frac{dz}{z-i}$$

Here $f(z) = 1$ and $a = i, -i$

$$\int_{\gamma} \frac{dz}{z^2+1} = -\frac{1}{2i}.2\pi i f(-i) + \frac{1}{2i}.2\pi i f(i)$$
$$\int_{\gamma} \frac{dz}{z^2+1} = 0.$$

## 3.4 Higher Derivatives

The representation formula

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

gives us an ideal tool for the study of the local properties of analytic functions. In particular we can now show that an analytic function has derivatives of all orders, which are then also analytic.

**Theorem 3.4.1.** ***Cauchy's Higher Derivative Formula.*** *Let $f(z)$ be analytic in the region $\Omega$ then it has derivatives of all orders in the region $\Omega$ which are also analytic function in the region $\Omega$.*

$$\begin{aligned}
i.e., \; f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
\text{then} \quad f'(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta \\
f''(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^3}\,d\zeta \\
\text{In general} \quad f^n(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta
\end{aligned}$$

*where $\gamma$ is a circle.*

**Proof.** By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{3.4.1}$$

Choose $|\Delta z|$ so small such that $z + \Delta z$ lies with in $\gamma$.

$$\therefore \Rightarrow \; f(z + \Delta z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - (z + \Delta z)}\,d\zeta.$$

Now

$$\begin{aligned}
f(z + \Delta z) - f(z) &= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{\zeta - (z + \Delta z)} - \frac{1}{\zeta - z}\right] d\zeta \\
&= \frac{\Delta z}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{[\zeta - (z + \Delta z)][\zeta - z]} \\
\frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{[\zeta - (z + \Delta z)][\zeta - z]}
\end{aligned}$$

Now,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}=\frac{1}{2\pi i}\int_\gamma\left[\frac{1}{[\zeta-(z+\Delta z)][\zeta-z]}-\frac{1}{(\zeta-z)^2}\right]f(\zeta)d\zeta$$

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}=\frac{\Delta z}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{[\zeta-(z+\Delta z)][\zeta-z]^2}$$

$$\therefore\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right|\leq\frac{|\Delta z|}{2\pi}\int_\gamma\frac{|f(\zeta)|\,|d\zeta|}{|\zeta-z|^2\,|\zeta-(z+\Delta z)|}\tag{3.4.2}$$

Let $\delta$ denote the mean distance of the point $\zeta$ and $\gamma$ from $z$. Since $f(\zeta)$ is analytic on $\gamma$ and therefore it is continuous on $\gamma$. Hence

$$|f(\zeta)|\leq M \text{ on } \gamma$$

$$\text{and } |\zeta-z|\geq\delta$$

$$\frac{1}{|\zeta-z|^2}\leq\frac{1}{\delta^2}$$

$$|\zeta-z-\Delta z|\geq|\zeta-z|-|\Delta z|$$

$$\frac{1}{|\zeta-z-\Delta z|}\leq\frac{1}{\delta-|\Delta z|}$$

$$(17.5.2)\Rightarrow\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right|\leq\frac{|\Delta z|}{2\pi}\int_\gamma\frac{M}{\delta^2}\frac{|d\zeta|}{\delta-|\Delta z|}$$

$$\leq\frac{|\Delta z|\,M}{2\pi\ \delta^2}\frac{1}{(\delta-|\Delta z|)}\int_\gamma|d\zeta|$$

$$\int_\gamma|d\zeta|=l$$

is the length of $\gamma$

$$\therefore\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right|\leq\frac{|\Delta z|}{2\pi}\frac{M}{\delta^2}\frac{1}{(\delta-|\Delta z|)}l$$

$$\therefore\left|\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right|\to 0\quad as\ |\Delta z|\to 0.$$

$$f'(z) - \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)d(\zeta)}{(\zeta - z)^2} = 0$$

$$f'(z) = \frac{1!}{2\pi i}\int_{\gamma} \frac{f(\zeta)d(\zeta)}{(\zeta - z)^2}$$

$$Similarly,\quad f''(z) = \frac{2!}{2\pi i}\int_{\gamma} \frac{f(\zeta)d(\zeta)}{(\zeta - z)^3}$$

$$In\ general\quad f^{n}(z) = \frac{n!}{2\pi i}\int_{\gamma} \frac{f(\zeta)d(\zeta)}{(\zeta - z)^{n+1}}$$

Hence the theorem is proved. ■

**Lemma 3.4.1.** *Suppose that $\phi(\zeta)$ is continuous on the arc $\gamma$. Then the function*

$$F_n(z) = \int_{\gamma} \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n}$$

*is analytic in each of the regions determined by $\gamma$, and its derivative is*

$$F'_n(z) = nF_{n+1}(z).$$

**Proof.** First we shall prove that $F_1(z)$ is continuous.

Let $z_0$ be a point not on $\gamma$ and choose the neighborhood $|z - z_0| < \delta$ so that it does not meet $\gamma$. Claim : $F_1(z)$ is continuous.

Let us restrict $z$ to the smallest neighborhood so that $|z - z_0| < \frac{\delta}{2}$.
Now

$$F_1(z) - F_1(z_0) = \int_{\gamma} \frac{\phi(\zeta)d\zeta}{\zeta - z} - \int_{\gamma} \frac{\phi(\zeta)d\zeta}{\zeta - z_0}$$

$$F_1(z) - F_1(z_0) = \int_{\gamma} \frac{z - z_0}{(\zeta - z)(\zeta - z_0)}\phi(\zeta)d\zeta \tag{3.4.3}$$

$$|F_1(z) - F_1(z_0)| \leq |z - z_0| \int_{\gamma} \frac{|\phi(\zeta)|\,|d\zeta|}{|\zeta - z|\,|\zeta - z_0|} \tag{3.4.4}$$

Since $|z - z_0| < \frac{\delta}{2}$ and $|\zeta - z_0| > \delta$

$$\begin{aligned}
|\zeta - z| &= |\zeta - z_0 + z_0 - z| \\
|\zeta - z| &\geq |\zeta - z_0| - |z - z_0| \\
&\geq \delta - \frac{\delta}{2} \\
|\zeta - z| &\geq \frac{\delta}{2} \\
\frac{1}{|\zeta - z|} &\leq \frac{2}{\delta}.
\end{aligned}$$

Since $|\phi(\zeta)| \leq M$ and $\int_\gamma |d\zeta| = l$.

$$\therefore (17.5.3) \Rightarrow |F_1(z) - F_1(z_0)| \leq \frac{\delta}{2} \cdot \frac{2}{\delta} \frac{1}{\delta} Ml \leq \frac{Ml}{\delta}.$$

Put $\delta = \frac{Ml}{\varepsilon}$

$$|F_1(z) - F_1(z_0)| \leq Ml \frac{\varepsilon}{Ml} \leq \varepsilon$$

Therefore $F_1(z)$ is continuous.

$$\begin{aligned}
(17.5.3) \Rightarrow \frac{F_1(z) - F_1(z_0)}{z - z_0} &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta \\
\lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} &= \lim_{z \to z_0} \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta \\
F_1'(z_0) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)(\zeta - z_0)} d\zeta \\
F_1'(z_0) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^2} d\zeta \\
F_1'(z_0) &= F_2(z_0)
\end{aligned}$$

$$\begin{aligned}
\therefore F_1'(z) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^2} d\zeta \\
F_1'(z) &= 1.F_2(z).
\end{aligned}$$

The general case is proved by induction. We have already proved by the case $n = 1$. We shall assume that the result is true for $n = n - 1$. *i.e.,* we shall assume that

$$F'_{n-1}(z) = (n-1)F_n(z).$$

To prove that

$$F'_n(z) = n.F_{n+1}(z).$$

Consider

$$\begin{aligned}
F_n(z) - F_n(z_0) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n}\,d\zeta \\
&= \int_\gamma \frac{(\zeta - z_0)\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n}\,d\zeta \\
&= \int_\gamma \frac{(\zeta - z + z - z_0)\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n}\,d\zeta \\
&= \int_\gamma \frac{(\zeta - z)\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta + (z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n}\,d\zeta \\
F_n(z) - F_n(z_0) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}\,d\zeta + (z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^{n-1}(\zeta - z_0)}\,d\zeta
\end{aligned}$$

$$\begin{aligned}
Define\ G_n(z) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta \\
G_{n-1}(z) &= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}\,d\zeta
\end{aligned}$$

$$\therefore F_n(z) - F_n(z_0) = G_{n-1}(z) - G_{n-1}(z_0) + (z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}\,d\zeta \qquad (3.4.5)$$

Since $\frac{\phi(\zeta)}{\zeta - z_0}$ is continuous on $\gamma$ ($\zeta \neq z_0$.) $G_{n-1}(z)$ is continuous on $\gamma$. Also $|z - z_0|$ is bounded. i.e., $|z - z_0| < \eta$.

$$\begin{aligned}
|F_n(z) - F_n(z_0)| &\leq |G_{n-1}(z) - G_{n-1}(z_0)| + |z - z_0|\int_\gamma \frac{|\phi(\zeta)|\,|d\zeta|}{|\zeta - z|^n\,|\zeta - z_0|} \\
&= \frac{\varepsilon}{2} + \eta.\frac{Ml}{\delta^n}.\frac{2^n}{\delta}
\end{aligned}$$

Choose

$$\eta = \frac{\delta^{n+1}\varepsilon}{2^{n+1}Ml}$$

$$\begin{aligned}
|F_n(z) - F_n(z_0)| &\le \frac{\varepsilon}{2} + \frac{\delta^{n+1}\varepsilon}{2^{n+1}Ml}\cdot\frac{Ml2^n}{\delta^{n+1}} \\
|F_n(z) - F_n(z_0)| &\le \varepsilon, \quad |z - z_0| < \eta.
\end{aligned}$$

$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta$$

$$\begin{aligned}
\lim_{z\to z_0}\frac{F_n(z) - F_n(z_0)}{z - z_0} &= \lim_{z\to z_0}\frac{G_{n-1}(z) - G_{n-1}(z_0)}{z - z_0} + G_n(z_0) \\
F_n'(z_0) &= G_{n-1}'(z_0) + G_n(z_0) \\
&= (n-1)G_n(z_0) + G_n(z_0) \\
F_n'(z_0) &= n.G_n(z_0) \\
&= n\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^{n+1}} \\
&= nF_{n+1}(z_0).
\end{aligned}$$

Hence by induction the theorem is proved. ■

**Note.1** We have proved that an analytic function has derivatives of all orders which are analytic and can be represented by the formula

$$f^n(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}.$$

**Note.2** As a consequence of the above result we have the following two classical theorems.

**Remark.** The integral $\int_\gamma f(z)dz$, with continuous $f$, depends only on the end points of $\gamma$ if and only if $f$ is the derivative of an analytic function in $\Omega$.

**Theorem 3.4.2.** *If $f(z)$ is defined and continuous in a region $\Omega$, and if $\int_\gamma f\,dz = 0$ for all closed curves $\gamma$ in $\Omega$, then $f(z)$ is analytic in $\Omega$.*

**Proof.** Since $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma$ in $\Omega$. The integral depends only on end points of $\gamma$.

Therefore by the above Remark, $f(z)$ is the derivative of an analytic function $F(z)$ in $\Omega$.
i.e., $f(z) = F'(z)$.
Since the derivative of an analytic function is analytic, we say that $f(z)$ is analytic. ■

**Theorem 3.4.3.** ***Cauchy's Inequality (or) Cauchy's Estimates*** *If $f(z)$ is analytic within and on a circle $C$ given by $|z - a| = r$ lying inside $\Delta$ and if $|f(z)| \le M$ for every $z$ on $C$ then*

$$|f^n(a)| \le \frac{n!\,M}{r^n}.$$

**Proof.** By Cauchy's higher derivative formula,

$$\begin{aligned} f^n(a) &= \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz \\ |f^n(a)| &\le \frac{n!}{|2\pi i|}\int_\gamma \frac{|f(z)|\,|dz|}{|z-a|^{n+1}} \\ &\le \frac{n!}{2\pi}\frac{M2\pi r}{r^{n+1}} \\ f^n(a) &\le \frac{Mn!}{r^n}. \end{aligned}$$

■

**Theorem 3.4.4.** ***Liouville's Theorem*** *A function which is analytic and bounded in the whole plane must reduce to a constant.*

**Proof.** Let $C$ be a circle with centre at $a$ and radius $r$. By Cauchy's integral formula for higher derivative, we have

$$f^n(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz.$$

Since $f(z)$ is bounded, we have $f(z) \le M, \forall z$.

$$\begin{aligned} |f'(a)| &\le \frac{1}{|2\pi i|}\int_\gamma \frac{|f(z)|\,|dz|}{|z-a|^2} \\ &\le \frac{1}{2\pi}\frac{M2\pi r}{r^2} \\ f'(a) &\le \frac{M}{r} \end{aligned}$$

This is true for any circle with radius $a$. We know that the complex plane is a circle with infinite radius.

$$\lim_{r\to\infty} |f'(a)| \leq \lim_{r\to\infty} \frac{M}{r} = 0$$

$$i.e., f'(a) = 0$$

for all points $a$ in the $z$ plane.

$$\therefore\ f'(z) = 0, \forall z.$$

$$f(z) = constant.$$

■

**Theorem 3.4.5.** ***Fundamental Theorem of Algebra.*** *Every polynomial in $z$ of degree $n > 0$ must have at lest one root.*

**Proof.** Consider the polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad a_0 \neq 0.$$

Suppose that $p(z) \neq 0$ in the whole complex plane then we say that $p(z)$ is analytic in the whole complex plane and therefore $\frac{1}{p(z)}$ is analytic in the whole complex plane since $p(z) \neq 0$.

$$\begin{aligned} |p(z)| &= \left|a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n\right| \\ &= |a_0 z^n| \cdot \left|1 + \frac{a_1 z^{n-1} + \dots + a_n}{a_0 z^n}\right| \\ &= |a_0|\,|z^n| \left|1 + \frac{a_1}{a_0 z} + \frac{a_2}{a_0 z^2} + \dots + \frac{a_n}{a_0 z^n}\right| \\ |p(z)| &\to \infty \quad as \quad |z| \to \infty. \end{aligned}$$

$$\therefore \frac{1}{p(z)} \to 0 \quad as \quad |z| \to \infty.$$

Therefore $\frac{1}{p(z)}$ is bounded in the whole plane. Hence $\frac{1}{p(z)}$ is analytic and bounded in the whole complex plane.

Therefore by Liouville's theorem, $\frac{1}{p(z)}$ is constant and hence $p(z)$ is constant which is a contradiction.

Our assumption $p(z) \neq 0$ is not true. Thus $p(z)$ has at least one root. ■

# BLOCK-I

# UNIT 4

# Local Properties of Analytic Functions

**Objectives**

Upon completion of this Unit, students will be able to

- express the analytic function as an infinite series.
- classify the singular points.
- identify the maximum principle.

## 4.1 Introduction

We have already proved that an analytic function has derivatives of all orders. In this section we will make a closer study of the local properties. It will include a classification of the isolated singularities of analytic functions.

## 4.2 Removable Singularities. Taylor's Theorem

In Theorem.1.2.2 we introduced a weaker condition which could be substituted for analyticity at a finite number of points without affecting the end result. We showed moreover, in Theorem 2.1.2, that Cauchy's theorem in a circular disk remains true under these weaker conditions. This was an essential point in our derivation of Cauchy's integral formula, for we were required to apply Cauchy's theorem to a function of the form

$$\frac{f(z) - f(a)}{z - a}.$$

Finally, it was pointed out that Cauchy's integral formula remains valid in the presence of a finite number of exceptional points, all satisfying the fundamental condition of Theorem 1.2.2, provided that none of them coincides with $a$. This remark is more important than it may seem on the surface. Indeed, Cauchy's formula provides us with a representation of $f(z)$ through an integral which in its dependence on $z$ has the same character at the exceptional points are such only by lack of information, and not by their intrinsic nature. Points with this character are called removable singularities. We shall prove the following precise theorem.

**Theorem 4.2.1.** *Suppose that $f(z)$ is analytic in the region $\Omega'$ obtained by omitting a point $a$ from a region $\Omega$. A necessary and suflcient condition that there exists an analytic function in $\Omega$ which coincides with $f(z)$ in $\Omega'$ is that*

$$\lim_{z\to a}(z-a)f(z) = 0.$$

*The extended function is uniquely determined.*

**Proof.** Since the extended function must be continuous at $a$, the necessary and the uniqueness are trivial.
To prove the sufficiency part:
Draw a circle $C$ about the point $a$ such that $C$ and its interior points are contained in $\Omega$. Then the Cauchy's integral formula is valid and we can write

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z}, \quad \forall\, z \neq a$$

in C. The integrand $\dfrac{f(\zeta)}{\zeta - z}$ is an analytic function throughout the inside of the circle $C$. Consequently, the function has the value which is $f(z)$ for $z \neq a$.

$$i.e., f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z}, \quad \forall\, z \neq a$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z}, \quad \forall\, z = a. \tag{4.2.1}$$

We apply this result to the function

$$F(z) = \frac{f(z) - f(a)}{z - a},$$

where $f(z)$ is not defined at $z = a$. $F(z)$ satisfied by the condition

$$\begin{aligned}
\lim_{z\to a}(z-a)F(z) &= \lim_{z\to a}[f(z)-f(a)]\\
\lim_{z\to a}(z-a)F(z) &= 0\\
\therefore \lim_{z\to a} F(z) &= \lim_{z\to a}\frac{f(z)-f(a)}{z-a}\\
i.e., \lim_{z\to a} F(z) &= f'(a).
\end{aligned}$$

Hence there exists an analytic function which is equal to $F(z)$ for $z \neq a$ and equal to $f'(a)$ for $z = a$. Hence the theorem is proved. ■

**Theorem 4.2.2.** ***Taylors's theorem*** *If $f(z)$ is analytic in a region $\Omega$, containing $a$, if it possible to write*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n, \quad (4.2.2)$$

*where $f_n(z)$ is analytic in $\Omega$.*

**Note.** This finite development must be well distinguished from the infinite Taylor's series which we will study later. It is however, the finite development (4.2.2) which is the most useful for the study of the local properties of $f(z)$.

**Proof.** Consider the function

$$F(z) = \frac{f(z)-f(a)}{z-a}$$

is defined and analytic in $\Omega$ except at $z = a$.

$$\therefore F(z) = f'(a) \quad for \quad z = a.$$

$F(z)$ can be denoted by the $F(z) = f_1(z)$.

$$i.e., \ f_1(z) = \frac{f(z)-f(a)}{z-a}, \quad z = a$$

and

$$f_1(z) = f'(a), \quad z = a.$$

where $f_1(z)$ is analytic in $\Omega$.

$$(z-a)f_1(z) = f(z) - f(a)$$

$$f(z) = f(a) + (z-a)f_1(z) \tag{4.2.3}$$

Repeating the above process we can obtain an analytic function

$$f_2(z) = \begin{cases} \dfrac{f_1(z) - f_1(a)}{z-a}, & z \neq a \\ f_1'(a), & z = a \end{cases}$$

$$(z-a)f_2(z) = f_1(z) - f_1(a)$$

$$f_1(z) = f_1(a) + (z-a)f_2(z) \tag{4.2.4}$$

Using (4.2.4) in (4.2.3), we get

$$f(z) = f(a) + f_1(a)(z-a) + (z-a)^2 f_2(z).$$

Continuing in this way, we get

$$f(z) = f(a) + f_1(a)(z-a) + (z-a)^2 f_2(z) + \cdots + f_{n-1}(a)(z-a)^{n-1} + f_n(z)(z-a)^n$$

Differentiating the above expression $n$ times and setting $z = a$ we get

$$f^n(a) = \frac{f^n(a)}{n!}$$

$$f_1(a) = \frac{f'(a)}{1!},\ f_2(a) = \frac{f''(a)}{2!},\ \cdots f_{n-1}(a) = \frac{f^{n-1}(a)}{(n-1)!}.$$

$$\therefore\ f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

Hence the proof of Taylor's theorem completed. ∎

**Example.** Express $f_n(z)$ as a simple line integral.

**Solution.** Let $C$ be a circle with centre at $a$ contained in the region $\omega$. Since $f_n(z)$ is analytic throughout $C$. We can use Cauchy's integral formula

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{\zeta - z}\,d\zeta \tag{4.2.5}$$

By Taylor's formula, we have

$$f(\zeta) = f(a) + \frac{f'(a)}{1!}(\zeta - a) + \frac{f''(a)}{2!}(\zeta - a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(\zeta - a)^{n-1} + f_n(\zeta)(\zeta - a)^n$$

$$(\zeta - a)^n f_n(\zeta) = f(\zeta) - f(a) - \frac{f'(a)}{1!}(\zeta - a) - \frac{f''(a)}{2!}(\zeta - a)^2 - \cdots - \frac{f^{n-1}(a)}{(n-1)!}(\zeta - a)^{n-1} \tag{4.2.6}$$

Using (4.2.6) in (4.2.5), we get

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)(\zeta - a)^n} - \frac{f(a)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)(\zeta - a)^n} - \cdots$$

Thus there is one main term containing $f(\zeta)$. The remaining terms are, except for constant factors of the form

$$F_\gamma(a) = \int_C \frac{d\zeta}{(\zeta - z)(\zeta - a)^\gamma}, \quad \gamma \geq 1.$$

Put $\gamma = 1$, we have

$$F_1(a) = \int_C \frac{d\zeta}{(\zeta - z)(\zeta - a)}$$

$$\begin{aligned}
\text{Consider} \frac{1}{(\zeta - z)(\zeta - a)} &= \frac{A}{\zeta - z} + \frac{B}{\zeta - a} \\
F_1(a) &= \frac{1}{z - a} \cdot \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta - a} \\
F_1(a) &= \frac{1}{z - a}\,[n(c, z) - n(c, a)]
\end{aligned}$$

Since $n(\gamma, a) = n(\gamma, b)$ where $a$ and $b$ are any two points inside $C$. Here $z$ and $a$ are points inside $C$.

$$\begin{aligned}
\therefore n(c, z) &= n(c, a) \\
n(c, z) - n(c, a) &= 0 \\
F_1(a) &= 0
\end{aligned}$$

$$\therefore F_1'(a) = 0, \;\; F_2''(a) = 0, \;\; F_2'''(a) = 0, \cdots$$

By Lemma 3.4.1, we have

$$F_n'(z) = n F_{n+1}(z)$$
$$\therefore F_{\gamma+1} = \frac{F_\gamma'(a)}{\gamma}.$$

Put $\gamma = 2$, we have $F_2(a) = 0$.

Similarly, $F_3(a) = 0 \cdots$ Hence

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - a)^n}.$$

## 4.3 Zeros and Poles

**Definition 4.3.1.** The zero of an analytic function $f(z)$ is a value of $z$ for which $f(z) = 0$.

**Example 4.3.1.** Let $f(z) = z \sin z$.
$\sin z = 0 \Rightarrow z = n\pi$, $n \in \mathbb{Z}$ are the zeros of $f(z)$.

**Theorem 4.3.1.** *If $f(z)$ is analytic in the region $\Omega$ and $f(a)$ together with all derivatives $f^\gamma(a)$ vanishes in $\Omega$ then $f(z) = 0$ in $\Omega$.*

**Proof.** Since $f(z)$ is analytic in the region $\Omega$. By Taylor's theorem, $f(z)$ can be expressed as

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

Since $f(a), f'(a), f''(a), \cdots$ vanishes. Therefore the above expression reduces to

$$f(z) = (z-a)^n f_n(z)$$

$$|f(z)| \leq |z-a|^n |f_n(z)| \tag{4.3.1}$$

Let $C$ be a circle centre at $a$ contained in $\Omega$.

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)}$$
$$|f_n(z)| \leq \frac{1}{2\pi}\int_C \frac{|f(\zeta)||d\zeta|}{|\zeta-a|^n|\zeta-z|}$$

$|f(\zeta)| \leq M$ and $|\zeta - a| = R$, where $R$ is the radius of the circle.

$$\begin{aligned}
|\zeta - z| &= |\zeta - a - (z - a)| \\
&\geq |\zeta - a| - |z - a| \\
|\zeta - a| &\geq R - |z - a| \\
\frac{1}{|\zeta - z|} &\leq \frac{1}{R - |z - a|} \\
\therefore |f_n(z)| &\leq \frac{M}{R^{n-1}(R - |z - a|)} \\
(4.3.1) \Rightarrow |f(z)| &\leq |z - a|^n \cdot \frac{M}{R^{n-1}(R - |z - a|)} \\
&\leq \left|\frac{z - a}{R}\right|^n \frac{MR}{R - |z - a|}
\end{aligned}$$

But $\frac{|z-a|^n}{R} \to 0$ for $n \to \infty$, since $|z - a| < R$. Hence $f(z) = 0$ inside of $C$. Let $E_1$ be the set on which $f(z)$ and all derivatives vanish and $E_2$ the set on which the function or one of the derivatives is different from zero. $E_1$ is open by the above reasoning and $E_2$ is open because the function and all derivatives are continuous. Since $\gamma$ is connected, one of the sets $E_1$ or $E_2$ must be empty.

$E_1 \quad \emptyset \Rightarrow E_2 = \emptyset$. Therefore $\Omega = E_1$, since $\Omega = E_1 \cup E_2$. Therefore $\Omega = E_1$. Hence $f(z) = 0$ in $\Omega$. ∎

**Theorem 4.3.2.** *The zero's of an analytic function which is not identically to zero are isolated points.*

**Proof.** Let $f(z)$ be an analytic function in the region $\Omega$ and $z = a$ be a zero of $f(z)$. Supposethat $f_n(a) \neq 0$. From Taylor's theorem, we have

$$f(z) = (z - a)^n f_n(z),$$

where $f_n(z)$ is analytic and $f_n(a) \neq 0$.

$\therefore z = a$ is a zero of order $n$ for $f(z)$.

If there exists any other zero of $f(z)$ it should arise from $f_n(z)$ only. But $f_n(z)$ is analytic at $a$ and $f_n(a)$. Therefore $f_n(z)$ is not equal to zero in the neighborhood of $a$. Hence $f(z)$ has no other zero in the neighborhood of $a$ except $a$. Thus the zero $f(z)$ are isolated. ∎

**Definition 4.3.2.** Let $f(z)$ be analytic in the region $\Omega$ and defined for $0 < |z - a| < \delta$. In otherwords, $f(z)$ should be analytic in the neighborhood of $a$ except at $a$ itself then $z = a$ is called isolated singularity.

**Example 4.3.2.** $f(z) = \frac{1}{z-a}$ has an isolated singularity at $z = a$.

**Definition 4.3.3. Pole.** If $\lim_{z\to a} f(z) = \infty$ then the point $z = a$ is called the pole of $f(z)$.

**Example 4.3.3.** $f(z) = \frac{1}{(z-a)^m}$

$\lim_{z\to a} f(z) = \infty$.

$\therefore z = a$ is a pole of $f(z)$.

**Note.** If $z = a$ is a pole of $f(z)$ then $z = a$ is zero of $\frac{1}{z}$. Also the function $g(z) = \frac{1}{f(z)}$ has a removable singularity at $z = a$. If $z = a$ is a zero of order $n$ for $g(z)$ then in the neighborhood of $a$, $g(z)$ can be expressed as

$$g(z) = (z-a)^h g_n(z)$$

where $g_n(z)$ is analytic and $g_n(a) \neq 0$. The number $h$ is the order of the pole.

$$f(z) = (z-a)^{-h} f_h(z)$$

where $f_n(z) = \frac{1}{g_h(z)}$ is analytic and different from zero in the neighborhood of $a$.

**Definition 4.3.4. Removable Singularity.**Let $f(z)$ be defined in a region $\Omega$ and if $\lim_{z\to a} f(z)$ exists finitely then $z = a$ is called removable singularity of $f(z)$. In otherwords if $\lim_{z\to a}(z-a)f(z) = 0$ then $z = a$ is called removable singularity.

**Example 4.3.4.** $\frac{\sin z}{z}$ has a removable singularity at $z = 0$.

**Example 4.3.5.** $f(z) = \frac{e^z - 1}{z}$ has removable singularity at $z = 0$.

**Definition 4.3.5. Meromorphic Function.** A single valued function $f(z)$ which is analytic except for poles in the region $\Omega$ is called meromorphic function. $\tan z$, $\cot z$, and any function $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomial function are meomorphic functions.

**Definition 4.3.6. Essential Singularity.** Let $f(z)$ be defined in the region $\Omega$. We consider the conditions (i) $\lim_{z\to a} |z-a|^{\alpha} . |f(z)| = 0$, $\alpha$ is real.

(ii) $\lim_{z\to a} |z-a|^{\beta} . |f(z)| = \infty$, $\beta$ is real.

If neither condition (i) nor (ii) holds for any real $\alpha$, $\beta$ is called essential singularity of $f(z)$.

**Note.** If $\lim_{z\to a} f(z)$ does not exist then $z = a$ is called an essential singularity.

**Example 4.3.6.**

$$\begin{aligned} f(z) &= e^{\frac{1}{z}} \\ f(z) &= 1 + \frac{\frac{1}{z}}{1!} + \frac{(\frac{1}{z})^2}{2!} + \cdots \\ f(z) &= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots \end{aligned}$$

$\therefore \lim_{z\to a} f(z)$ does not exist at $z = 0$.

$\therefore z = 0$ is an essential singularity.

**Theorem 4.3.3.** ***Weierstrass Theorem.*** *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

**Proof.** Suppose that the assertion is not true then there exists a complex value $A$ and a $\delta > 0$ such that

$$|f(z) - A| > s \quad for \quad |z - a| < \delta.$$

For any $\alpha < 0$, we have

$$\lim_{z\to a} |z - a|^{\alpha} \, |f(z) - A| = \infty.$$

$\therefore z = a$ cannot be an essential singularity of $f(z) - A$. Accordingly, there exist a $\beta$ with

$$\lim_{z\to a} |z - a|^{\beta} \, |f(z) - A| = 0,$$

and we are free to choose $\beta > 0$.

$$\begin{aligned} \lim_{z\to a} |z - a|^{\beta} \, |f(z)| &= \lim_{z\to a} |z - a|^{\beta} \, |f(z) - A + A| \\ &\leq \lim_{z\to a} |z - a|^{\beta} \, [\,|f(z) - A| + |A|\,] \\ \lim_{z\to a} |z - a|^{\beta} \, |f(z)| &= 0. \end{aligned}$$

This implies that $z = a$ cannot be an essential singularity of $f(z)$. This is a contradiction to the statement. Therefore our assumption is wrong.

$$\therefore |f(z) - A| < s, \ \forall \ |z - a| < \delta.$$

Hence $f(z)$ comes arbitrarily close to any complex value $A$ in every neighborhood of an essential singularity. ■

**Note.** From the above theorem we observed that the behavior of an function in the neighborhood of essential singularity is very complicated.

**Singular part of a function.** Let $z = a$ be a pole of order $h$ for a function $f(z)$ then the neighborhood of $z = a$ we can write

$$f(z) = \frac{f_h(z)}{(z-a)^h}$$

where $f_h(z)$ is at $a$ and $f_h(a) \neq 0$ By Taylor's theorem,

$$f_h(z) = B_h + (z-a)B_{h-1} + \cdots + (z-a)^{h-1}B_1 + (z-a)^h \varphi(z)$$

where $\varphi(z)$ is analytic in the neighborhood of $a$. Substituting this in $f(z)$, we get

$$\begin{aligned} f(z) &= \frac{1}{(z-a)^h}\left[B_h + (z-a)B_{h-1} + \cdots + (z-a)^{h-1}B_1 + (z-a)^h\varphi(z)\right] \\ f(z) &= \frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \cdots + \frac{B_1}{(z-a)} + \varphi(z) \end{aligned}$$

The part

$$\frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \cdots + \frac{B_1}{(z-a)}$$

is called the singular part or principal part of $f(z)$ in the neighborhood of the pole $z = a$ of order $h$.

## 4.4 Local Mapping

We begin with the proof of a general formula which enables us to determine the number of zeros of an analytic function.

**Theorem 4.4.1.** *Let $z_j$ be the zeros of a function $f(z)$ which is analytic in a disk $\Delta$ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed* *curve* $\gamma$ *in* $\Delta$ *which does not pass through a zero*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz, \tag{4.4.1}$$

*where the sum has only a finite number of terms* $\neq 0$.

**Proof.** Consider a function $f(z)$ which is analytic and not identically zero in an open disk $\Delta$. Let $z_1, z_2, \cdots, z_n$ be the finite number of zeros of $f(z)$ inside $\Delta$, each zero being counted according to its degree of multiplicity. Then we can write

$$f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)g(z)$$

where $g(z)$ is analytic and not equal to zero in $\Delta$.

$$\log(f(z)) = \log(z - z_1) + \log(z - z_2) + \cdots + \log(z - z_n) + \log(g(z))$$

Differentiating with respect to $z$, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

Multiply by $\frac{1}{2\pi i}$ and integrating each term along with $\gamma$, we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_1} + \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_2} + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_n} + \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$$

Since $g(z)$ is analytic and non-null in $\Delta$ and $g'(z)$ is also analytic and hence $\frac{g'(z)}{g(z)}$ must be analytic.

$$\therefore \text{ by Cauchy's theorem } \int_\gamma \frac{g'(z)}{g(z)}dz = 0.$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = n(\gamma, z_1) + n(\gamma, z_2) + \cdots + n(\gamma, z_n) + 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{j=1} n(\gamma', z_j)$$

Hence the theorem. ■

**Note.1** The function $w = f(z)$ maps $\gamma$ onto a closed curve $\Gamma$ in the $w-$ plane, and we find

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)} dz.$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-0} = \sum_{j=1}^{n} n(\gamma', z_j) \quad by \ (4.4.1)$$

$$n(\Gamma, 0) = \sum_{j=1}^{n} n(\gamma', z_j)$$

**Note.2** Let

$$n(\gamma, z_j) = \begin{cases} 1, & \text{if } z_j \textit{ lies inside } \gamma \\ 0, & \text{if } z_j \textit{ lies outside } \gamma \end{cases}$$

Therefore the above result reduces to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = N,$$

where $N$ is the total number of zeros of $f(z)$.

**Note.3** Let $z_j(a)$ denote the number of zeros of the function $f(z) - a$ inside $\gamma$, then replacing $f(z)$ by $f(z) - a$ in the above result, we get

$$\sum_{j=1} n(\gamma, z_j(a)) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} dz$$

If $\Gamma$ is the image of $\gamma$ under the mapping $w = f(z)$ then we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} dz = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-a}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} dz = n(\Gamma, a)$$

If $a$ and $b$ lie in the same region determined by $\Gamma$, we have

$$\sum_{j=1}^{n} n(\gamma, z_j(a)) = \sum_{j=1}^{n} n(\gamma, z_j(b))$$
$$n(\Gamma, a) = n(\Gamma, b)$$

Thus if $\gamma$ is a circle, it follows that $f(z)$ takes the value $a$ and $b$ equally many times of $\gamma$.

The following theorem on local correspondence is an immediate consequence of this result.

**Theorem 4.4.2.** *Suppose that $f(z)$ is analytic at $z_0$, $f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order $n$ at $z_0$. If $\epsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all $a$ with $|a - w_0| < \delta$ the equation $f(z) = a$ has exactly $n$ roots in the disk $|z - z_0| < \epsilon$.*

**Proof.** Let $\Gamma$ be the circle $|z - z_0| = \epsilon$ and $f(z)$ defined and analytic for $|z - z_0| \leq \epsilon$ and given that $f(z_0) = w_0$. $f(z) - w_0$ has a zero of order $n$ at $z_0$. The image of $\gamma$ under $f(z)$ be the closed curve $\Gamma$ in the $w-$ plane. Now

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}dz = Total\ number\ of\ zeros\ of\ f(z)$$
$$n(\gamma, a) = n(\gamma, w_0)$$
$$n(\gamma, a) = n$$

i.e., the function takes all the values in the neighborhood of the point $w_0$ equally many times inside $\gamma$.

$\therefore$ the equation $f(z) - w_0$ has exactly $n$ roots. Thus every value $a$ is taken $n$ times inside $\gamma$. ■

**Corollary 4.4.1.** *A non-constant analytic function maps open sets onto open sets.*

**Proof.** Let $z = z_0$ be a zero of order $n$ for the analytic function $f(z) - w_0$. Consider the disc $\gamma$ which is $|z - z_0| \leq \epsilon$. Let $G$ denote the image of the disk. Let $U$ be an open subset of the region $\Omega$ and let $z_0 \in U$ such that $f(z_0) = w_0$ then there exist an $\epsilon > 0$ as above and a region $G$ containing $w_0$ such that $|z - z_0| < \epsilon$ is a subset of open set $U$. We know that each $w \in G$ is assumed by $f(z)$ at $n$ points in $|z - z_0| < \epsilon$.

Since $G$ is an open set and $w_0 \in G$ there exists a $\delta > 0$ such that $|w - w_0| < \delta$ is a subset of $G$. But $G$ is contained in $f(U)$. Hence $|w - w_0| < \delta$ is completely contained in $f(U)$. We can state that image of every sufficiently small disk $|z - z_0| < \epsilon$ contains a neighborhood $|w - w_0| < \delta$. ■

**Corollary 4.4.2.** *If $f(z)$ is analytic at $z_0$ with $f'(z_0) \neq 0$, it maps a neighborhood of $z_0$ conformally and topologically onto a region.*

## 4.5 The Maximum Principle

Corollary 4.4.1 of Theorem 4.4.2 has a very important analytical consequence known as the maximum principle for analytic functions. Because of its simple and explicit formulation in the theory of functions. As a rule all proofs based on the maximum principle are very straightforward and preference is quite justly given to proofs of this kind.

**Theorem 4.5.1.** ***The Maximum Principle.*** *If $f(z)$ is analytic and non-constant in a region $\Omega$ then its absolute value $|f(z)|$ has no maximum in $\Omega$.*

**Proof.** Let $z_0 \in \Omega$. Suppose that $|f(z)|$ takes maximum value $|f(z_0)|$ in $\Omega$ corresponding to $\delta > 0$. There exists a neighborhood $|w - w_0| < \delta$ in which there exists a point $w$ so that $|w| > |w_0|$. Hence $|f(z_0)|$ is not the maximum value of $f(z)$ in $\Omega$. Therefore maximum value of $|f(z)|$ cannot occur in $\Omega$.

Let $\gamma$ be a circle $|z - z_0| \leq \delta$ in the region $\Omega$ and $f(z_0) = w_0$. We know that a non-constant analytic function maps open sets onto open sets.

Let $\Gamma$ be a image of $\gamma$ under the mapping $w = f(z)$. Suppose that $|f(z_0)|$ is the maximum value of $|f(z)|$ in the region bounded by $\Gamma$ then we can say that there is at least one point $w$ in the $w-$ plane such that $|w| > |w_0|$ . That is $f(z) > f(z_0)$. Therefore our assumption is wrong and hence $|f(z)|$ has no maximum in the region $\Omega$. ■

In a positive formulation essentially the same theorem can be stated in the form:

**Theorem 4.5.2.** ***Maximum Modulus Theorem.*** *If $f(z)$ defined and continuous on a closed bounded set $E$ and analytic on the interior of $E$, then the maximum of $|f(z)|$ on $E$ is assumed on the boundary of $E$.*

**Proof.** Consider the closed disk $\Gamma$ and $|\zeta - z_0| = r$ which is contained in a set $E$. Since $f(z)$ is analytic in the interior of $E$ it must be analytic in the closed disk.

$\therefore$ by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \quad on\ \gamma.$$

$$|\zeta - z_0| = r$$
$$\zeta = z_0 + re^{i\theta}, \quad 0 \le \theta \le 2\pi$$
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Hence the value of the function $f(z)$ at centre $z_0$ is equal to the arithmetic mean values of the function $f(z)$ on a circle $\gamma$.

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| d\theta$$

Suppose that $|f(z_0)|$ is a maximum. Then we have $|f(z_0 + re^{i\theta})| \le |f(z_0)|$ and if the strict inequality held for a single value $\theta$ it would hold, by continuity, on a whole arc. But then the mean value of $|f(z_0 + re^{i\theta})| < |f(z_0)|$.

$$\therefore |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta$$
$$|f(z_0)| < |f(z_0)|,$$

which is a contradiction.

$$\therefore |f(z_0 + re^{i\theta})| = |f(z_0)|$$

Hence $f(z)$ reduces to a constant function in the neighborhood of $f(z_0)$. Since $|f(z)|$ is constant equal to $|f(z_0)|$ on a circle $\gamma$ of radius $r$ and it is arbitrary. Thus the maximum value of $|f(z)|$ occurs on the boundary of $E$. ■

Consider now the case of a function $f(z)$ which is analytic in the open disk $|z| < R$ and continuous on the closed disk $|z| \le R$. If it is known that $|f(z)| \le M$ on $|z| = R$, then $|f(z)| \le M$ in the whole disk. The equality can hold only if $f(z)$ is a constant of absolute value $M$. Therefore, if it is known that $f(z)$ takes some value of modulus $< M$, it may be expected that a better estimate can be given. Theorems to this effect are very useful. The following particular result is known as the lemma of Schwarz:

**Theorem 4.5.3.** *If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $f'(0) \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ with a constant $c$ of absolute value $1$.*

**Proof.** We apply the maximum principle to the following function.

$$f_1(z) = \begin{cases} \dfrac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases}$$

$$\begin{aligned} |f_1(z)| &= \frac{|f(z)|}{|z|} \\ &\leq \frac{1}{|z|}, \textit{ since } |f(z)| \leq 1 \textit{ on the circle } |z| = r < 1 \\ |f_1(z)| &\leq \frac{1}{r} \end{aligned}$$

Letting $r \to 1$ we have $|f_1(z)| \leq 1$. (i) If $z \neq 0 \Rightarrow \left|\dfrac{f(z)}{z}\right| \leq 1 \Rightarrow |f(z)| \leq |z|$
If $z = 0 \Rightarrow |f'(0)| \leq 1$.
(ii) If $|f(z)| = |z|$

$$\begin{aligned} \left|\frac{f(z)}{z}\right| &= 1 \\ \frac{f(z)}{z} &= c, \textit{ where } |c| = 1 \\ f(z) &= cz, \textit{ when } |c| = 1. \end{aligned}$$

■

# BLOCK-II

# UNIT 5

# The General Form of Cauchy's Theorem

**Objectives**

Upon completion of this Unit, students will be able to

- identify the general statement of Cauchy's theorem.
- understand the concept of multiply connected region.

## 5.1 Introduction

In our preliminary treatment of Cauchy's theorem and the integral formula we considered only the case of a circular region. For the purpose of studying the local properties of analytic functions this was quite adequate, but from a more general point of view we cannot be satisfied with a result which is so obviously incomplete. The generalization can proceed in two directions. For one thing we can seek to characterize the regions in which Cauchy's theorem has universal validity. Secondly, we can consider an arbitrary region and look for the curves $\gamma$ for which the assertion of Cauchy's theorem is true.

## 5.2 Chains and Cycles

In the first place we must generalize the notion of line integral. Consider an arc $\gamma$. Divide the arc $\gamma$ into subdivision $\gamma_1, \gamma_2, \cdots, \gamma_n$ then we can write

$$\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz \tag{5.2.1}$$

Since the right hand member of (15.2.1) has a meaning for any finite collection, nothing prevents us from considering an arbitrary formal sum of finite collection $\gamma_1 + \gamma_2 + \cdot \cdot \cdot + \gamma_n$ which need not be an arc and we define the corresponding integral by means of equation (15.2.1). Such formal sums of arcs are called *chains.* Also the following operations do not change the identity of the chains.

1. Permutations of two arcs.
2. Subdivision of an arc.
3. Fusion of sub arcs to a single arc.
4. Reparametrization of an arc.
5. Cancellation of opposite arcs.

The sum of two chains is defined by way of positions. Therefore it is clear that the additive property (15.2.1) of line integrals remains valid for a arbitrary sum of chains. When identical chains are added, it is convenient to denote the sum as a multiple. With this notation every chain can be written in the form

$$\gamma = a_1\gamma_1 + a_2\gamma_2 + \cdot \cdot \cdot + a_n\gamma_n \tag{5.2.2}$$

where the $a_j$ are positive integers and the $\gamma_j$ are all different. For opposite arcs we can write $a(-\gamma) = a\gamma$.

**Zero Chain.** The *zero chain* is either an empty sum or sum with all coefficients equal to zero.

**Cycle.** A *chain* is a cycle if it can be represented as a sum of closed curves. That is a chain is a cycle if and only if any representation the initial and end points of the individual arcs are identical in pairs.

## 5.3 Simple Connectivity.

**Definition 5.3.1.** A region is simply connected if its complement with respect to the extended plane is connected.

**Example 5.3.1.** Any half - plane is simply connected and any open disk is simply connected.

**Theorem 5.3.1.** *A region $\Omega$ is simply connected if and only if $n(\gamma, a) = 0$ for all cycles $\gamma$ in $\Omega$ and all points $a$ which do not belong to $\Omega$.*

**Proof. Necessary part.**The necessary part of the theorem is almost trivial. Let $\gamma$ be any cycle in $\Omega$. If the complement of $\Omega$ is connected, it must be contained in one of the regions determined by $\gamma$, and since the point $\infty$ belongs to the complement this must be unbounded region. Hence $n(\gamma, a) = 0$ for all finite points in the complement.

**Sufficiency part.** Assume that the complement of $\Omega$ can be represented as the union of two disjoint closed sets $A$ and $B$. One of these two sets contains $\infty$ and the other one is bounded set. Let $A$ be the bounded set. The sets $A$ and $B$ have the shortest distance $\delta > 0$ cover the whole plane with a net of squares $Q$ of side less than $\frac{\delta}{\sqrt{2}}$.

We are free to choose the net so that a certain point $a \in A$ lies at the centre of a square. The boundary curve of $Q$ is denoted by $\partial Q$. We assume that the squares $Q$ are closed and the interior of $Q$ lies to the left of the directed line segments which make up $\partial Q$. Consider the cycle

$$\gamma = \sum_j \partial Q_j \tag{5.3.1}$$

where the sum ranges over all squares $Q_j$ in the net which have a point in common point in $A$. Because $a$ is contained in one and only one of these squares, it is evident that $n(\gamma, a) = 1$. Furthermore, it is clear that $\gamma$ does not meet $B$. But if the cancellations are carried out, it is equally clear that $\gamma$ does not meet $A$.

Indeed, any side which meets $A$ is common side of two squares included in the sum (5.3.1), and since the directions are opposite the side does not appear in the reduced expression of $\gamma$. Hence $\gamma$ is contained in $\Omega$ and therefore the theorem is proved. ■

## 5.4 Homology

The characterization of simple connectivity by Theorem 1.3.1 singles out a property that is common to all cycles in a simply connected region, but which a cycle in an arbitrary region or open set may or may not have. This property plays an important role in topology and therefore has a special name.

**Definition 5.4.1.** A cycle $\gamma$ in an open set $\Omega$ is said to be homologous to zero with respect to $\Omega$ if $n(\gamma, a) = 0,$ for all points $a$ in the complement of $\Omega.$

In symbols we write $\gamma \sim$ $(\mathrm{mod}\ \Omega)$ . When it is clear to what open set we are referring, $\Omega$ need not be mentioned. The notation $\gamma_{\sim}\gamma_2$ shall be equivalent to $\gamma_1 - \gamma_2 \sim 0.$ Homologies can be added and subtracted, and $\gamma \sim$ $(\mathrm{mod}\ \Omega)$ implies $\gamma \sim 0$ $(\mathrm{mod}\ \Omega)'$ for all $\Omega' \supset \Omega.$

## 5.5 The General Statement of Cauchy's Theorem

The definitive form of Cauchy's theorem is now very easy to state.

**Theorem 5.5.1.** *If* $f(z)$ *is analytic in* $\omega,$ *then*

$$\int_{\gamma} f(z)dz = 0 \tag{5.5.1}$$

*for every cycle* $\gamma$ *which is homologous to zero in* $\Omega.$

**Proof.** Assume that the region $\Omega$ is bounded, but otherwise arbitrary. Given $\delta > 0$ we cover the plane by a net of squares of side $\delta$ and we denote by $Q_j,\ j \in J,$ the closed squares in the net which are contained in $\Omega.$ Since $\Omega$ is bounded must get finite number of closed squares and therefore the set $J$ is finite, and if $\delta$ is sufficiently small it is also non - empty. The union of the squares $Q_j, j \in J,$ consists of closed regions whose oriented boundaries make up the cycle

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j$$

where $\Gamma_\delta$ is the sum of the oriented line segments which are the sides of exactly one $Q_j.$ We denote the interior of the union $\cup Q_j$ by $\Omega_\delta.$

Let $\gamma$ be a cycle which is homologous to zero in $\Omega$; we choose $\delta$ so small such that $\gamma$ is contained in $\Omega_\delta$. Consider a point $\zeta \in \Omega - \Omega_\delta$. it belongs to at least partial square $Q$ which is not a $Q_j$ a full square. There is a point $\zeta_0$ belonging to $Q$ which is not in $\omega$. It is possible to join $\zeta$ and $\zeta_0$ by a line segment which lies in $Q$ and therefore does not meet $\Omega_\delta$.

Since $\gamma$ is contained in $\Omega_\delta$ it follows that $n(\gamma, \zeta) = n(\gamma, \zeta_0) = 0$. Since $\zeta$ and $\zeta_0$ are points not belonging to $\Omega_\delta$ in particular $n(\gamma, \zeta) = 0$ for all points $\zeta$ on $\Gamma_\delta$. Suppose that $f(z)$ is analytic in $\Omega$. If $z$ lies in the interior of $Q_{j_0}$, then by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\zeta)d\zeta}{\zeta - z} = \begin{cases} f(z) & \text{if } j = j_0 \\ 0, & \text{if } j \neq j_0. \end{cases}$$

and hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{5.5.2}$$

Since both sides are continuous functions of $z$, this equation will hold for all $z \in \omega_\delta$. As a consequence we obtain

$$\int_\gamma f(z)dz = \int_\gamma \left(\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z}\right) dz. \tag{5.5.3}$$

The integrand of the iterated integral is a continuous function of both integration variables namely $\zeta$ and $z$ which are parameters of $\Gamma_\delta$ and $\gamma$. Therefore the order of integration can be reversed. In other words,

$$\int_\gamma f(z)dz = \int_\gamma \left(\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)d\zeta}{\zeta - z}\right) dz = \int_{\Gamma_\delta} \left(\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}\right) f(\zeta)d\zeta.$$

By the index number,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{\zeta - z} = -\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - \zeta} = -n(\gamma, \zeta) = 0.$$

$$\therefore (5.5.3) \Rightarrow \int_\gamma f(z)dz = 0.$$

Thus we proved the theorem for bounded region $\Omega$.

If $\Omega$ is unbounded, we replace it by its intersection $\Omega'$ with a disk $|z| < R$ which is large enough to contain $\gamma$. Any point $a$ in the complement of $\Omega'$ is either in the complement of $\Omega$ or lies outside the disk. In either case $n(\gamma, a) = 0$, so that $\gamma \sim 0 \pmod{\Omega'}$. The proof is applicable to $\Omega'$, and we conclude that the theorem is valid for arbitrary $\Omega$. ■

**Corollary 5.5.1.** *If $f(z)$ is analytic in a simply connected region $\Omega$, then*

$$\int_\gamma f(z)dz = 0$$

*holds for all cycles $\gamma$ in $\Omega$.*

**Corollary 5.5.2.** *If $f(z)$ is analytic and $\neq 0$ in a simply connected region $\Omega$, then it is possible to define single-valued analytic branches of* $\log f(z)$ *and* $\sqrt[n]{f(z)}$ *in $\Omega$.*

## 5.6 Locally Exact Differentiable

A differential $pdx + qdy$ is said to be *locally exact* in $\Omega$ if it is exact in some neighborhood of each point in $\Omega$ which is possible if and only if

$$\int_\gamma pdx + qdy = 0 \tag{5.6.1}$$

for all $\gamma = \partial R$ where $R$ is a rectangle contained in $\Omega$. This condition is fulfilled if $pdx + qdy = f(z)dz$ with $f$ analytic in $\Omega$, and by Theorem 1.5.1, (5.6.1) is true for any cycle $\gamma \sim 0 \pmod{\Omega}$.

**Theorem 5.6.1.** *If $pdx + qdy$ is locally exact in $\Omega$, then*

$$\int_\gamma pdx + qdy = 0$$

*for every cycle $\gamma \sim 0$ in $\Omega$.*

**Proof.** It is sufficient if we prove the theorem for polygon $\sigma$ with sides parallel to the axis. We construct $\sigma$ as an approximation of $\gamma$. Let the distance form $\gamma$ to the complement of $\Omega$ be $\rho$. If $\gamma$ is given by $z = z(t)$ where $z(t)$ is uniformly continuous on the closed interval $[a, b]$. We determine $\delta > 0$ so that $|z(t) - z'(t)| < \rho$ for $|t - t'| < \delta$ and divide $[a, b]$ into subintervals of length $< \delta$.

The corresponding subarcs $\gamma_i$ of $\gamma$ have the property that each is contained in a disk of radius $\rho$ which lies entirely in $\Omega$. The end points of $\gamma_i$ can be joined within that disk by a polygon $\sigma_i$

consisting of a horizontal and a vertical segment. Since the differential is exact in the disk,

$$\int_{\sigma_i} pdx + qdy = \int_{\gamma_i} pdx + qdy$$

and if $\sigma = \sum \sigma_i$, we obtain

$$\sum \int_{\sigma_i} pdx + qdy = \sum \int_{\gamma_i} pdx + qdy$$
$$\int_{\sigma} pdx + qdy = \int_{\gamma} pdx + qdy$$

We extend all segments that make up $\sigma$ to infinite lines. They divide the plane into some finite rectangles $R_i$ and some unbounded regions $R'_j$ which may be regarded as infinite rectangles.

Choose a points $a_i$ from the interior of each $R_i$ and form the cycle

$$\sigma_0 = \sum_i n(\sigma, a_i)\partial R_i \tag{5.6.2}$$

where the sum ranges over all finite rectangles and the coefficients $n(\sigma, a_i)$ are well determined, for no $\sigma_i$ lies on $\sigma$. We can also make use of points $a'_j$ chosen from the interior of each $R'_j$. It is clear that

$$n(\partial R^i, a^k) = \begin{cases} 1, & \text{for } k = i \\ 0 & \text{for } k \neq i \end{cases}$$

Similarly, $n(\partial R_i, a'_j) = 0$ for all $j$. With this in mind, it follows from (5.6.2) that

$$n(\sigma_0, a_i) = n(\sigma, a_i)$$

and

$$n(\sigma_0, a'_j) = 0.$$

It is also true that $n(\sigma, a'_j) = 0$, for the interior of $R'_j$ belongs to the unbounded region determined by $\sigma$. Hence we have proved that

$$n(\sigma - \sigma_0, a) = 0, \ for \ all \ a = a_i \ and \ a = a'_j.$$

Therefore from this property of $\sigma - \sigma_0$ we wish to conclude that $\sigma_0$ is identical with $\sigma$ up to the segments that cancelled against each other. Let $\sigma_i k$ be the common side of two adjacent rectangles $R_i, R_k$; we choose the orientation so that $R_i$ lies to the left of $\sigma_i k$.

Suppose that the reduced expression of $\sigma - \sigma_0$ contains the multiple $c\sigma_i k$. Then the cycle $\sigma - \sigma_0 - c\partial R_i$ does not contain $\sigma_i k$ and it follows that $a_i$ and $a_k$ must have the same index with respect to this cycle. On the other hand, these indices are $-c$ and $0$, respectively; we conclude that $c = 0$. The same reasoning applies if $\sigma_i j$ is the common side of a finite rectangle $R_i$ and an infinite rectangle $R'_j$. Thus every side of a finite rectangle occurs with coefficient zero in $\sigma - \sigma_0$, proving that

$$\sigma = \sum_i n(\sigma, a_i)\partial R_i \tag{5.6.3}$$

whose corresponding coefficient $n(\sigma, a_i)$ is different from zero are actually contained in $\Omega$. Suppose that a point $a$ in the closed rectangle $R_i$ were not in $\Omega$. Then $n(\sigma, a) = 0$ because $\sigma \sim 0 \pmod{\Omega}$ . On the other hand, the line segment between $a$ and $a_i$ does not intersect $\sigma$, and hence

$$n(\sigma, a_i) = n(\sigma, a) = 0.$$

Thus we conclude by the local exactness that the integral of $pdx + qdy$ over any $\partial R_i$ occurs in (5.6.3) is zero and hence

$$\int_\gamma (pdx + qdy) = \int_\sigma (pdx + qdy) = 0.$$

Hence the theorem is proved. ■

## 5.7 Multiply Connected Regions

A region which is not simply connected is called multiply connected. More precisely, $\Omega$ is said to have the finite connectivity $n$ if the complement of $\Omega$ has exactly $n$ components.

Similarly $\Omega$ is said to have infinite connectivity if the complement has infinitely many components.

A region is said to have connectivity $n$ if there exists $n$ holes in the Riemann sphere. In the case of finite connectivity, let $A_1, A_2, \cdots, A_n$ be the components of the complement $\Omega$ and assume that $\infty$ belongs to $A_n$. If $\gamma$ is an arbitrary cycle in $\Omega$, we can prove, just as in Theorem 1.5.1, that $n(\gamma, a)$ is constant when $a$ varies over any one of the components $A_i$ and that $n(\gamma, a) = 0$ in $A_n$. Moreover, duplicating the construction used in the proof of the Theorem 1.5.1, we can find cycles $\gamma_i$, $i = 1, 2, \cdots, n-1$, such that $n(\gamma, a) = 1$ for $a \in A_i$ and $n(\gamma_i, a) = 0$ for all other points outside $\Omega$.

For a given cycle $\gamma$ in $\Omega$, let $c_i$ be the constant value of $n(\gamma, a)$ or $a \in A_i$. We find that any point outside of $\Omega$ has the index zero with respect to the cycle $\gamma - c_1\gamma_1 - c_2\gamma_2 \cdots - c_{n-1}\gamma_{n-1}$. In other words,

$$\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \cdots + c_2\gamma_{n-1}.$$

Thus every cycle is homologous to a linear combination of the cycles $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$. This linear combination is uniquely determined, for if there are two linear combinations namely $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}$ and $c_1'\gamma_1 + c_2'\gamma_2 + \cdots + c_{n-1}'\gamma_{n-1}$ each linear combination is homologous to zero.

$$i.e., \gamma \sim c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1} \text{ and } c_1'\gamma_1 + c_2'\gamma_2 + \cdots + c_{n-1}'\gamma_{n-1}$$

$$\therefore c_1 = c_1', \quad c_2 = c_2', \cdots \quad c_{n-1} = c_{n-1}'$$

It is clear that the cycle $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}$ winds $c_1$ times around the points in $A_1$ and $c_2$ times around the points in $A_2$ and so on. Hence it cannot be homologous to zero unless all the $c_i$ vanish.

In view of these circumstances the cycles $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ are said to form a homology basis for the region $\Omega$. It is not the only homology basis, but by an elementary theorem in linear algebra we may conclude that every homology basis has the same number of elements. Also every region with a finite homology basis has a finite connectivity and the number of basis element is one less than the connectivity.

By Theorem we obtain, for any analytic function $f(z)$ in $\Omega$,

$$\int_\gamma f(z)dz = c_1\int_{\gamma_1} f dz + c_2\int_{\gamma_2} f dz + \cdots + c_{n-1}\int_{\gamma_{n-1}} f dz.$$

The numbers

$$P_i = \int_{\gamma_i} f dz$$

depend only on the function, and not on $\gamma$. They are called *modules of periodicity* of the differential $f dz$ or the *periods* of the indefinite integral. We have found that the integral of $f(z)$

over any cycle is a linear combination of the periods with integers as coefficients, and the integral along an arc from $z_0$ to $z$ is determined up to additive multiples of the periods. The vanishing of the periods is a necessary and sufficient condition for the existence of a single valued indefinite integral.

**Illustration.** Consider the extremely simple case of an annulus, defined by $r_1 < |z| < r_2$. The component of this annulus has two components namely $|z| \le r_1$ and $|z_2| \ge r_2$. We include the degenerate cases $r_1 = 0$ and $r_2 = \infty$. The annulus is doubly connected and a homology basis is formed by any circle $|z| = r$, $r_1 < r < r_2$. If this circle is denoted by $C$, any cycle in the annulus satisfies $\gamma \sim nC$ where $n = n(\gamma, 0)$. Therefore the integral of an analytic function over a cycle is a multiple of the single period

$$P = \int_C f\,dz$$

whose values is independent of the radius.

# BLOCK-II

# UNIT 6

# The Calculus of Residues

| **Objectives** |
|---|
| After completion of this Unit, students will be able to |
| • prove argument principle. |
| • identify Rouche's theorem. |
| • solve definite integral by the method of residues. |

## 6.1 Introduction

The results of the preceding section have shown that the determination of line integrals of analytic functions over closed curves can be reduced to the determination of periods. Under certain circumstances it turns out that the periods can be found without or with very little computation. We are thus in possession of a method which in many cases permits us to evaluate integrals without resorting to explicit calculation. This is of great value for practical purposes as well as for the further development of the theory.

In order to make this method more systematic a simple formalism, known as the calculus of residues, was introduced by Cauchy, the founder of complex integration theory.

**Definition 6.1.1.** The residue of $f(z)$ at an isolated singularity $a$ is the unique complex number $R$ which makes $f(z) - \frac{R}{z-a}$ the derivative of a single valued analytic function in an annulus $0 < |z-a| < \delta$.

**Theorem 6.1.1.** *Let $f(z)$ be analytic except for isolated singularities $a_j$ in a region $\Omega$. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j)Res_{z=a_j} f(z) \tag{6.1.1}$$

**Proof.** Let $f(z)$ has finite number of singularities at $a_j,\ \ j = 1, 2, \cdots, n$ in the region $\Omega$. Let $P_j\left(\frac{1}{z-a_j}\right)$ be the singular point of $f(z)$ with respect to the isolated singularity $a_j$.

$$\therefore \sum_{j=1}^{n} P_j\left(\frac{1}{z-a_j}\right)$$

is not analytic at $z = a_j,\ j = 1, 2, \cdots, n$. The function $f(z)$ is analytic in the region $\Omega'$ obtained excluding the points $a_j$ from $\Omega$.

$$i.e., \Omega' = \Omega - \{a_1, a_2, \cdots, a_n\}.$$

By defining

$$g(z) = f(z) - \sum_{j=1}^{n} P_j\left(\frac{1}{z-a_j}\right).$$

The function $g(z)$ is analytic in $\Omega$, where $\gamma \sim 0 \pmod{\Omega}$, we have

$$\int_\gamma \left( f(z) - \sum_{j=1}^{n} P_j\left(\frac{1}{z-a_j}\right)\right) dz = 0$$

$$i.e., \int_\gamma f(z)dz = \sum_{j=1}^{n} \int_\gamma P_j\left(\frac{1}{z-a_j}\right).$$

But $P_j\left(\frac{1}{z-a_j}\right)$ is a polynomial in $\frac{1}{z-a_j}$ and therefore

$$\int_\gamma \frac{dz}{(z-a_j)^m} = \begin{cases} 0, & \text{if } m > 1 \\ 2\pi i\ \ n(\gamma, a_j), & \text{if } m = 1. \end{cases}$$

Hence

$$\begin{aligned} \int_\gamma f(z)dz &= \sum_{j=1}^{n} 2\pi i\ \ n(\gamma, a_j) \\ &= 2\pi i \left(\sum_{j=1}^{n} n(\gamma, a_j)(Res\ f(z)\ at\ z = a_j)\right) \\ i.e., \frac{1}{2\pi i} &= \sum_{j=1}^{n} n(\gamma, a_j)(Res\ f(z)\ at\ \ z = a_j). \end{aligned}$$

Hence the proof is complete. ■

**Definition 6.1.2.** A cycle $\gamma$ is said to bound the region $\Omega$ if and only if $n(\gamma, a)$ is defined and equal to $1$ for all points $a \in \Omega$ and either undefined or equal to zero for all points $a$ not in $\Omega$.

## 6.2 The Argument Principle

Cauchy's integral formula can be considered as a special case of the residue theorem. Indeed, the function $\frac{f(z)}{z-a}$ has a simple pole at $z = a$ with the residue $f(a)$, and when we apply (9.2.1), the integral formula results.

Another application of the residue theorem occurred in the proof of Theorem 4.4.1 which served to determine the number of zeros of an analytic function. For a zero of order of $h$, we can write $f(z) = (z-a)^h f_h(z)$, with $f_h(a) \neq 0$, and obtain $f'(z) = h(z-a)^{h-1} f_h(z) + (z-a)^h f_h'(z)$. Consequently $\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \frac{f_h'(z)}{f_h(z)}$, and we see that $\frac{f'}{f}$ has a simple pole with the residue $h$. In the formula

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz,$$

this residue is accounted for by a corresponding repetition of terms.

Now we can generalize Theorem 4.4.1 to the case of meromorphic function. If $f$ has a pole of order $h$, we find by the same calculation as above, with $-h$ replacing $h$, that $\frac{f'}{f}$ has the residue $-h$.

**Theorem 6.2.1.** *If $f(z)$ is meromorphic in $\Omega$ with the zeros $a_j$ and the poles $b_k$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k) \tag{6.2.1}$$

*for every cycle $\gamma$ which is homologous to zero in $\Omega$ and does not pass through any of the zeros or poles.*

**Proof.** Let $z = a_j$ be the zeros of order $h$ for the meromorphic function of $f(z)$, then $f(z)$ can be expressed as

$$f(z) = (z - a_j)^h f_h(z),$$

where $f(z)$ is analytic in $\Omega$ and $f_h(a_j) \neq 0$.

$$\log f(z) = h \log(z - a_j) = \log f_h(z)$$

Differentiating with respect to $z$, we get

$$\frac{f'(z)}{f(z)} = h\frac{1}{z - a_j} + \frac{f_h'(z)}{f_h(z)}$$

$z = a_j$ is a simple pole of $\dfrac{f'(z)}{f(z)}$ with residue $h$. Since $\dfrac{f_h'(z)}{f_h(z)}$ is analytic and non-null at $z = a_j$,

$$\int_\gamma \frac{f_h'(z)}{f_h(z)} dz = 0.$$

Let us assume that $z = b_k$ be a poles of order $m$ for $f(z)$ then we can write $f(z)$ as

$$f(z) = \frac{g_m(z)}{(z - b_k)^m},$$

where $g_m(z)$ is analytic and $g_m(b_k) \neq 0$.

$$\log f(z) = \log(g_m(z)) - m \log(z - b_k)$$

Differentiating with respect to $z$, we get

$$\frac{f'(z)}{f(z)} = \frac{g'_m(z)}{g_m(z)} - m\frac{1}{z - b_k}$$

But

$$\int_\gamma \frac{g'_m(z)}{g_m(z)} dz = 0$$

as $\dfrac{g'_m(z)}{g_m(z)}$ is analytic.

From the above equation, we see that $z = b_j$ is a simple pole of $\dfrac{f'(z)}{f(z)}$ with residue $-m$. Therefore by the residue theorem, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

where each $a_j$ and $b_k$ are counted according to its degree of multiplicity. ■

**Corollary 6.2.1.** ***Rouche's Theorem.****Let $\gamma$ be homologous to zero in $\Omega$ and such that $n(\gamma, z)$ is either $0$ or $1$ for any point $z$ not on $\gamma$. Suppose that $f(z)$ and $g(z)$ are analytic in $\Omega$ and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on $\gamma$. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by $\gamma$.*

**Proof.** Let us prove $f(z)$ and $g(z)$ are zero free on $\gamma$. Suppose that $g(a) = 0$, $a$ belongs to the boundary of $\gamma$.

$$\therefore\ f(a) = f(a) - g(a) + g(a)$$

$$|f(a)| \geq |f(a) - g(a)| - |g(a)| \quad or$$
$$|f(a)| \geq |g(a)| - |f(a) - g(a)|$$
$$|f(a)| \geq |g(a)| - |f(a)|$$
$$2|f(a)| \geq |g(a)|$$

Since $g(a) = 0 \Rightarrow |f(a)| > 0$.

$$|f(a) - g(a)| < |f(a)|$$
$$|f(a)| < |f(a)|$$

which is a contradiction. Also, $|f(z) - g(z)| < |f(z)|$ on $\gamma$.

$$\therefore \left|\frac{f(z) - g(z)}{f(z)}\right| < 1$$
$$\left|1 - \frac{g(z)}{f(z)}\right| < 1$$
$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

Put $F(z) = \frac{g(z)}{f(z)}$, $f(z) = 0$ on $\gamma$.

$$\therefore\ |F(z) - 1| < 1.$$

Let $w = F(z)$. As $z$ moves on $\gamma$ $w$ moves $\Gamma$ such that $|w - 1| < 1$. That is $w$ moves on the unit circle $\Gamma$ with 1 as centre and radius is $1$ unit. $w$ moves on $\Gamma$ which lies only inside the unit circle $|w - 1| = 1$ with centre $1$ and radius $a$. Therefore $n(\Gamma, 0) = 0$. By applying Theorem

2.2.1 to $F(z)$, we get

$$\frac{1}{2\pi i}\int_{\gamma}\frac{F'(z)}{F(z)}dz = 0$$

$$\Longrightarrow \frac{1}{2\pi i}\int_{\gamma}\left(\frac{g'}{g}-\frac{f'}{f}\right)dz = 0$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f}dz = \frac{1}{2\pi i}\int_{\gamma}\frac{g'}{g}dz$$

Number of zeros of $f(z)$ inside $\gamma$ = Number of zeros of $g(z)$ inside $\gamma$. This completes the proof. ■

## 6.3 Evaluation of Definite Integrals

The calculus of residues provides a very efficient tool for the evaluation of definite integrals. It is particularly important when it is impossible to find the indefinite integral explicitly, but even if the ordinary methods of calculus can be applied the use of residues is frequently a laborsaving device. The fact that the calculus of residues yields complex rather than real integrals is no disadvantage, for clearly the evaluation of a complex integral is equivalent to the evaluation of two definite integrals.

**Methods for the evaluation of residues**

**1. (a)** If $z = a$ is a simple pole of $f(z)$ then the residue of $f(z)$ at $z = a$ is given by

$$Res.\, f(z)|_{z=a} = \lim_{z\to a}(z-a)f(z).$$

**1. (b)** If $z = a$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$ then the residue of $f(z)$ at $z = a$ is given by

$$Res.\, f(z)|_{z=a} = \lim_{z\to a}\frac{P(z)}{Q'(z)}.$$

**2.** If $z = a$ is a pole of order $m$ of $f(z)$ then the residue of $f(z)$ at $z = a$ is given by

$$Res.\, f(z)|_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).$$

**Type . I** All integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta \tag{6.3.1}$$

where the integrand is a rational function of $\cos\theta$ and $\sin\theta$ can be easily evaluated by means of residues.

Put $z = e^{i\theta}$. $dz = izd\theta$, $d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z} \; and \; \sin\theta = \frac{z^2-1}{2iz}.$$

Substituting these in (16.4.1), the integral reduces to the line integral is of the form

$$\int_\gamma f(z)dz$$

where $\gamma$ is the unit circle. By residue theorem, we have

$$\int_\gamma f(z)dz = 2\pi i \sum_i R_i,$$

where $\sum_i R_i$ denotes the sum of all residues at the poles of $f(z)$ that lies within the unit circle $\gamma$.

**Example 6.3.1.** Evaluate $\int_0^\pi \frac{d\theta}{a+\cos\theta}$, $a > 1$

**Solution.**

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta},$$

since

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x). \end{cases}$$

Put $z = e^{i\theta}$ then $dz = ie^{i\theta}d\theta$, $\frac{dz}{iz} = d\theta$, $|z| = 1$. As $\theta$ varies from 0 to $2\pi$, $z$ varies through the circle $|z| = 1$.

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

$$= \frac{1}{2}\int_{|z|=1} \frac{\frac{dz}{iz}}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= \frac{1}{i}\int_{|z|=1} \frac{dz}{z^2+2az+1}$$

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{i}\int_{|z|=1} f(z)dz$$

where $f(z) = \dfrac{1}{z^2+2az+1}$. The poles of $f(z)$ is given by $z^2 + 2az + 1 = 0$.

$$i.e., z = -a \pm \sqrt{a^2-1}.$$

Let $\alpha = -a + \sqrt{a^2-1}$ and $\beta = -a - \sqrt{a^2-1}$. Since $a > 1$, the pole $z = \alpha$ lies inside $\gamma$ and the pole $z = \beta$ lies outside $\gamma$.

$\therefore$ residue of $f(z)$ at $z = \alpha$ is

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{z^2+2az+1}$$

$$= \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\alpha-\beta}$$

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \frac{1}{2\sqrt{a^2-1}}$$

Therefore by Cauchy's residue theorem, we have

$$\int_\gamma f(z)dz = 2\pi i \sum R_i = 2\pi i.\frac{1}{2\sqrt{a^2-1}} = \frac{\pi i}{\sqrt{a^2-1}}$$

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{i}\cdot\frac{\pi i}{\sqrt{a^2-1}}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

**Example 6.3.2.** Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x}$, $|a| > 1$.

**Solution.**

$$\begin{aligned} Let\ I &= \int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} \\ &= \int_0^{\frac{\pi}{2}} \frac{2dx}{2a+1-\cos 2x} \end{aligned}$$

Put $t = 2x \Rightarrow dt = 2dx$
when $x = 0$, $t = 0$
when $x = \frac{\pi}{2}$, $t = \pi$.

$$\begin{aligned} I &= \int_0^{\pi} \frac{dt}{1+2a-\cos t} \\ &= \frac{1}{2}\int_0^{2\pi} \frac{dt}{1+2a-\cos t} \end{aligned}$$

Put $z = e^{it} \Rightarrow \frac{dz}{iz} = dt$
$\cos t = \frac{z^2+1}{2z}$
As $t$ varies from $0$ to $2\pi$, $z$ varies over the circle $\gamma : |z| = 1$.

$$I = \frac{1}{2}\int_\gamma \frac{\frac{dz}{iz}}{1+2a-\frac{1}{2}(z+\frac{1}{z})}$$

$$I = -i\frac{1}{}\int_\gamma \frac{dz}{z^2-(2+4a)z+1}$$

$$I = -\frac{1}{i}\int_\gamma f(z)dz, \tag{6.3.2}$$

where $f(z) = \frac{1}{z^2-(2+4a)z+1}$. The poles of $f(z)$ is given by

$$\begin{aligned} z^2 - (2+4a)z + 1 &= 0 \\ z &= 1+2a \pm 2\sqrt{a+a^2} \end{aligned}$$

Let $\alpha = 1 + 2a + 2\sqrt{a + a^2}$ and $\beta = 1 + 2a - 2\sqrt{a + a^2}$. The pole $z = \beta$ lies inside $\gamma$.

$$\begin{aligned}
Res.\, f(z)|_{z=\beta} &= \lim_{z\to\beta}(z-\beta)\frac{1}{(z-\alpha)(z-\beta)} \\
&= \frac{1}{\beta-\alpha} \\
Res.\, f(z)|_{z=\beta} &= -\frac{1}{4\sqrt{a+a^2}}
\end{aligned}$$

Therefore by Cauchy's residue theorem, we have

$$\int_\gamma f(z)dz = 2\pi i \cdot \sum R_i \cdot = 2\pi i \cdot -\frac{1}{4\sqrt{a+a^2}} = -\frac{\pi i}{2\sqrt{a+a^2}}.$$

$$\begin{aligned}
\therefore (16.4.2) \Rightarrow I &= -\frac{1}{i} \cdot -\frac{\pi i}{2\sqrt{a+a^2}} \\
\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} &= \frac{\pi}{2\sqrt{a+a^2}}.
\end{aligned}$$

**Type. 2** An integral of the form $\int_{-\infty}^{\infty} R(x)dx$, where $R(x)$ is the rational function in $x$. This integral converges if and only if the degree of the denominator of $R(x)$ is at least two units higher than the degree of the numerator, and if no poles lies on the real axis.

To evaluate this integral, we evaluate $\int_C R(z)\, dz$ where $C$ is the closed curve consisting of a line segment $(-R, R)$ and the semicircle from $R$ to $-R$ in the upper half plane. If $R$ is large enough this curves encloses all poles in the upper half plane, and the corresponding integral is equal to $2\pi i$ times the sum of the residues in the upper half plane. As $R \to \infty$ obvious estimate show that the integral over the semicircle tends to $0$, and we obtain

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{y>0} Res.\, R(z).$$

**Example 6.3.3.** Evaluate $\int_0^\infty \frac{x^2dx}{(x^2+a^2)^3}$ where $a$ is real , by the method of residues.

**Solution.** Consider
$$\int_C \frac{z^2}{(z^2+a^2)^3}dz = \int_C f(z)dz,$$
where $f(z) = \frac{z^2}{(z^2+a^2)^3}$ and $C$ is the upper half of the semicircle $|z| = R$ along the diameter on the real axis from $-R$ to $R$, where $R$ is sufficiently large. The poles of $f(z)$ are given by

$$(z^2+a^2)^3 = 0 \Rightarrow z = \pm ia \ \ thrice$$

The pole $z = ia$ of order 3 lies inside $C$ and the pole $z = -ia$ lies outside $C$.

To find the residue of $f(z)$ at $z = ai$

$$\begin{aligned}
Res.\ f(z)|_{z=ai} &= \frac{1}{2!}\lim_{z\to ai}\frac{d^2}{dz^2}\cdot(z-ai)^3 f(z)\cdot \\
&= \frac{1}{2}\lim_{z\to ai}\frac{d^2}{dz^2}\cdot\frac{z^2}{(z+ai)^3}\cdot \\
Res.\ f(z)|_{z=ai} &= \frac{1}{16ia^3}
\end{aligned}$$

Therefore by Cauchy's residue theorem, we have

$$\begin{aligned}
\int_C f(z)dz &= 2\pi i\cdot\sum R_i\cdot \\
&= 2\pi i\cdot\frac{1}{16ia^3}\cdot \\
\int_C f(z)dz &= \frac{\pi}{8a^3}
\end{aligned}$$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{8a^3} \tag{6.3.3}$$

$$f(z) = \frac{z^2}{(z^2+a^2)^3} = \frac{z^2}{z^6(1+\frac{a^2}{z^2})^3} \to 0 \ \ as \ \ |z| = R \to \infty$$

$$\therefore \int_{C^R} f(z)dz = 0$$

Letting $R \to \infty$ in (16.4.3), we get

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{8a^3}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}dx = \frac{\pi}{8a^3}$$

$$\therefore \int_0^{\infty} \frac{x^2}{(x^2+a^2)^3}dx = \frac{\pi}{16a^3}.$$

**Example 6.3.4.** Evaluate $\frac{x^2 - x + 2}{x^4 + 10x^2 + 9}dx$ by the method of residues.

**Solution.** Consider

$$\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}dz = \int_C f(z)dz,$$

where $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$ and $C$ is the upper half of the semicircle $|z| = R$ along the diameter on the real axis from $-R$ to $R$, where $R$ is sufficiently large.

The poles of $f(z)$ are given by

$$z^4 + 10z^2 + 9 = 0 \Rightarrow z = \pm i, \ z = \pm 3i.$$

The simple poles $z = i$ and $z = 3i$ are lies inside $C$ and the poles $z = -i$ and $z = -3i$ lies outside $C$.

$$\begin{aligned} R_1 &= Res. f(z)|_{z=i} \\ &= \lim_{z\to i} \frac{P(z)}{Q'(z)} \\ &= \lim_{z\to i} \frac{z^2 - z + 2}{4z^3 + 20z} \\ R_1 &= \frac{1-i}{16i} \end{aligned}$$

$$
\begin{aligned}
R_2 &= Res.\, f(z)|_{z=3i} \\
&= \lim_{z\to 3i} \frac{P(z)}{Q'(z)} \\
&= \lim_{z\to 3i} \frac{z^2 - z + 2}{4z^3 + 20z} \\
R_2 &= \frac{7+3i}{48i}
\end{aligned}
$$

Therefore by Cauchy's residue theorem, we have

$$
\begin{aligned}
\int_C f(z)dz &= 2\pi i \left(\sum R_i\right) \\
&= 2\pi i \left(R_1 + R_2\right) \\
&= 2\pi i \left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right) \\
\int_C f(z)dz &= \frac{5\pi}{12}
\end{aligned}
$$

$$
\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{5\pi}{12} \tag{6.3.4}
$$

Now

$$
\left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{|z^2 - z + 2|}{|z^2+1|\,|z^2+9|}|dz|
$$

On $C_R$, $|z| = R$

$$
\begin{aligned}
|z^2+1| &\geq |z|^2 - 1 \\
|z^2+1| &\geq R^2 - 1 \\
\frac{1}{|z^2+1|} &\leq \frac{1}{R^2-1}
\end{aligned}
$$

and

$$
\begin{aligned}
|z^2+9| &\geq |z|^2 - 9 \\
|z^2+9| &\geq R^2 - 9 \\
\frac{1}{|z^2+9|} &\leq \frac{1}{R^2-9}.
\end{aligned}
$$

$$\left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{R^2+R+2}{(R^2-1)(R^2-9)}|dz|$$

$$\leq \frac{R^2+R+2}{(R^2-1)(R^2-9)} \int_{C_R} |dz|$$

$$\leq \frac{R^2+R+2}{(R^2-1)(R^2-9)} . \pi R$$

$$\left|\int_{C_R} f(z)dz\right| \to 0 \ as \ R \to \infty$$

$$\therefore \int_{C_R} f(z)dz = 0 \ as \ R \to \infty.$$

Letting $R \to \infty$ in (6.3.4), we get

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}.$$

**Type. 3** Integrals of the form

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx,$$

where $R(x)$ is the rational function in $x$. We can use Type.2 method to evaluate this integral.

The real and imaginary parts determine the important integrals

$$\int_{-\infty}^{\infty} R(x)\cos x\, dx, \qquad \int_{-\infty}^{\infty} R(x)\sin x\, dx.$$

Since $|e^{iz}| = e^{-y}$ is bounded in the upper half plane, we can again conclude that the integral over the semicircle tends to zero, provided that the rational function $R(z)$ has a zero of at least order two at infinity. We obtain

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum_{y>0} Res.\, R(z)e^{iz}.$$

**Example 6.3.5.** Evaluate $\int_0^{\infty} \frac{\cos x}{x^2+a^2}dx$, $a$ is real.

**Solution.** Consider

$$\int_C \frac{e^{iz}}{z^2+a^2}dz = \int_C f(z)dz,$$

where $f(z) = \frac{e^{iz}}{z^2+a^2}$ and $C$ is the upper half of the semicircle $|z| = R$ along the diameter on the real axis from $-R$ to $R$, where $R$ is sufficiently large.

The poles of $f(z)$ are given by

$$z^2 + a^2 = 0 \Rightarrow z = \pm ia$$

The simple pole $z = ia$ lies inside $C$ and the pole $z = -ia$ lies outside $C$.

$$\begin{aligned} Res.f(z)|_{z=ia} &= \lim_{z\to ia}(z - ia)f(z) \\ &= \lim_{z\to ia}(z - ia)\frac{e^{iz}}{z^2+a^2} \end{aligned}$$

$$Res.f(z)|_{z=ia} = \frac{e^{-a}}{2ai}$$

Therefore by residue theorem, we have

$$\int_C f(z)dz = 2\pi i \left(\sum R_i\right) = 2\pi i\left(\frac{e^{-a}}{2ai}\right) = \frac{\pi e^{-a}}{a}.$$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi e^{-a}}{a} \tag{6.3.5}$$

Now,

$$\left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{|e^{iz}|}{|z^2 + a^2|}|dz|$$

$$|e^{iz}| = e^{-y} \leq 1$$

$$\begin{aligned}
|z^2+a^2| &\geq |z|^2-|a|^2 \\
&\geq R^2-a^2 \\
\frac{1}{|z^2+a^2|} &\leq \frac{1}{R^2-|a^2|} \\
\left|\int_{C_R} f(z)dz\right| &\leq \int_{C_R}\frac{|dz|}{|R^2-a^2|} \\
&\leq \frac{1}{R^2-|a^2|}\int_{C_R}|dz| \\
&\leq \frac{\pi R}{R^2-a^2}.
\end{aligned}$$

$$\therefore \left|\int_{C_R} f(z)dz\right| \to 0 \text{ as } R\to\infty.$$

Letting $R\to\infty$ in (6.3.5), we get

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+a^2}dx &= \frac{\pi e^{-a}}{a} \\
\int_{-\infty}^{\infty}\frac{\cos x+i\sin x}{x^2+a^2}dx &= \frac{\pi e^{-a}}{a}
\end{aligned}$$

Equating real parts on both sides, we get

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx &= \frac{\pi e^{-a}}{a} \\
\therefore \int_{0}^{\infty}\frac{\cos x}{x^2+a^2}dx &= \frac{\pi e^{-a}}{2a}.
\end{aligned}$$

**Example 6.3.6.** Evaluate $\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx$, where $a$ is real.

**Solution.** Consider

$$\int_C \frac{ze^{iz}}{z^2+a^2}dz = \int_C f(z)dz,$$

where $f(z)=\frac{ze^{iz}}{z^2+a^2}$ and $C$ is the upper half of the semicircle $|z|=R$ along the diameter on the real axis from $-R$ to $R$, where $R$ is sufficiently large.

The poles of $f(z)$ are given by $z^2+a^2=0 \Rightarrow z=\pm ia$.

The simple pole $z=ia$ lies inside $C$ and the pole $z=-ia$ lies outside $C$.

$$Res.f(z)|_{z=ia} = \lim_{z\to ia}(z-ia)\frac{ze^{iz}}{z^2+a^2}$$

$$Res.f(z)|_{z=ia} = \frac{e^{-a}}{2}$$

Therefore by Cauchy's residue theorem, we have

$$\int_C f(z)dz = 2\pi i \sum R_i = \pi i e^{-a}.$$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \pi i e^{-a} \tag{6.3.6}$$

$$\left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{|z|\,|e^{iz}|}{|z^2+a^2|}|dz|$$

On $C_R$, $|z| = R$, $z = Re^{i\theta} \Rightarrow dz = Re^{i\theta} i d\theta \Rightarrow |dz| = Rd\theta$, $0 \leq \theta \leq \pi$.

$$\begin{aligned}
\left|\int_{C_R} f(z)dz\right| &\leq \int_{C_R} \frac{|z|\,|e^{i(R\cos\theta+iR\sin\theta)}|}{|R^2-|a^2||}Rd\theta \\
&\leq \frac{R^2}{R^2-|a^2|}\int_0^{\pi} |e^{iR\cos\theta}|\,e^{-R\sin\theta}d\theta \\
&\leq \frac{2R^2}{R^2-|a^2|}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}d\theta \\
&\leq \frac{\pi R}{R^2-|a^2|}[1-e^{-R}] \\
&\leq \frac{\pi R}{R^2\left(1-\frac{|a^2|}{R^2}\right)}(1-e^{-R})
\end{aligned}$$

$$\left|\int_{C_R} f(z)dz\right| \to 0 \text{ as } R \to \infty.$$

Letting $R \to \infty$ in (6.3.6), we get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+a^2}dx = \pi i e^{-a}$$

Equating the imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \pi e^{-a}$$

$$\therefore \int_{0}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-a}.$$

**Example 6.3.7.** Prove that $\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$

**Solution.** Consider

$$\int_C \frac{e^{iz}}{z}dz = \int_C f(z)dz$$

where $f(z) = \frac{e^{iz}}{z}dz$ and $C$ is the upper half of the semicircle $|z| = R$ along the diameter on the real axis from $-R$ to $R$, where $R$ is sufficiently large, but with an indent i.e., a small semicircle at the origin, which is introduced to avoid the singularity $z = 0$, which lies on the real axis.
The poles of $f(z)$ is given by $z = 0$.
The closed curve does not include the singularity of $f(z) = \frac{e^{iz}}{z}$.
$\therefore$ by Cauchy's residue theorem, we get

$$\int_C f(z)dz = 0$$

$$i.e., \int_{-R}^{-\rho} f(x)dx + \int_{BDE} f(z)dz + \int_{\rho}^{R} f(x)dx + \int_{RGA} f(z)dz = 0 \qquad (6.3.7)$$

The equation of $BDE$ is $|z| = \rho$.
$\therefore z = \rho e^{i\theta}$ and $dz = \rho e^{i\theta} i d\theta$.
As $BDE$ is described in the clockwise sense, $\theta$ varies from $\pi$ to $0$. Thus

$$\int_{BDE} \frac{e^{iz}}{z}dz = \int_{\pi}^{0} \frac{e^{ire^{i\theta}}}{re^{i\theta}} re^{i\theta} i d\theta$$

$$\lim_{r\to 0}\int_{BDE} \frac{e^{iz}}{z}dz = \int_{\pi}^{0} \lim_{r\to 0}\left(e^{ire^{i\theta}}\right) i d\theta$$

$$\lim_{r\to 0}\int_{BDE} \frac{e^{iz}}{z}dz = -i\pi$$

$$\left|\int_{RGA} f(z)dz\right| = \left|\int_{RGA} \frac{e^{iz}}{z}dz\right| \to 0 \ as \ |z| = R \to \infty.$$

Letting $R \to \infty$ and $\rho \to 0$ in (6.3.7), we get

$$\int_{-\infty}^{0} \frac{e^{ix}}{x} dx - i\pi + \int_{0}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$

Equating the imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$$
$$\therefore \int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}.$$

**Type. 4** The next category of integrals have the form

$$\int_{0}^{\infty} x^{\alpha} R(x) dx$$

where the exponent $\alpha$ is real and may be supposed to lie in the interval $0 < \alpha < 1$. For convergence $R(z)$ must have a zero of at least order two at $\infty$ and at most a simple pole at the origin. The new feature is the fact that $R(z)z^{\alpha}$ is not single-valued. This, however, is just the circumstance which makes possible to find the integral from $0$ to $\infty$.

**Example 6.3.8.** Evaluate $\int_{0}^{\infty} \frac{x^{\frac{1}{3}}}{1 + x^2} dx$.

**Solution.** Consider
$$\int_{C} \frac{z^{\frac{1}{3}}}{1 + z^2} dz = \int_{C} f(z) dz,$$
where $f(z) = \frac{z^{\frac{1}{3}}}{1 + z^2}$ and $C$ be the simple closed contour consisting of the circles $|z| = R$ ($R$ is large) and $|z| = \rho$ ($\rho$ is small) and position of real axis between them. The poles of $f(z)$ are given by $1 + z^2 = 0 \Rightarrow z = \pm i$ and $z = 0$ is a branch point for $f(z)$.
The simple pole $z = i$ lies inside $C$.

$\therefore$ the residue of $f(z)$ is given by

$$\begin{aligned} Res. f(z)|_{z=i} &= \lim_{z\to i} \frac{P(z)}{Q'(z)} \\ &= \lim_{z\to i} \frac{z^{\frac{1}{3}}}{2z} \\ Res. f(z)|_{z=i} &= \frac{e^{i\frac{\pi}{6}}}{2i} \end{aligned}$$

$\therefore$ by Cauchy's residue theorem, we have

$$\int_C f(z)dz = 2\pi i \sum R_i.$$

$$\int_C f(z)dz = \pi e^{i\frac{\pi}{6}}$$

$$i.e., \int_{-R}^{\rho} f(x)dx + \int_{BDE} f(z)dz + \int_{\rho}^{R} f(x)dx + \int_{FGA} f(z)dz = \pi e^{i\frac{\pi}{6}} \qquad (6.3.8)$$

Now,

$$\left| \int_{BDE} f(z)dz \right| \leq \int_{BDE} \frac{|z|^{\frac{1}{3}}}{|1+z^2|} |dz|$$

On $BDE$, $|z| = \rho$.

$$\begin{aligned} |1+z^2| &\geq 1 - |z|^2 \\ \frac{1}{|1+z^2|} &\leq \frac{1}{1-\rho^2} \end{aligned}$$

$$\begin{aligned} \left| \int_{BDE} f(z)dz \right| &\leq \frac{\rho^{\frac{1}{3}}}{1-\rho^2} \int_{BDE} |dz| \\ &\leq \frac{\rho^{\frac{1}{3}}}{1-\rho^2} \pi\rho \\ &\leq \frac{\rho^{\frac{4}{3}}}{1-\rho^2} \end{aligned}$$

$$\therefore \left| \int_{BDE} f(z)dz \right| \to 0 \ \ as \ \ |z| = \rho \to 0 \qquad (6.3.9)$$

Now,

$$\left|\int_{FGA} f(z)dz\right| \le \int_{FGA} \frac{|z|^{\frac{1}{3}}}{|1+z^2|}|dz|$$

On FGA,

$$|1+z^2| \ge |z|^2 - 1$$

$$\frac{1}{|1+z^2|} \le \frac{1}{R^2-1}$$

$$\begin{aligned}\left|\int_{FGA} f(z)dz\right| &\le \frac{R^{\frac{1}{3}}}{R^2-1}\int_{FGA}|dz| \\ &\le \frac{R^{\frac{1}{3}}}{R^2-1}\pi R \\ &\le \frac{R^{\frac{4}{3}}}{R^2-1}\end{aligned}$$

$$\therefore \left|\int_{FGA} f(z)dz\right| \to 0 \ \ as \ \ |z| = R \to \infty. \tag{6.3.10}$$

Letting $R \to \infty$ and $\rho \to 0$ in (6.3.8) and using (6.3.9) and (6.3.10) in (6.3.8), we get

$$\int_{-\infty}^{0} \frac{x^{\frac{1}{3}}}{1+x^2}dx + 0 + \int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx + 0 = \pi e^{i\frac{\pi}{6}}$$

Put $x = -y$ in the first integral, we have

$$\begin{aligned}\int_{\infty}^{0} \frac{(-y)^{\frac{1}{3}}(-dy)}{1+y^2} + \int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx &= \pi e^{i\frac{\pi}{6}} \\ \int_0^\infty \frac{(-1)^{\frac{1}{3}}y^{\frac{1}{3}}}{1+y^2}dy + \int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx &= \pi e^{i\frac{\pi}{6}} \\ e^{i\frac{\pi}{3}}\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx + \int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx &= \pi e^{i\frac{\pi}{6}} \\ (1+e^{i\frac{\pi}{3}})\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx &= \pi e^{i\frac{\pi}{6}}\end{aligned}$$

Equating the real parts on both sides, we get

$$\left(1+\cos\frac{\pi}{3}\right)\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx = \pi\frac{\sqrt{3}}{2}$$

$$\therefore \int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx = \frac{\pi}{\sqrt{3}}.$$

**Example 6.3.9.** Evaluate $\int_0^\infty \frac{\log x}{1+x^2}dx$

**Solution.** Consider

$$\int_C \frac{\log z}{1+z^2}dz = \int_C f(z)dz,$$

where $f(z) = \frac{\log z}{1+z^2}$ and $C$ be the simple closed contour consisting of the circles $|z| = R$ ($R$ is large) and $|z| = \rho$ ($\rho$ is small) and position of real axis between them. The poles of $f(z)$ are given by $1+z^2 = 0 \Rightarrow z = \pm i$ and $z = 0$ is a branch point of the function $\log z$.
The simple pole $z = i$ lies inside $C$.

$\therefore$ the residue of $f(z)$ is given by

$$\begin{aligned} Res. f(z)|_{z=i} &= \lim_{z\to i}\frac{P(z)}{Q'(z)} \\ &= \lim_{z\to i}\frac{\log z}{2z} \\ Res. f(z)|_{z=i} &= \frac{\pi}{4} \end{aligned}$$

$\therefore$ by Cauchy's residue theorem, we have

$$\begin{aligned} \int_C f(z)dz &= 2\pi i\left(\sum R_i\right) \\ \int_C f(z)dz &= \frac{\pi^2 i}{2} \end{aligned}$$

$$i.e., \int_{-R}^{\rho} f(x)dx + \int_{BDE} f(z)dz + \int_{\rho}^{R} f(x)dx + \int_{FGA} f(z)dz = \frac{\pi^2 i}{2} \qquad (6.3.11)$$

Now,

$$\left|\int_{BDE} f(z)dz\right| \le \int_{BDE} \left|\frac{\log z}{1+z^2}\right| |dz|.$$

On $BDE$, $|z| = \rho$.

$$\left|1+z^2\right| \geq 1-|z|^2$$

$$\frac{1}{\left|1+z^2\right|} \leq \frac{1}{1-\rho^2}$$

$$z = \rho e^{i\theta}$$

$$\log z = \log\rho + i\theta$$

$$\left|\log z\right| \leq \left|\log\rho\right| + |i\theta|$$

$$\leq \log\rho + \theta$$

$$\left|\log z\right| \leq \log\rho + \pi, \quad 0 \leq \theta \leq \pi$$

$$\left|\int_{BDE} f(z)dz\right| \leq \int_{BDE} \frac{\log\rho+\pi}{1-\rho^2}|dz|$$

$$\leq \frac{\pi+\log\rho}{1-\rho^2}\int_{BDE}|dz|$$

$$\leq \frac{\pi+\log\rho}{1-\rho^2}\pi\rho$$

$$\leq \frac{\pi^2\rho+\pi\rho\log\rho}{1-\rho^2}$$

$$Thus \quad \left|\int_{BDE} f(z)dz\right| \to 0 \quad as \quad |z| = \rho \to 0 \tag{6.3.12}$$

Now,

$$\left|\int_{FGA} f(z)dz\right| \leq \int_{FGA} \frac{\left|\log z\right|}{\left|1+z^2\right|}|dz| .$$

On FGA,

$$\left|1+z^2\right| \geq |z|^2 - 1$$

$$\frac{1}{\left|1+z^2\right|} \leq \frac{1}{R^2-1}$$

$$
\begin{aligned}
\log z &= \log R + i\theta \\
|\log z| &\leq |\log R| + |i\theta| \\
&\leq \log R + \theta \\
|\log z| &\leq \log R + \pi, \quad 0 \leq \theta \leq \pi, \quad \textit{maximum value of } \theta \textit{ is } \pi.
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{FGA} f(z)dz\right| &\leq \int_{FGA} \frac{\log R + \pi}{R^2 - 1}|dz| \\
&\leq \frac{\pi + \log R}{R^2 - 1}\pi R \\
&\leq \frac{\pi^2 R + \pi R \log R}{R^2 - 1}
\end{aligned}
$$

$$
\textit{Thus } \left|\int_{FGA} f(z)dz\right| \to 0 \ \textit{as} \ |z| = R \to \infty. \tag{6.3.13}
$$

Letting $\rho \to 0$ and $R \to \infty$ in (6.3.11) and using in (6.3.12) and (6.3.13), we get

$$
\int_{-\infty}^{0} \frac{\log x}{1 + x^2}dx + \int_{0}^{\infty} \frac{\log x}{1 + x^2}dx = \frac{\pi^2 i}{2}
$$

Put $x = -y$ in the first integral, we get

$$
\begin{aligned}
\int_{0}^{\infty} \frac{\log(-y)}{1 + y^2}dy + \int_{0}^{\infty} \frac{\log x}{1 + x^2}dx &= \frac{\pi^2 i}{2} \\
\int_{0}^{\infty} \frac{\log(-1) + \log y}{1 + y^2}dy + \int_{0}^{\infty} \frac{\log x}{1 + x^2}dx &= \frac{\pi^2 i}{2} \\
i\pi \int_{0}^{\infty} \frac{dy}{1 + y^2} + \int_{0}^{\infty} \frac{\log y}{1 + y^2}dy + \int_{0}^{\infty} \frac{\log x}{1 + x^2}dx &= \frac{\pi^2 i}{2}
\end{aligned}
$$

$$
i\pi \int_{0}^{\infty} \frac{dy}{1 + y^2} + 2\int_{0}^{\infty} \frac{\log x}{1 + x^2}dx = \frac{\pi^2 i}{2}
$$

Equating real parts on both sides, we get

$$
2\int_{0}^{\infty} \frac{\log x}{1 + x^2}dx = 0
$$

$$
\textit{Hence } \int_{0}^{\infty} \frac{\log x}{1 + x^2}dx = 0.
$$

## 6.4 Exercises

1. Find the poles and residues of the following functions:

$(a)\ \frac{1}{z^2+5+6}$, $(b)\ \frac{1}{(z^2-1)^2}$, $(c)\ \frac{1}{\sin z}$, $(d)\ \cot z$, $(e)\ \frac{1}{\sin^2 z}$,

$(f)\ \frac{1}{z^m(1-z)^m}$ ($m$, $n$ are positive integers).

2. Evaluate the following integrals by the method of residues:

$(a)\ \int_0^\infty \frac{x^2dx}{x^4+5x^2+6}dx$, $(b)\ \int_0^\infty (1+x^2)^{-1}\log x dx$

# BLOCK-II

# UNIT 7

# Harmonic Functions

**Objectives**

Upon completion of this Unit, students will be able to

- prove the properties of harmonic functions.
- identify the Poisson integral formula.
- understand the concept of the mean - value property.

## 7.1 Introduction

The real and imaginary parts of an analytic functions are conjugate harmonic functions. Therefore, all theorems on analytic functions are also theorems on pairs of conjugate harmonic functions. However, harmonic functions are important in their own right, and their treatment is not always simplified by the use of complex methods. This is particularly true when the conjugate harmonic functions is not single-valued. In this section we discuss some facts about harmonic functions that are intimately connected with Cauchy's theorem.

## 7.2 Definition and Basic Properties

**Definition 7.2.1. Harmonic Function.** A real - valued function $u(z)$ or $u(x, y)$, defined and single-valued in a region $\Omega$, is said to be harmonic in $\Omega$, or a potential function, if it is continuous together with its partial derivatives of the first two orders and satisfies *Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{7.2.1}$$

**Example 7.2.1.** The simplest harmonic functions are the linear functions $ax + by$.

**Note.** In polar coordinates $(r, \theta)$ equation (14.2.1) takes the form

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

**Example 7.2.2.** The function $\log r$ is a harmonic function $r > 0$.

**Example 7.2.3.** The function $a \log r + b$ is a harmonic function.

**Example 7.2.4.** The argument $\theta$ is a harmonic function.

**Properties of Harmonic Functions**

1. The sum of two harmonic functions is also harmonic function.

**Proof.** Let $u_1(x, y)$ and $u_2(x, y)$ are harmonic functions. Let $U = u_1 + u_2$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 (u_1 + u_2)}{\partial x^2} + \frac{\partial^2 (u_1 + u_2)}{\partial y^2} = 0.$$

■

2. A constant multiple of a harmonic function is also a harmonic function.

**Proof.** Leu $u(x, y)$ be a harmonic function.

$$\frac{\partial^2 cu}{\partial x^2} + \frac{\partial^2 cu}{\partial y^2} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

$\therefore$ $cu$ is harmonic. ■

. 3. If $u(x, y)$ is harmonic in $\Omega$ then $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic.

**Proof.** Let $U = \frac{\partial u}{\partial x}$ and $V = -\frac{\partial u}{\partial y}$

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial V}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial V}{\partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

Since $u$ is harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}.$$

Also

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$\therefore$ $U, V$ satisfy Cauchy's - Riemann equations. Also $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial x}$, and $\frac{\partial V}{\partial y}$ are continuous.

$$\therefore f(z) = U + iV = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \text{ is analytic}.$$

∎

**Theorem 7.2.1.** *If $u_1$ and $u_2$ are harmonic in a region $\Omega$, then*

$$\int_\gamma u_1 *du_2 - u_2 *du_1 = 0 \tag{7.2.2}$$

*for every cycle $\gamma$ which is homologous to zero in $\Omega$.*

**Proof.** Let $v_1, v_2$ denote the conjugate harmonic functions of $u_1, u_2$ in the region $\Omega$. Let us choose cycle $\gamma$ to be boundary of the rectangle $R$ contained in $\Omega$.

$$\gamma = \partial R.$$

$$*du_1 = dv_1 \text{ and } *du_2 = dv_2$$

$$\begin{aligned}
\therefore u_1^* du_2 - u_2^* du_1 &= u_1 dv_2 - u_2 dv_1 \\
&= u_1 dv_2 + v_1 du_2 - v_1 du_2 - u_2 dv_1 \\
&= u_1 dv_2 + v_1 du_2 - (v_1 du_2 + u_2 dv_1) \\
&= u_1 dv_2 + v_1 du_2 - d(u_2 v_1)
\end{aligned}$$

$$\therefore \int_\gamma u_1^* du_2 - u_2^* du_1 = \int_\gamma (u_1 dv_2 + v_1 du_2) - \int_\gamma d(u_2 v_1)$$

$$\int_\gamma u_1^* du_2 - u_2^* du_1 = \int_{\partial R} (u_1 dv_2 + v_1 du_2) - \int_{\partial R} d(u_2 v_1) \tag{7.2.3}$$

Since $d(u_2v_1)$ is an exact differential,

$$\int_{\partial R} d(u_2v_1) = 0.$$

Now $u_1dv_2 + v_1du_2$ represents the imaginary points of an analytic function for

$$\begin{aligned}(u_1 + iv_1)(du_2 + idv_2) &= u_du_2 - v_1dv_2 + i(u_1dv_2 + v_1du_2)\\ F_1(z)\frac{d}{dz}(u_2 + iv_2)dz &= u_1du_2 - v - 1dv_2 + i(u_1dv_2 + v_1du_2).\end{aligned}$$

Here the product $F_1.f_1$ being analytic inside and on $R$.

$$\begin{aligned}\therefore \int_{\partial R} F_1.f_1dz &= 0\\ \int_{\partial R}(u_1du_2 - v_1dv_2) + i\int_{\partial R}(u_1dv_2 + v_1du_2) &= 0\end{aligned}$$

Equating the real parts and imaginary parts on both sides, we get

$$\int_{\partial R}(u_1du_2 - v_1dv_2) = 0 \text{ and } \int_{\partial R}(u_1dv_2 + v_1du_2) = 0$$

$$\therefore (7.2.3) \Rightarrow \int_\gamma u_1^*du_2 - u_2^*du_1 = 0.$$

∎

**Note.** In the classical notation (10.2.2) would be written as

$$\int_\gamma \left( u_1\frac{\partial u_2}{\partial n} - u_2\frac{\partial u_1}{\partial n}\right) |dz| = 0.$$

## 7.3 The Mean-Value Property

**Theorem 7.3.1.** *The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of* $\log r$,

$$\frac{1}{2\pi}\int_{|z|=r} ud\theta = \alpha\log r + \beta, \qquad (7.3.1)$$

*and if $u$ is harmonic in a disk $\alpha = 0$ and the arithmetic mean is constant.*

**Proof.** By Theorem 3.1.1,

$$\int_\gamma (u_1 *du_2 - u_2 *du_1) = 0.$$

Put $u_1 = \log r$, $u_2 = u$

For $\Omega$ we choose the punctured disk $0 < |z| < \rho$, and for $\gamma$ we take the cycle $C_1 - C_2$ where $C_i$ $(i = 1, 2)$ is a circle $|z| = r_i < \rho$ described in the positive sense.

$$\int_{C_1 - C_2} (u_1 *du_2 - u_2 *du_1) = 0$$

$$\int_{C_1 - C_2} \left(\log r \, r\frac{\partial u}{\partial r} d\theta - u d\theta\right) = 0$$

$$\int_{C_1} \left(\log r_1 \, r_1\frac{\partial u}{\partial r} d\theta - u d\theta\right) - \int_{C_2} \left(\log r_2 \, r_2\frac{\partial u}{\partial r} d\theta - u d\theta\right) = 0$$

$$\log r_1 \int_{C_1} r_1\frac{\partial u}{\partial r} d\theta - \int_{C_1} u d\theta = \log r_2 \int_{C_2} r_2\frac{\partial u}{\partial r} d\theta - \int_{C_2} u d\theta$$

Each value is constant, say $\beta$.

$$\log r \int_{|z|=r} r\frac{\partial u}{\partial r} d\theta - \int_{|z|=r} u d\theta = constant = \beta.$$

$$\int_{|z|=r} u d\theta - \log r \int_{|z|=r} r\frac{\partial u}{\partial r} d\theta = constant = \beta.$$

$$\frac{1}{2\pi}\int_{|z|=r} u d\theta - \log r \frac{1}{2\pi}\int_{|z|=r} r\frac{\partial u}{\partial r} d\theta = \beta \tag{7.3.2}$$

Since $u$ is harmonic,

$$\int_{C_1 - C_2} *du = 0$$

$$\int_{C_1} *du - \int_{C_2} *du = 0$$

$$\int_{C_1} r_1\frac{\partial u}{\partial r} d\theta = \int_{C_2} r_2\frac{\partial u}{\partial r} d\theta = constant$$

$$\therefore \frac{1}{2\pi}\int_{|z|=r<\rho} r\frac{\partial u}{\partial r} d\theta = \alpha \tag{7.3.3}$$

Using (7.3.3) in (14.3.2), we get

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta - (\log r)\alpha = \beta$$
$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \alpha\log r + \beta.$$

∎

**Note.**

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \alpha\log r + \beta$$

Let us choose the circle $|z - z_0| = r \Rightarrow z = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$.
If $u$ is harmonic in the whole disk of radius $r < (\rho)$, then $\alpha = 0$.

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = \beta$$
$$u(z_0) = \beta \text{ as } r \to 0$$
$$\therefore \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = \beta = u(z_0).$$

**Theorem 7.3.2.** ***Maximum Principle.*** *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently, the maximum and the minimum on a closed bounded set $E$ are taken on the boundary of $E$.*

**Proof.** Let $f(z)$ represents an analytic function $u = Real\ \ part\ \ of\ \ f(z)$. Consider the closed disk $|z - z_0| \le r$ contained in a region $\Omega$.
$\therefore$ by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz, \quad on\ \ C, \quad |z - z_0| = r \Rightarrow z = z_0 + re^{i\theta}$$
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$
$$\therefore u(z_0) + iv(z_0) = \frac{1}{2\pi}\int_0^{2\pi} [u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta})]d\theta$$

Equating the real parts on both sides, we get

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

$$|u(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|u(z_0 + re^{i\theta})\right|.|d\theta|$$

Suppose that $|u(z)| \le |u(z_0)|$ throughout $|z - z_0| \le r$, then

$$|u(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |u(z_0)|\, d\theta$$

$$\le u(z_0)\frac{1}{2\pi}\int_0^{2\pi} d\theta \le |u(z_0)|$$

which is a contradiction. $|u(z)| = |u(z_0)|$ , $|z - z_0| \le r$. On concentric circles $\Omega$, $|u(z)|$ is also a constant. Therefore $u(z)$ reduces to a constant in $\Omega$. $u(z)$ cannot attain maximum value in $\Omega$. Since $u$ is continuous on a closed set $E$ and hence $u$ attains into maximum value $|u(z)|$ on the boundary of $E$.

The minimum principle can be obtained by applying the above result to the harmonic function $(-u)$. ■

## 7.4 Poisson's Formula

**Theorem 7.4.1.** ***Poisson Formula for Harmonic Functions.*** *Suppose that* $u(z)$ *is harmonic for* $|z| < R$, *continuous for* $|z| \le R$. *Then*

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, u(z)d\theta \tag{7.4.1}$$

*for all* $|a| < R$.

**Proof.** Let $u(z)$ be harmonic for $|z| \le R$. The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

maps the circle $|\zeta| \le 1$ onto $|z| \le R$ with $\zeta = 0$ corresponding to $z = a$. The function

$u(S(\zeta)) = u(z)$ is harmonic in $|\zeta| \leq 1$.

$$u(s(0)) = \frac{1}{2\pi} \int_{|\zeta|=1} u(s(\zeta))\, d(\arg(\zeta))$$

by mean value property.

$s(0) = a$,

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(s(\zeta))\, d(\arg(\zeta)) \tag{7.4.2}$$

$$|\zeta| = 1 \Rightarrow \zeta = e^{i\varphi} \Rightarrow d\varphi = -i\frac{d\zeta}{\zeta} \Rightarrow d(\arg(\zeta)) = -i\frac{d\zeta}{\zeta}$$

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(s(\zeta)) \cdot \left(-i\frac{d\zeta}{\zeta}\right) \tag{7.4.3}$$

$$\begin{aligned} z &= \frac{R(R\zeta + a)}{R + \bar{a}\zeta} \\ R(z-a) &= \zeta(R^2 - \bar{a}z) \end{aligned}$$

$$\begin{aligned} \zeta &= \frac{R(z-a)}{R^2 - \bar{a}z} \\ \log\zeta &= \log R = \log(z-a) + \log(R^2 - \bar{a}z) \\ \frac{d\zeta}{\zeta} &= \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right) dz \end{aligned}$$

$|z|^2 = R^2 \Rightarrow dz = Re^{i\theta} i\, d\theta \Rightarrow dz = zi\, d\theta$.

$$\begin{aligned} \frac{d\zeta}{\zeta} &= \left(\frac{1}{z-a} + \frac{\bar{a}}{z\bar{z} - \bar{a}z}\right) iz\, d\theta \\ -i\frac{d\zeta}{\zeta} &= \left(\frac{z}{z-a} + \frac{\bar{a}}{\bar{z} - \bar{a}}\right) d\theta \\ &= \frac{|z|^2 - |a|^2}{|z-a|^2}\, d\theta \\ -i\frac{d\zeta}{\zeta} &= \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta \end{aligned}$$

$$\therefore (17.4.3) \Rightarrow u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta$$

∎

**Note.(i)** In (17.4.1), put $u(z) = 1$ (which is analytic everywhere), we get

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta$$

$$2\pi = \int_0^{2\pi} \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta$$

**(ii.)** Put $z = Re^{i\theta},\ a = re^{i\varphi},\ r < R$.

$$\therefore u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(Re^{i\theta})}{|Re^{i\theta} - re^{i\varphi}|^2}\, d\theta.$$

Now

$$\begin{aligned} \left|Re^{i\theta} - re^{i\varphi}\right|^2 &= (Re^{i\theta} - re^{i\varphi})(\overline{Re^{i\theta} - re^{i\varphi}}) \\ &= R^2 - 2Rr\cos(\theta - \varphi) + r^2 \end{aligned}$$

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(Re^{i\theta})}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, d\theta,\ r < R.$$

This is called polar form of Poisson's formula.

**(iii).** Another form of Poisson's formula:

The other form of Poisson formula is

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \operatorname{Re}\left(\frac{z+a}{z-a}\right) u(z)\, d\theta.$$

**Proof.** Now

$$
\begin{aligned}
\frac{z+a}{z-a} &= \frac{z+a}{z-a}\cdot\frac{\bar z-\bar a}{\bar z-\bar a}\\
&= \frac{|z|^2-|a|^2+(a\bar z-\bar a z)}{|z-a|^2}\\
&= \frac{R^2-|a|^2+a\bar z-\overline{(a\bar z)}}{|z-a|^2}\\
&= \frac{R^2-|a|^2+2i\,Im(a\bar z)}{|z-a|^2}
\end{aligned}
$$

$$\therefore Re\left(\frac{z+a}{z-a}\right) = \frac{R^2-|a|^2}{|z-a|^2}$$

From Poisson's formula,

$$
\begin{aligned}
u(a) &= \frac{1}{2\pi}\int_{|z|=R}\frac{R^2-r^2}{|z-a|^2}\,u(z)\,d\theta\\
u(a) &= \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right)u(z)\,d\theta.
\end{aligned}
$$

∎

# BLOCK-III

# UNIT 8

# Partial Fractions and Factorization

| **Objectives** |
|---|
| Upon completion of this Unit, students will be able to |
| x prove Mittag - Leffler theorem. |
| x understand the concept of infinite products. |
| x identify Weierstrass theorem on an entire function. |

## 8.1 Introduction

A rational function has standard representations, one by partial fractions and the other by factorization of the numerator and the denominator. The present section is devoted to similar representations of arbitrary meromorphic functions.

## 8.2 Partial Fractions

**Theorem 8.2.1.** ***Mittag-Leffler Theorem.*** *Let $\{b_\nu\}$ be a sequence of complex numbers with $\lim_{\nu\to\infty} b_\nu = \infty$, and let $P_\nu(\zeta)$ be a polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at the points $b_\nu$ and the corresponding singular parts $P_\nu\left(\frac{1}{z-b_\nu}\right)$. Moreover, the most general meromorphic function of this kind can be written in the form*

$$f(z) = \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right] + g(z) \qquad (8.2.1)$$

*where the $p_\nu(z)$ are suitably chosen polynomials and $g(z)$ is analytic in the whole plane.*

**Proof.** Let us suppose that no $b_\nu$ equal to zero. Consider a circle with centre at the origin and radius less than $|b_\nu|$. Then from Taylor's series we get

$$P_\nu\left(\frac{1}{z-b_\nu}\right) = a_{0\nu} + a_{1\nu}z + a_{2\nu}z^2 + \cdots + a_{n\nu}z^{n_\nu} + a_{(n+1)\nu}z^{n_\nu+1} + \cdots$$

$$P_\nu\left(\frac{1}{z-b_\nu}\right) - a_{0\nu} + a_{1\nu}z + a_{2\nu}z^2 + \cdots + a_{n\nu}z^{n_\nu} = a_{(n+1)\nu}z^{n_\nu+1} + \cdots$$

Choose

$$p_\nu(z) = a_{0\nu} + a_{1\nu}z + \cdots + a_{n\nu}z^{n_\nu}$$

$$P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu = a_{(n+1)\nu} z^{n_\nu+1} + \cdots$$

$$\therefore \left|P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu\right| \le \left(\frac{|z|}{R}\right)^{n_\nu+1}\frac{M_\nu R}{(R-|z|)}$$

where $M_\nu$ = max value of $P_\nu$ on the circle of radius $R$. Put $R = \frac{1}{2}|b_\nu|$.

$$|P_\nu(z) - p_\nu| \le \left(\frac{2|z|}{|b_\nu|}\right)^{n_\nu+1}\frac{M_\nu \frac{1}{2}|b_\nu|}{\frac{1}{2}|b_\nu| - |z|}$$

where $|z| < \frac{1}{2}|b_\nu|$. The above result is also valid for $|z| \le \frac{|b_\nu|}{4}$.

$$\therefore f(z) = \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right]$$

represents function in the whole complex plane without pole. That is it represents an entire function say $g(z)$.

$$\therefore f(z) - \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right] = g(z)$$

$$\therefore f(z) = \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right] + g(z)$$

where $g(z)$ is analytic in the whole plane. If some $b_\nu = 0$, we choose $P_\nu(z) = 0$. ■

**Example 8.2.1.** Prove that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$.

**Solution.** Let $f(z) = \frac{\pi^2}{\sin^2 \pi z}$. The poles of $f(z)$ are given by

$$\sin^2 \pi z = 0 \Rightarrow \pi z = n\pi \ (twice) \Rightarrow z = n \ (twice.)$$

$$\begin{aligned}
\frac{\pi^2}{\sin^2 \pi z} &= \frac{\pi^2}{\pi^2 z^2 \left(1 - \frac{1}{3!}\pi^2 z^2 + \cdots\right)^2} \\
&= \frac{1}{z^2}\left(1 - \frac{1}{3!}\pi^2 z^2 + \cdots\right)^{-2}
\end{aligned}$$

$$\begin{aligned}
\frac{\pi^2}{\sin^2 \pi z} &= \frac{1}{z^2}\left[1 + 2\left(\frac{1}{3!}\pi^2 z^2 + \cdots\right) + 3\left(\frac{1}{3!}\pi^2 z^2 + \cdots\right)^2 + \cdots\right] \\
&= \frac{1}{z^2} + powers\ of\ z \\
&= \frac{1}{(z-0)^2} + powers\ of\ z
\end{aligned}$$

$\therefore z = 0$ is a double pole with singular part $\frac{1}{z^2}$. Singular part with respect to double pole $z = n$ is $\frac{1}{(z-n)^2}$.

$$\begin{aligned}
P_n\left(\frac{1}{z - b_n}\right) &= \frac{1}{(z-n)^2} \\
&= \frac{1}{(n-z)^2} \\
&= \frac{1}{n^2}\left(1 - \frac{z}{n}\right)^{-2}
\end{aligned}$$

$$\frac{1}{(z-n)^2} = \frac{1}{n^2} + \frac{2}{n^3}z + \frac{3}{n^4}z^2 + \cdots$$

Since $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ etc., are convergent, we choose $P_n(z) = 0$. By Mittag - Leffler theorem, we have

$$
\begin{aligned}
\frac{\pi^2}{\sin^2 \pi z} &= \sum_n \left[ P_n\left(\frac{1}{z-b_n}\right) - P_n(z) \right] + g(z) \\
&= \sum_{n=-\infty}^{\infty} \left[ \frac{1}{(z-n)^2} - 0 \right] + g(z) \\
&= \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)
\end{aligned}
$$

where $g(z)$ is analytic in the whole plane. Since $\csc^2 \pi z$ is periodic of period $1.$
Consider the strip $0 \leq x \leq 1.$

$$
\begin{aligned}
\csc^2 \pi z &= \frac{1}{(\sin \pi z)^2} \\
\sin \pi z &= \left[ \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \right] \\
|\sin \pi z| &= \left| \frac{1}{2} \left( e^{i\pi z} - e^{-i\pi z} \right) \right| \\
&\geq \frac{1}{2} \left| e^{-\pi y} - e^{\pi y} \right| \\
|\csc \pi z| &\leq \frac{2}{e^{-\pi y} - e^{\pi y}} \\
|\csc^2 \pi z| &\leq \frac{4}{(e^{\pi y} - e^{-\pi y})^2} \\
|\pi^2 \csc^2 \pi z| &\leq 4\pi^2 \frac{1}{e^{2\pi y}(1 - e^{-2\pi y})^2} \\
&\leq 4\pi^2 \frac{e^{-2\pi y}}{(1 - e^{-2\pi y})^2} \\
|\pi^2 \csc^2 \pi z| &\to 0 \quad as \ y \to \pm\infty.
\end{aligned}
$$

Thus $\pi^2 \csc^2 \pi z \to 0$ uniformly in the strip as $|y| \to \infty.$ Also $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ has the same property. Indeed, the convergence is uniform for $|y| \geq 1,$ say, and the limit for $|y| \to \infty$ can thus be obtained by taking the limit in each term.

$\therefore g(z) \to 0$ uniformly for $|y| \to \infty.$ This is sufficient to infer that $|g(z)|$ is bounded in a period strip $0 \leq x \leq 1,$ and because of the periodicity $|g(z)|$ will be bounded in the whole plane.

$\therefore$ by Liouville's theorem $g(z)$ must reduce to a constant.

$$\therefore g(z) = k.$$

$$\pi^2 \csc^2 \pi z = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} + k$$

As $|z| \to \infty$ in the strip, both sides of the above equation tends to $0$.
$\therefore k = 0$. Hence

$$\pi^2 \csc^2 \pi z = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

**Example 8.2.2.** Prove that $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$

**Solution.** Since $\pi^2 \csc^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

$$\pi^2 \csc^2 \pi z = \frac{1}{z^2} + \sum_{n \neq 0} \frac{1}{(z-n)^2}$$

$$\frac{d}{dz}(-\pi \cot \pi z) = \frac{d}{dz} \cdot \left(-\frac{1}{z}\right) + \sum_{n \neq 0} \frac{d}{dz} \cdot \left(-\frac{1}{(z-n)}\right)$$

$$= -\frac{d}{dz} \cdot \frac{1}{z} - \sum_{n \neq 0} \frac{d}{dz} \cdot \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

$$-\frac{d}{dz}(\pi \cot \pi z) = -\frac{d}{dz} \cdot \frac{1}{z} - \frac{d}{dz} \sum_{n \neq 0} \cdot \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Integrating on both sides, we get

$$\begin{aligned}
\pi\cot\pi z &= \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) + c\\
&= \frac{1}{z} + \sum_{n\neq 0}\frac{1}{z-n} + c\\
&= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z-n} + \sum_{n=-1}^{-\infty}\frac{1}{z-n} + c\\
&= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z-n} + \sum_{n=1}^{\infty}\frac{1}{z+n} + c \quad \text{by replacing } n \text{ by } -n
\end{aligned}$$

$$\pi\cot\pi z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2} + c \tag{8.2.2}$$

Replacing $z$ by $-z$, we have

$$-\pi\cot\pi z = -\frac{1}{z} - 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2} + c \tag{8.2.3}$$

$$(12.2.2) + (8.2.3) \Rightarrow 2c = 0 \Rightarrow c = 0$$

Hence

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

## 8.3 Infinite Products

**Definition 8.3.1.** Consider a sequence of non - zero complex numbers $p_1, p_2, \cdots, p_n, \cdots$. A product of the form

$$\prod_{n=1}^{\infty} p_n = p_1p_2\ldots p_n\ldots$$

is called an infinite product.

To see the convergence of the product, let us define

$$P_n = p_1p_2\cdots p_n.$$

It is said to converge to the value $P$, if

$$P = \lim_{n\to\infty} P_n$$

if the limit exists and is different from zero.

Now,

$$\begin{aligned} p_n &= \frac{P_n}{P_{n-1}} \\ \lim_{n\to\infty} p_n &= \lim_{n\to\infty} \frac{P_n}{P_{n-1}} \\ \lim_{n\to\infty} &= 1, \quad \textit{when the product is convergent} \end{aligned}$$

In view of this fact it is preferable to write all infinite products in the form

$$\prod_{n=1}^{\infty} (1 + a_n),$$

where $a_n$ are complex numbers.This product is convergent if $a_n \to 0$. This condition is only necessary. Then converse is not true.

**Theorem 8.3.1.** *The infinite product* $\prod_{1}^{\infty}(1 + a_n)$ *with* $(1 + a_n) \neq 0$ *converges simultaneously with the series* $\sum_{1}^{\infty} \log(1 + a_n)$ *whose terms represent the values of the principal branch of the logarithm.*

**Proof.** Let us write

$$\begin{aligned} P_n &= (1 + a_1)(1 + a_2) \cdots (1 + a_n) \\ \therefore \log(P_n) &= \log(1 + a_1) + \log(1 + a_2) + \cdots + \log(1 + a_n) \\ \log(P_n) &= \sum_{k=1}^{n} (1 + a_k) \end{aligned}$$

We suppose that the series $\sum_{n=1}^{\infty} \log(1+a_n)$ is convergent.

$$\begin{aligned}
\therefore \log P_n &= S_n \\
P_n &= e^{S_n} \\
\lim_{n\to\infty} P_n &= e^{\lim_{n\to\infty} S_n} \\
\lim_{n\to\infty} P_n &= e^{S}.
\end{aligned}$$

$\sum_{n=1}^{\infty} \log(1+a_n)$ is convergent implies that the sequence of the $n^{th}$ partial sum of the given series $S_n$ is convergent.

$$\therefore \lim_{n\to\infty} P_n = e^{S} (\neq 0) \ since \ S_n \to S \ \ as \ n \to \infty.$$

Hence $\prod_{n=1}^{\infty}$ is convergent.

Conversely, let us assume that the product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent then $a_n \to 0$.
Now

$$\begin{aligned}
P_n &= (1+a_1)(1+a_2)\cdots(1+a_n)\cdots \\
\log(P_n) &= \log(1+a_1)+\log(1+a_2)+\cdots+\log(1+a_n)+\cdots \\
\log(P_n) &= \sum_{n=1}^{\infty} \log(1+a_n)+2\pi h_n
\end{aligned}$$

where $h_n$ is well determined integers. Equating imaginary parts on both sides,

$$\arg P_n = \arg(1+a_1)+\cdots+\arg(1+a_n)+2\pi h_n.$$

Put $\beta_n = \arg P_n$ and $\alpha_n = \arg(1+a_n)$.

$$\beta_n = \alpha_1+\alpha_2+\cdots+\alpha_n+2\pi h_n.$$

$$\therefore \beta_{n+1} = \alpha_1+\alpha_2+\cdots+\alpha_n+\alpha_{n+1}+2\pi h_{n+1}$$

$$\beta_{n+1}-\beta_n = \alpha_{n+1}+2\pi(h_{n+1}-h_n)$$

Let us assume that as $n \to \infty,\ P_n \to p$.

$$\therefore \beta_{n+1} = \arg P_{n+1} \ and\ \ \beta_n = \arg P_n$$

$$\alpha_{n+1} = \arg(1 + a_{n+1}) = \arg 1 = 0$$

$$\arg P_1 - \arg P = 0 + 2\pi(h_{n+1} - h_n),\ n\ \ is\ \ large.$$

$$\therefore h_{n+1} - h_n \to 0\ n\ \ is\ \ large.$$

$$\therefore h_{n+1} = h_n\ where\ \ n\ \ is\ \ large.$$

Hence $h_n$ becomes a unique integer $h$. Now,

$$\log P_n = \sum_{n=1}^{\infty} \log(1 + a_n) + 2\pi h_n$$

Let $n \to \infty,\ P_n \to P(\neq 0)$.

$$\therefore \log P = \sum_{n=1}^{\infty} \log(1 + a_n) + 2\pi h$$

$$\therefore \sum_{n=1}^{\infty} \log(1 + a_n) = \log P - 2\pi h$$

Hence

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

is convergent. ■

**Definition 8.3.2.** An infinite product $\prod_{n=1}^{\infty} \log(1+a_n)$ is said to be absolutely convergent if and only if the series $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges absolutely.

**Theorem 8.3.2.** *A necessary and sufficient condition for the absolute convergence of the product* $\prod_{1}^{\infty} (1 + a_n)$ *is the convergence of the series* $\sum_{1}^{\infty} |a_n|$.

**Proof.** We know that

$$\lim_{z\to 0} \frac{\log(1 + z)}{z} = 1.$$

If either the series $\sum_{n=1}^{\infty} \log(1 + a_n)$ or $\sum_{n=1}^{\infty} |a_n|$ converges, we have $a_n \to 0,\ n \to \infty$ and we have

$\sum |a_n|$ is convergent as $a_n \to 0$. Let $\epsilon$ be any positive given number.

$$\lim_{a_n \to 0} \frac{\log(1+a_n)}{a_n} = 1$$

$$\lim_{n\to\infty} \frac{\log(1+a_n)}{a_n} = 1$$

$$\lim_{n\to\infty} \left|\frac{\log(1+a_n)}{a_n}\right| = 1$$

$$\left|\frac{\log(1+a_n)}{a_n} - 1\right| < \epsilon \quad for\ large\ n$$

$$1-\epsilon < \left|\frac{\log(1+a_n)}{a_n}\right| < 1+\epsilon \quad for\ large\ n$$

$$(1-\epsilon)\,|a_n| < \log(1+a_n) < (1+\epsilon)\,|a_n| \quad for\ large\ n$$

Since $\sum |a_n|$ is convergent, it follows that $\sum \log(1+a_n)$ is convergent.

$\therefore \sum \log(1+a_n)$ is absolutely convergent and hence $\prod (1+a_n)$ is converges absolutely. ■

**Example 8.3.1.** Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$.

**Solution.** Here

$$P_n = 1 - \frac{1}{n^2} = \frac{n-1}{n}\cdot\frac{n+1}{n}.$$

$$\therefore P_n = p_2 p_3 \cdots p_n = \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{n-3}{n-2}\cdot\frac{n-1}{n-2}\right)\left(\frac{n-2}{n-1}\cdot\frac{n}{n-1}\right)\left(\frac{n-1}{n}\cdot\frac{n+1}{n}\right).$$

$$P_n = \frac{1}{2}\cdot\frac{n+1}{n} = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$P_n \to \frac{1}{2}\ as\ n \to \infty.$$

$$\therefore P = \frac{1}{2}.$$

$\therefore$ the product is convergent.

**Example 8.3.2.** Prove that for $|z| < 1$,

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}.$$

**Solution.**

$$\prod_{n=0}^{\infty}(1+z^{2^n}), \quad |z| < 1.$$

Let $a_n = z^{2^n} \Rightarrow |a_n| = |z^{2^n}| = |z|^{2^n}$

Since $|z| < 1$, $\sum |z|^{2^n}$ is convergent.

$\therefore \sum a_n$ is convergent.

Here

$$P_n = p_1 \cdot p_2 \cdots p_n = (1+z)(1+z^2)\cdots(1+z^{2^n})$$

For $|z| < 1$,

$$\begin{aligned}(1-z)P_n &= (1-z)(1+z)(1+z^2)\cdots(1+z^{2^n})\\ &= (1-z^2)(1+z^2)\cdots(1+z^{2^n})\\ &= (1-z^4)(1+z^4)\cdots(1+z^{2^n})\end{aligned}$$

$$\begin{aligned}(1-z)P_n &= (1-z^{2^n})(1+z^{2^n})\\ &= 1-(z^{2^n})^2\\ &= 1-z^{2.2^n}\\ (1-z)P_n &= 1-z^{2^{n+1}}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}(1-z)P_n &= \lim_{n\to\infty}(1-z^{2^{n+1}}), \quad |z|<1\\ \lim_{n\to\infty}(1-z)P_n &= 1\\ \lim_{n\to\infty} P_n &= \frac{1}{1-z}, \quad |z|<1\\ \therefore \prod (1+z^{2^n}) &= \frac{1}{1-z}, \quad |z|<1.\end{aligned}$$

## 8.4 Canonical Products

**Definition 8.4.1. Entire Function.** A function which is analytic in the whole plane is called an entire function or an integral function. The simplest entire functions which are not polynomials are $e^z$, $\sin z$, and $\cos z$.

**Theorem 8.4.1.** *If $g(z)$ is an entire function then $f(z) = e^{g(z)}$ is entire and $\neq 0$.*

**Proof.** Since $f(z) \neq 0$, it follows that $\dfrac{f'(z)}{f(z)}$ is analytic in the whole plane.

$\therefore \dfrac{f'(z)}{f(z)}$ represents an entire function say $h(z)$.
Integrating along a simple path from $z_0$ to $z$.

$$\begin{aligned}
\int_{z_0}^{z} \frac{f'(z)}{f(z)}dz &= \int_{z_0}^{z} h(z)dz \\
\log f(z) - \log f(z_0) &= H(z), \ \ where \ \ H(z) = \int_{z_0}^{z} h(z)dz \\
\frac{f(z)}{f(z_0)} &= e^{H(z)} \\
f(z) &= f(z_0)e^{H(z)} \\
&= e^k e^{H(z)}, \ \ where \ \ f(z_0) = a \ \ constant \ \ = e^k \\
\therefore f(z) &= e^{g(z)}
\end{aligned}$$

where $g(z) = k + H(z)$ is an entire function. ■

**Theorem 8.4.2.** ***Weierstrass Theorem on an Entire Function.*** *There exists an entire function with arbitrarily prescribed zeros $a_n$ provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{8.4.1}$$

*where the product is taken over all $a_n \neq 0$, the $m_n$ are certain integers, and $g(z)$ is an entire function.*

**Proof.** Consider an arbitrary sequence of complex numbers $a_n \neq 0$ with $\lim_{n\to\infty} a_n = \infty$. Let us prove the existence of polynomials $p_n(z)$ such that

$$\prod_{1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{8.4.2}$$

converges to an entire function. The above product converges absolutely and uniformly if the corresponding series $\sum_{n=1}^{\infty} \left[\log\left(1 - \frac{z}{a_n}\right) + p_n(z)\right]$ converges absolutely and uniformly.
For a given $R$ we consider only the terms with $|a_n| > R$. In the disk $|z| \leq R$ the principal branch

of $\log\left(1-\frac{z}{a_n}\right)$ can be developed in a Taylor's series

$$\log\left(1-\frac{z}{a_n}\right)=-\frac{z}{a_n}-\frac{1}{2}\left(\frac{z}{a_n}\right)^2-\frac{1}{3}\left(\frac{z}{a_n}\right)^3-\cdots.$$

We reserve the signs and choose $p_n(z)$ as a partial sum

$$p_n(z)=\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}.$$

Then

$$\log\left(1-\frac{z}{a_n}\right)+p_n(z)=-\frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1}-\frac{1}{m_n+2}\left(\frac{z}{a_n}\right)^{m_n+2}-\cdots$$

$$\left|\log\left(1-\frac{z}{a_n}\right)+p_n(z)\right|\le\frac{1}{m_n+1}\left|\frac{z}{a_n}\right|^{m_n+1}\left(1+\frac{m_n+1}{m_n+2}\left|\frac{z}{a_n}\right|+\left|\frac{z}{a_n}\right|^2+\cdots\right)$$

$$\le\frac{1}{m_n+1}\left|\frac{z}{a_n}\right|^{m_n+1}\left(1+\left|\frac{z}{a_n}\right|+\left|\frac{z}{a_n}\right|^2+\cdots\right)$$

$$\left|\log\left(1-\frac{z}{a_n}\right)+p_n(z)\right|\le\frac{1}{m_n+1}\left(\frac{|z|}{|a_n|}\right)^{m_n+1}\left(1-\frac{|z|}{|a_n|}\right)^{-1}$$

$$\left|\log\left(1-\frac{z}{a_n}\right)+p_n(z)\right|\le\frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1-\frac{R}{|a_n|}\right)^{-1}\tag{8.4.3}$$

Suppose that the series

$$\sum_{n=1}^{\infty}\frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\tag{8.4.4}$$

converges.

$$\therefore\sum_{n=1}^{\infty}\left[\log\left(1-\frac{z}{a_n}\right)+p_n(z)\right]$$

is absolutely and uniformly convergent for $|z|\le R$ and therefore the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{p_n(z)}$$

is uniformly convergent for $|z|\le R$. Thus the product (8.4.2) represents an analytic function in $|z|<R$.

It remains only to show that the series (8.4.4) can be made convergent for all $R$. But this is obvious, for if we take $m_n=n$, it is clear that (8.4.4) has a majorant geometric series with ratio $<1$ for any fixed value $R$.

Since $R$ is arbitrary, $\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{p_n(z)}$ is an entire function

$$f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}.$$

Let $f(z)$ be an entire function with zero's $a_1, a_2, \cdots$ and zero of order $m$ at the origin then the quotient

$$\frac{f(z)}{z^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}}$$

is an entire function without zero's and it is equal to $e^{g(z)}$, $g(z)$ is entire. Thus

$$\frac{f(z)}{z^m\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}} = e^{g(z)}.$$

$$\therefore\ f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}.$$

■

**Corollary 8.4.1.** *Every function which is meromorphic in the whole plane is the quotient of two entire functions.*

**Proof.** If $F(z)$ is meromorphic function in the whole plane, we can find an entire function $g(z)$ with the poles of $F(z)$ for zeros. The product $F(z)g(z)$ is then an entire function $f(z)$, and we obtain $F(z) = \frac{f(z)}{g(z)}$. ■

**Definition 8.4.2. Genus of the canonical product.** From the Weierstrass theorem, we have

$$\therefore\ f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}.$$

Consider the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{8.4.5}$$

which is convergent and represents an entire function provided that the series

$$\sum_{n=1}^{\infty} \frac{1}{h+1} \left( \frac{R}{|a_n|} \right)^{h+1}$$

converges for all $R$.
That is provided the series

$$\sum \frac{1}{|a_n|^{h+1}} < \infty.$$

Assume that $h$ is the smallest positive integer for which the series $\sum \frac{1}{|a_n|^{h+1}}$ converges. Then the product (8.4.5) is called the *canonical product* associated with the sequence $\{a_n\}$, and $h$ is the *genus* of the canonical product.

# BLOCK-III

# UNIT 9

# Entire Functions

| **Objectives** |
|---|
| Upon completion of this Unit, students will be able to |
| x identify Poisson - Jensen formula. |
| x prove Hadamard's theorem. |

## 9.1 Introduction

We have already considered the representation of entire function as infinite products, and, in special cases, as canonical products. In this unit we study the connection between the product representation and the rate of growth of the functions. Such questions were first investigated by Hadamard who applied the results to his celebrated proof of the Prime Number Theorem. Space does not permit us to include this application, but the basic importance of Hadamard's factorization theorem will be quite evident.

## 9.2 Jensen's Formula.

**Theorem 9.2.1.** ***Poisson - Jensen Formula.*** *Let $f(z)$ be analytic in $|z| \leq \rho$ and suppose that the non - null zeros of $f(z)$ inside the circle $|z| = \rho$ are $a-1, a_2, \cdots, a_n$ and each zero being counted according to its degree of multiplicity. Then*

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{\rho^2-\bar{a}_i z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)\log\left|f(\rho e^{i\theta})\right| d\theta. \qquad (9.2.1)$$

*where* $\log|f(z)|$ 0, *and* $f(z) \neq 0$.

**Proof.** Let $C$ be the circle $|z| = \rho$. Consider the function

$$F(z) = f(z)\prod_{i=1}^{n}\frac{\rho^2 - \overline{a_i}z}{\rho(z - a_i)}.$$

The zeros of $f(z)$ exactly cancels with the factors of the denominator on the right hand side of $F(z)$.

$\therefore F(z)$ is analytic throughout the circle $C$. Zeros of $F(z)$ are given by

$$\rho^2 - \overline{a_i}z = 0 \Rightarrow z = \frac{\rho^2}{\overline{a_i}}, \quad (i = 1, 2, \cdots, n.)$$

The point $\dfrac{\rho^2}{\overline{a_i}}$ $(i = 1, 2, \cdots, n)$ being inverse point as $a_i$ with respect to the circle $C$, $\dfrac{\rho^2}{\overline{a_i}}$ lies outside $C$. Therefore $F(z)$ has no zeros inside and on $C$. Thus $\log f(z)$ is analytic on and inside $C$. This implies that $\log|f(z)|$ is harmonic on and inside $C$. Applying Poisson formula for $\log|f(z)|$, we get

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right]\log\left|F(\rho e^{i\theta})\right| d\theta.$$

But on $C$, $|F(\rho e^{i\theta})| = |f(\rho e^{i\theta})|$. When $z$ is any point of $C$, $f(z) \neq 0$

$$\log\left|f(z)\prod_{i=1}^{n}\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| = \frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right]\log\left|f(\rho e^{i\theta})\right| d\theta$$

$$\log|f(z)| + \sum_{i=1}^{n}\log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| = \frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right]\log\left|f(\rho e^{i\theta})\right| d\theta$$

$$\log|f(z)| = -\sum_{i=1}^{n}\log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right]\log\left|f(\rho e^{i\theta})\right| d\theta,$$

when $f(z) \neq 0$. ■

**Note.** Put $z = 0$ in the above formula,

$$\log|f(0)| = -\sum_{i=1}^{n}\log\frac{\rho}{|a_i|} + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\rho e^{i\theta})\right| d\theta.$$

which is called Jensen's formula. Its importance lies in the fact that it relates the modulus $|f(z)|$

on a circle to the moduli of the zeros.

The Jensen and Poisson - Jensen formulas have important applications in the theory of entire functions.

## 9.3 Hadamard's Theorem.

Let $f(z)$ be an entire function with zeros $a_1, a_2, \cdots, a_n$, $a_n \to \infty$ as $n \to \infty$. For the sake of simplicity we will assume that $f(0) \neq 0$. If $h$ is the smallest positive integer such that $f$ can be represented in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

where $g(z)$ is a polynomial of degree $\leq h$. Then $f$ is of finite genus $h$. If there is no such representation the genus is infinite.

**Order of an entire function.** Denote by $M(r)$ the maximum of $|f(z)|$ on $|z| = r$. The *order* of the entire function $f(z)$ is defined by $\lim\limits_{r\to\infty} \log \dfrac{\log M(r)}{\log r}$ and we denote it by $\lambda$.

$$i.e., \ \lambda = \lim_{r\to\infty} \log \frac{\log M(r)}{\log r}.$$

According to this definition $\lambda$ is the smallest number such that

$$M(r) \leq e^{r^{\lambda+\varepsilon}}$$

for any given $\varepsilon > 0$ as soon as $r$ sufficiently large.

**Theorem 9.3.1.** *The genus and the order of an entire function satisfy the double inequality* $h \leq \lambda \leq h + 1$.

**Proof.** Assume that $f(z)$ is of finite genus $h$. Then $f(z)$ can be represented in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n}$$

where $g(z)$ is a polynomial of degree $\leq h$. Since $g(z)$ is a polynomial of degree $\leq h$, we have $e^{g(z)}$ is an entire function of order $\leq h$. Also the order of a product cannot exceed the orders of both factors. Hence it is sufficient to show that the canonical product is of order $\leq h+1$.
The convergence of the canonical product gives that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty.$$

Let

$$P(z) = \prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right),$$

where

$$E_h(u) = (1-u)\, e^{u + \frac{1}{2}u^2 + \cdots + \frac{1}{h}u^h}$$

and $E_0(u) = 1 - u$. We shall show that

$$\log |E_h(u)| \leq (2h+1)\, |u|^{h+1} \tag{9.3.1}$$

for all $u$.
If $|u| < 1$, we have by power series development

$$\begin{aligned}
\log E_h(u) &= \log(1-u) + \left(u + \frac{1}{2}u^2 + \cdots + \frac{1}{h}u^h\right) \\
&= -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots - \frac{u^h}{h} - \frac{u^{h+1}}{h+1} - \cdots + \left(u + \frac{1}{2}u^2 + \cdots + \frac{1}{h}u^h\right) \\
&= -\frac{u^{h+1}}{h+1} - \frac{u^{h+2}}{h+2} - \cdots \\
\left|\log E_h(u)\right| &\leq \frac{|u|^{h+1}}{h+1} + \frac{|u|^{h+2}}{h+2} + \cdots \\
&= \frac{|u|^{h+1}}{h+1}\left(1 + |u| + |u|^2 + \cdots\right) \\
\left|\log E_h(u)\right| &\leq \frac{|u|^{h+1}}{(h+1)(1-|u|)}
\end{aligned}$$

$$\begin{aligned}
Re[\log E_h(u)] &\leq \left|\log E_h(u)\right| \\
i.e., \log|E_h(u)| &\leq \left|\log E_h(u)\right|. \\
\therefore \log|E_h(u)| &\leq \frac{|u|^{h+1}}{(h+1)(1-|u|)} \\
&\leq \frac{|u|^{h+1}}{1-|u|}
\end{aligned}$$

$$(1-|u|)\log|E_h(u)| \leq |u|^{h+1} \tag{9.3.2}$$

For arbitrary $u$ and $h \geq 1$,

$$\frac{E_h(u)}{E_{h-1}(u)} = \frac{(1-u)e^{u+\frac{1}{2}u^2+\cdots+\frac{1}{h}u^h}}{(1-u)\,e^{u+\frac{1}{2}u^2+\cdots+\frac{1}{h-1}u^{h-1}}}$$

$$\left|\frac{E_h(u)}{E_{h-1}(u)}\right| = \left|e^{\frac{u^h}{h}}\right|$$

$$\left|\frac{E_h(u)}{E_{h-1}(u)}\right| \leq e^{|u|^h}$$

$$\log|E_h(u)| - \log|E_{h-1}(u)| \leq |u|^h$$

$$\therefore \log|E_h(u)| \leq \log|E_{h-1}(u)| + |u|^h. \tag{9.3.3}$$

Let us prove (16.3.2) by mathematical induction. For $h = 0$,

$$\begin{aligned}
(16.3.2) \Rightarrow \log|E_0(u)| &\leq |u| \\
Also\ \ |E_0(u)| &\leq 1-u \leq 1+|u| \\
i.e., \log|E_0(u)| &\leq \log(1+|u|) \leq |u|.
\end{aligned}$$

Therefore (16.3.2) is true when $h = 0$. Now we assume that (16.3.2) is true when we replace $h$ by $h-1$.

$$\log|E_{h-1}(u)| \leq (2h-1)\,|u|^h. \tag{9.3.4}$$

If $|u| \geq 1$, from (16.3.3), we have

$$\begin{aligned}
\log|E_h(u)| &\leq (2h-1)\,|u|^h + |u|^h \\
\log|E_h(u)| &\leq (2h+1)\,|u|^h
\end{aligned}$$

If $|u| < 1$, we have proved that

$$\begin{aligned}
(1-|u|)\log|E_h(u)| &\leq |u|^{h+1} \\
\log|E_h(u)| &\leq |u|\log|E_{h-1}(u)| + |u| + |u|^{h+1} \\
&= |u|\log|E_{h-1}(u)| + 2\,|u|^{h+1} \\
\log|E_h(u)| &\leq |u|\,(2h-1)\,|u|^h + 2\,|u|^{h+1}
\end{aligned}$$

$$\therefore \log|E_h(u)| \leq (2h+1)\,|u|^{h+1}.$$

This completes the proof of the induction.
Now consider the canonical product

$$\begin{aligned}
P(z) &= \prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right) \\
|P(z)| &= \prod_{n=1}^{\infty} \left|E_h\left(\frac{z}{a_n}\right)\right| \\
\log|P(z)| &= \sum_{n=1}^{\infty} \log\left|E_h\left(\frac{z}{a_n}\right)\right| \\
&= \sum_{n=1}^{\infty} \log|E_h(u)| \\
&\leq \sum_{n=1}^{\infty} (2h+1)\,|u|^{h+1} \\
&\leq \sum_{n=1}^{\infty} (2h+1)\frac{|z|^{h+1}}{|a_h|^{h+1}} \\
\log|P(z)| &\leq (2h+1)\,|z|^{h+1}\sum_{n=1}^{\infty}\frac{1}{|a_h|^{h+1}}
\end{aligned}$$

and it follows that $P(z)$ is at most of order $h + 1$.

$$\log |P(z)| \leq (2h+1)\,|z|^{h+1}\, C, \quad where \quad C = \sum_{n=1}^{\infty} \frac{1}{|a_h|^{h+1}}.$$

On $|z| = \gamma$, $\max |P(z)| = M(\gamma)$, we have

$$\begin{aligned}
\log M(\gamma) &\leq C(2h+1)\gamma^{h+1} \\
\lim_{\gamma\to\infty} \frac{\log\log M(\gamma)}{\log\gamma} &\leq \lim_{\gamma\to\infty} \left[ \frac{\log C(2h+1)}{\log r} + \frac{(h+1)\log r}{\log r} \right].
\end{aligned}$$

For the opposite inequality, assume that $f(z)$ is of finite order $\lambda$ and let $h$ be the largest integer $\leq \lambda$. Then $h + 1 > \lambda$. First we have to prove

$$\sum_{n=1}^{\infty} \frac{1}{|a_h|^{h+1}}$$

converges.

Let us denote $\nu(\rho)$ the number of zeros $a_n$ with $|a_n| \leq \rho$. In order to find an upper bound for $\nu(\rho)$, we apply Jensen's formula.

$$\log |f(0)| = -\sum_{n=1}^{\nu(\rho)} \log \left|\frac{\rho}{a_n}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log \left|f(\rho e^{i\theta})\right| d\theta$$

$$\begin{aligned}
\sum_{1}^{\nu(\rho)} \log \left|\frac{2\rho}{a_n}\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log \left|f(2\rho e^{i\theta})\right| d\theta - \log |f(0)| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log \left|f(\rho e^{i\theta})\right| d\theta, \quad since \quad \log |f(0)| > 0
\end{aligned}$$

$$\begin{aligned}
We\ know\ that\ \left|f(2\rho e^{i\theta})\right| &\leq M(2\rho) \leq e^{2\rho^{\lambda+s}} \\
\Rightarrow \log \left|f(2\rho e^{i\theta})\right| &= (2\rho)^{\lambda+s}
\end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2\rho e^{i\theta})\right| d\theta \leq \frac{(2\rho)^{\lambda+\epsilon}}{2\pi}\int_0^{2\pi} d\theta$$

$$\therefore \sum_{1}^{\nu(\rho)} \log\left|\frac{2\rho}{|a_n|}\right| = \frac{(2\rho)^{\lambda+\epsilon}}{2\pi}.2\pi$$

$$\therefore \sum_{1}^{\nu(\rho)} \log\left|\frac{2\rho}{|a_n|}\right| + \sum_{\nu(\rho+1)}^{\nu(2\rho)} \log\frac{2\rho}{|a_n|} \leq (2\rho)^{\lambda+\epsilon}$$

We have $|a_n| \leq \rho \Rightarrow \frac{2\rho}{|a_n|} \geq 2$.

$$\sum_{1}^{\nu(\rho)} \log\left|\frac{2\rho}{|a_n|}\right| \geq \sum_{1}^{\nu(\rho)} \log 2 = \log 2 \sum_{1}^{\nu(\rho)} = \nu(\rho)\log 2$$

$$\Rightarrow \nu(\rho)\log 2 = (2\rho)^{\lambda+\epsilon}$$

$$\therefore \lim_{\rho\to\infty} \frac{\nu(\rho)\log 2}{(2\rho)^{\lambda+\epsilon}} \to 0 \; for \;\; every \;\; \epsilon > 0$$

$$\therefore \lim_{\rho\to\infty} \frac{\nu(\rho)}{\rho^{\lambda+\epsilon}} \to 0 \; for \;\; every \;\; \epsilon > 0.$$

$$i.e., \nu(\rho) < \rho^{\lambda+\epsilon} \quad for \;\; every \;\; \rho.$$

We assume that the zeros $a_n$ are ordered according to absolute values.

$$|a_1| \leq |a_2| \leq \cdots |a_n| \leq \cdots$$

Then it is clear that,

$$n \leq \nu(|a_n|) < |a_n|^{\lambda+\epsilon}.$$

Then, it is clear that $n \leq \nu(|a_n|)^{\lambda+\epsilon} \Rightarrow \frac{1}{n^{\lambda+\epsilon}} < |a_n|$

$$\frac{1}{|a_n|} < \frac{1}{h^{\frac{h+1}{\lambda+\epsilon}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \sum_{n=1}^{\infty} \frac{1}{h^{\frac{h+1}{\lambda+\epsilon}}}.$$

Since $h \le \lambda$, we choose $s$, so that $\lambda + s < h + 1$.

$$\therefore \sum_{n=1}^{\infty} \frac{1}{h^{\frac{h+1}{\lambda+s}}} \text{ is convergent.}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} \text{ is convergent.}$$

Thus we have proved that $f(z)$ can be written in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$

where $g(z)$ is an entire function. It remains to prove that $g(z)$ is a polynomial of degree $\le h$. It is enough to prove that $g^{(h+1)}(z) = 0$. For this purpose it is easiest to use the Poisson - Jensen formula. Apply Poisson - Jensen formula to $f(z)$ :

$$\log|f(z)| = -\sum_{n=1}^{\nu(\rho)} \log\left|\frac{\rho^2 - \overline{a_n} z}{\rho(z - a_n)}\right| + \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z}\right) \log\left|f(\rho e^{i\theta})\right| d\theta.$$

If the operation $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ is applied to both sides, we obtain

$$\frac{f'(z)}{f(z)} = \sum_{1}^{\nu(\rho)} (z - a_n)^{-1} + \sum_{1}^{\nu(\rho)} \overline{a_n}(\rho^2 - \overline{a_n} z)^{-1} + \frac{1}{2\pi}\int_0^{2\pi} 2\rho e^{i\theta}(\rho e^{i\theta} - z)^{-2} \log\left|f(\rho e^{i\theta})\right| d\theta.$$

Differentiating with respect to $z$, for $h$ times, we get

$$D^{(h)}(z)\frac{f'(z)}{f(z)} = -h! \sum_{1}^{\nu(\rho)} (a_n - z)^{-h-1} + \sum_{1}^{\nu(\rho)} \overline{a_n}^{h+1}(\rho^2 - \overline{a_n} z)^{-h-1}$$
$$+ (h+1)! \frac{1}{2\pi}\int_0^{2\pi} 2\rho e^{i\theta}(\rho e^{i\theta} - z)^{-h-2} \log\left|f(\rho e^{i\theta})\right| d\theta. \quad (9.3.5)$$

It is our intention to let $\rho$ tend to $\infty$. In order to estimate the integral in (16.3.5), we observe that

$$\int_0^{2\pi} \rho e^{i\theta}(\rho e^{i\theta} - z)^{-h-2} d\theta = 0.$$

Therefore nothing changes if we subtract $\log M(\rho)$ from $\log|f|$. If $\rho > 2|z|$ it follows that the last term in (16.3.5) has a modulus at most equal to

$$(h+1)!\, 2^{h+2}\rho^{-h-1}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{M(\rho)}{|f(\rho e^{i\theta})|}d\theta,$$

for $\log\dfrac{M(\rho)}{|f(\rho e^{i\theta})|} \geq 0$. But

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f|\,d\theta \geq \log|f(0)|$$

by Jensen's formula, and $\rho^{-h-1}\log M(\rho) \to 0$, since $\lambda < h+1$. We conclude that the integral in (16.3.5) tends to zero.

The second sum in (16.3.5), the same preliminary inequality $\rho > 2|z|$ together with $|a_n| \leq \rho$ makes each term absolutely less than $\dfrac{2^{h+1}}{\rho}$, and the whole sum has modulus at most $2^{h+1}\nu(\rho)\rho^{-h-1}$. We have already proved that this tends to zero.

$$\therefore D^{(h)}\frac{f'(z)}{f(z)} = -h!\sum_{n=1}(a_n - z)^{-h-1}.$$

If we take $f(z) = e^{g(z)}P(z)$, we find that

$$g^{h+1}(z) = D^{(h)}\frac{f'}{f} - D^{(h)}\frac{P'}{P}.$$

By Weierstrass's theorem the quantity $D^{(h)}\dfrac{P'}{P}$ can be found by separate differentiation of each factor, and in this way we obtain precisely the right hand side of $D^{(h)}\dfrac{f'}{f}$. Consequently, $g^{h+1}(z)$ is identically zero, and $g(z)$ must be polynomial of degree $\leq h$. Hence the proof is complete. ■

**Corollary 9.3.1.** *An entire function of fractional order assumes every finite value infinitely many times.*

**Proof.** It is clear that $f$ and $f - a$ have the same order for any constant $a$. Therefore it is enough to prove that $f$ has finitely many zeros. If $f$ has only finite number of zeros we can divide by a polynomial and obtain a function of the same order without zeros. By the theorem it must be of the form $e^g$ where $g$ is polynomial. But it is evident that the order of $e^g$ is exactly the degree of $g$, and hence an integer. The contradiction proves the corollary. ■

# BLOCK-III

# UNIT 10

# The Riemann Zeta Function

**Objectives**

Upon completion of this Unit, students will be able to

- understand the concept of Riemann zeta function.
- extend the Riemann zeta function to the whole plane.
- prove the Riemann zeta function satisfies functional equation.

## 10.1 Introduction

The series $\sum_{n=1}^{\infty} n^{-\sigma}$ converges uniformly for all real $\sigma$ greater than or equal to a fixed $\sigma_0 > 1$ *i.e.*, $\sigma > \sigma_0 > 1$. Here $s = \sigma + it$.

$$\sigma = Re\, s$$

$$\sigma < Re\, |s|$$

$$n^{-\sigma} > n^{-|s|}$$

$$As \ \frac{1}{|n^s|} \leq \frac{1}{n^{\sigma}}$$

the series $\sum_{n=1}^{\infty} n^{-\sigma}$ is a majorant of these series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-\sigma}, \ s = \sigma + it.$$

The series $\zeta(s)$ is convergent and represents an analytic function of $s$ in the half plane $Re\, s > 1$. The function $\zeta(s)$ is known as Riemann zeta function. It plays a central role in the applications of complex analysis to number theory.

## 10.2 The Product Development

The number - theoretic properties of $\zeta(s)$ are inherent in the following connection between the $\zeta-$ function and the ascending sequence of primes $p_1, p_2, \cdots, p_n, \cdots$

**Theorem 10.2.1.** *For $\sigma = Re\, s > 1$,*

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}(1 - p_n^{-s}) \tag{10.2.1}$$

*where $p_1.p_2, \cdots, p_n, \cdots.$ are ascending sequence of primes.*

**Proof.** The infinite product $\prod_{n=1}^{\infty}(1 - p_n^{-s})$ converges absolutely if and only if $\sum_{n=1}^{\infty} p_n^{-s}$ converges. Under the assumption $\sigma > 1$, it is seen at once that

$$\begin{aligned}
\zeta(s)(1 - 2^{-s}) &= \zeta(s) - 2^{-s}\zeta(s) \\
&= \sum_{n=1}^{\infty} n^{-s} - 2^{-s}\sum_{n=1}^{\infty} n^{-s} \\
&= \sum_{n=1}^{\infty}[n^{-s} - (2n)^{-s}] \\
\zeta(s)(1 - 2^{-s}) &= \sum_{m} m^{-s}
\end{aligned}$$

where $m$ runs through the odd integers.
By the same reasoning,

$$\begin{aligned}
\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) &= \sum_{m} m^{-s} - 3^{-s}\sum_{m} m^{-s} \\
&= \sum_{m} m^{-s} - \sum_{m}(3m)^{-s} \\
\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) &= \sum_{m} m^{-s}
\end{aligned}$$

where the sum runs through the integers that are neither divisible by 2 nor by 3. More generally,

$$\zeta(s)(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p_N^{-s}) = \sum_{m=1}^{\infty} m^{-s} \qquad (10.2.2)$$

where the sum of the right being over all integers that contains none of the prime factors $2, 3, 5, 7, \cdots p_N$.

$$\zeta(s)\prod_{n=1}^{N}(1-p_n^{-s}) = 1 + p_{N+1}^{-s} + p_{N+2}^{-s} + \cdots$$

$$\zeta(s)\prod_{n=1}^{N}(1-p_n^{-s}) = 1 + \sum_{n=N+1}^{\infty} p_n^{-s}$$

the first term in the sum is 1 and the next term is $p_{N+1}^{-s}$.

$$\sum_{m=1}^{\infty} m^{-s} = 1 + p_{N+1}^{-s} + p_{N+2}^{-s} + \cdots$$

Therefore the sum of all terms except the first tends to zero as $N \to \infty$. Hence

$$\lim_{N\to\infty} \zeta(s)\prod_{n=1}^{\infty}(1-p_n^{-s}) = 1.$$

This proves the theorem. ■

**Result.** The number of primes is infinite.

**Proof.** Suppose on the contrary that, the number of primes is finite. Let the largest prime be $p_N$. Then (10.2.2) reduces to

$$\zeta(s)(1-2^{-s})(1-3^{-s})\cdots(1-p_N^{-s}) = 1.$$

Replace $s$ by $\sigma$ we get

$$\zeta(s)(1-2^{-\sigma})(1-3^{-\sigma})\cdots(1-p_N^{-\sigma}) = 1.$$

As $\sigma \to 1$, we have

$$\zeta(1)(1-2^{-1})(1-3^{-1})\cdots(1-p_N^{-1}) = 1$$
$$\zeta(1) = \frac{1}{\prod_{N=1}^{n}\left(1-\frac{1}{p_N}\right)} \infty$$

$\sum_{n=1}^{\infty}\frac{1}{n}$ $\infty$ which is a contradiction to the fact that $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.
Therefore our assumption is wrong. Hence the number of prime is infinite. ■

## 10.3 Extension of $\zeta(s)$ to the whole plane

The Gamma function is

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx \; for \; \sigma > 1.$$

On replacing $x$ by $nx$, in the integral, we obtain

$$\begin{aligned}
\Gamma(s) &= \int_0^{\infty}(nx)^{s-1}e^{-nx}d(nx)\\
n^{-s}\Gamma(s) &= \int_0^{\infty}x^{s-1}e^{-nx}dx\\
\sum_{n=1}^{\infty} n^{-s}\Gamma(s) &= \sum_{n=1}^{\infty}\int_0^{\infty} x^{s-1}e^{-nx}dx\\
&= \int_0^{\infty}x^{s-1}\cdot\sum_{n=1}^{\infty}e^{-nx}dx\\
\zeta(s)\Gamma(s) &= \int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx
\end{aligned}$$

Because $\sigma > 1$, the integral is absolutely convergent at both ends and this justifies the interchange of integration and summation.

**Theorem 10.3.1.** *For $\sigma > 1$,*

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z-1}dz \qquad (10.3.1)$$

*where $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi <$*

$Im \log(-z) < \pi$.

**Proof.** Here there are two infinite paths $C_n$ and $C$ both begins and ends near the positive real axis. Let us consider only $C$, its precise shape is irrelevent as large as the radius of the circle about the origin is less than $2\pi$. We have

$$\int_C \frac{(-z)^{s-1}}{e^z - 1} dz = \int_{circle} \frac{(-z)^{s-1}}{e^z - 1} dz - \int_{Backpositive} \frac{(-z)^{s-1}}{e^z - 1} dz + \int_{Frontpositive} \frac{(-z)^{s-1}}{e^z - 1} dz$$

The integral is obviously convergent. By Cauchy's theorem its value does not depend on the shape of $C$ as long as $C$ does not enclose any multiples of $2\pi i$. In particular, we are free to let $r \to 0$. It is readily seen that the integral over the circle tends to zero with $r$.

$$i.e., \int_{circle} \frac{(-z)^{s-1}}{e^z - 1} dz \to 0 \ as \ r \to 0.$$

$$\therefore \int_C \frac{(-z)^{s-1}}{e^z - 1} dz = -\int_{Backpositive} \frac{(-z)^{s-1}}{e^z - 1} dz + \int_{Frontpositive} \frac{(-z)^{s-1}}{e^z - 1} dz$$

On the upper edge,

$$(-z)^{s-1} = (-1)^{s-1}(z)^{s-1} = e^{-i\pi(s-1)} x^{s-1}$$

and the lower edge

$$(-z)^{s-1} = (-1)^{s-1}(z)^{s-1} = e^{i\pi(s-1)} x^{s-1}$$

∎

$$\begin{aligned}
\int_C \frac{(-z)^{s-1}}{e^z - 1} dz &= -\int_0^\infty \frac{e^{-i\pi(s-1)}}{e^x - 1} x^{s-1} dx + \int_0^\infty \frac{e^{i\pi(s-1)}}{e^x - 1} x^{s-1} dx \\
&= -e^{-i\pi(s-1)}\zeta(s)\Gamma(s) + e^{i\pi(s-1)}\zeta(s)\Gamma(s) \\
&= -\zeta(s)\Gamma(s) 2i \sin \pi(s-1) \\
&= -\zeta(s)\Gamma(s) 2i \sin \pi s \\
&= -\frac{2\pi i}{\Gamma(1-s)}\zeta(s), \quad since \ \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}
\end{aligned}$$

$$\therefore \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$$

**Note.** The importance of the formula (14.3.1) lies in the fact that the right - hand side is defined and meromorphic for all values of $s$, so the formula can be used to extend $\zeta(s)$ to a meromorphic function in the whole plane. It is indeed quite obvious that the integral in (14.3.1) is an entire function of $s$, while $\Gamma(1-s)$ is meromorphic with poles at $s = 1, 2, \cdots$.

**Corollary 10.3.1.** *The $\zeta-$ function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at $s = 1$ with the residue $1$.*

**Proof.** The integral in (14.3.1) is an entire function of $s$, in the whole plane, $\Gamma(1-s)$ is meromorphic with poles at $s = 1, 2, \cdots$. For,

$$\Gamma(1-s) = \frac{e^{-\gamma(1-s)}}{1-s} \prod_{n=1}^{\infty} \left(1 + \frac{1-s}{n}\right)^{-1} e^{\frac{1-s}{n}}$$

and the poles of $\Gamma(1-s)$ are given by $1 - s = 0$ and $1 + \frac{1-s}{n} = 0$

$$i.e., s = 1 \text{ and } s = n+1, \quad n = 1, 2, \cdots$$

$\therefore s = 1, 2, \cdots$ are the poles of $\Gamma(1-s)$. Since $\zeta(s)$ is already known to be analytic for $\sigma > 1$, the poles at the integers $n \geq 2$ must cancel against the zeros of the integral. At $s = 1$, $-\Gamma(1-s)$ has a simple pole.

To find the residue of $-\Gamma(1-s)$ at $s = 1$ :

$$\begin{aligned}
Res.of \ \Gamma(1-s)|_{s=1} &= \lim_{s\to 1}(s-1)\Gamma(1-s) \\
&= \lim_{s\to 1} \frac{e^{-\Gamma(1-s)}}{1-s} \prod_{n=1}^{\infty} \left(1 + \frac{1-s}{n}\right)^{-1} e^{\frac{1-s}{n}} \\
Res.of \ \Gamma(1-s)|_{s=1} &= -1 \\
\therefore Res.of \ (-\Gamma(1-s)) &= 1.
\end{aligned}$$

On the other hand,

$$\frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i} \ Sum \ of \ the \ residues \ of \ f(z),$$

where $f(z) = \frac{1}{e^z - 1}$. The poles of $f(z)$ are given by

$$e^z = 0 \Rightarrow z = 2n\pi i.$$

The pole $z = 0$ lies inside $C$.

$$\begin{aligned} Res.\,of\ f(z)|_{z=0} &= \lim_{z\to 0} z\, f(z) \\ &= \lim_{z\to zero} \frac{z}{z + \frac{z^2}{2!} + \cdots} \\ Res.\,of\ f(z)|_{z=0} &= 1 \end{aligned}$$

Hence the zeta function can be extended to a meromorphic function in the whole plane whose only pole is at $s = 1$ with residue $1$. ■

**Note.** The values $\zeta(-n)$ at the negative integers and zero can be evaluated explicitly.
We have

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k!)} z^{2k-1} \tag{10.3.2}$$

From (14.3.1)

$$\zeta(-n) = \frac{(-1)^n\, n!}{2\pi i} \int_C \frac{(-z)^{-(n+1)}}{e^z - 1}\, dz$$

Hence $\zeta(-n)$ is equal to $(-1)^n n!$ times the coefficient of $z^n$ in (14.3.2)

$$i.e., \zeta(-n) = (-1)^n n!$$

coefficient of $z^n$. in (14.3.2). We also have

$$\zeta(0) = \frac{1}{2}\zeta(-2m) = 0$$

and

$$\zeta(-2m+1) = \frac{(-1)^n B_n}{2^m}$$

for positive integers $m$. We also have the following values: $\zeta(0) = -\frac{1}{2}$, $\zeta(-2m) = 0$. The points $-2m$ are called the *trivial zeros* of the $\zeta-$ function.

## 10.4 The Functional Equation

In the half plane $\sigma > 1$ the $\zeta-$ function is given explicitly by the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and it therefore subject to the estimate $|\zeta(s)| \leq \zeta(\sigma)$. Riemann recognised that there is a rather simple relationship between $\zeta(s)$ and $\zeta(1-s)$. As a consequence, one has good control of the behavior of the $\zeta-$ function also in the half plane $\sigma < 0$. We shall reproduce one of the standard proofs of the *functional equations*, as it is commonly called.

**Theorem 10.4.1.** *Prove that the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\,\zeta(1-s).$$

**Proof.** We assume that the square part lies on the lines $t = \pm(2n+1)\pi$ and $\sigma = \pm(2n+1)\pi$. The cycle $C_n - C$ has winding number one about the points $\pm 2m\pi i$ with $m = 1, 2, \cdots, n$. At these points the function $\frac{(-z)^{s-1}}{e^z - 1}$ has simple pole with residues $(\pm 2m\pi i)^{s-1}$. For,

$$e^z - 1 = 0 \Rightarrow z = \pm 2m\pi i,\quad m = 0, \pm 1, \cdots \Rightarrow \frac{(-z)^{s-1}}{e^z - 1}$$

has poles at $z = \pm 2m\pi i$, $m = 1, 2, \cdots, n$.

$$\begin{aligned} Res.f(z)|_{z=\pm 2m\pi i} &= \lim_{z\to \pm 2m\pi i} \frac{(-z)^{s-1}}{d(e^z - 1)} \\ Res.f(z)|_{z=\pm 2m\pi i} &= (\pm 2m\pi i)^{s-1} \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\,dz &= \sum_{m=1}^{n} \left[(-2m\pi i)^{s-1} + (2m\pi i)^{s-1}\right] \\ &= \sum_{m=1}^{n} (2m\pi)^{s-1}\left[(e^{-i\frac{\pi}{2}})^{s-1} + (e^{i\frac{\pi}{2}})^{s-1}\right] \\ &= \sum_{m=1}^{n} (2m\pi)^{s-1}\, 2\cos(s-1)\frac{\pi}{2} \\ \frac{1}{2\pi i}\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\,dz &= 2\sum_{m=1}^{n} (2m\pi)^{s-1}\cos\left(\frac{\pi}{2} - \frac{\pi s}{2}\right) \end{aligned}$$

$$\frac{1}{2\pi i}\int_{C_n-C}\frac{(-z)^{s-1}}{e^z-1}\,dz = 2\sum_{m=1}^{n}(2m\pi)^{s-1}\sin\frac{\pi s}{2} \tag{10.4.1}$$

We divide $C_n$ into $C_n' + C_n''$ where $C_n'$ is the part of the square and $C_n''$ the part outside the square. It is easy to see that $|e^z - 1|$ is bounded below on $C_n$ by a fixed positive constant, independent of $n$ while $(-z)^{s-1}$ is bounded by a multiple of $n^{\sigma-1}$. The length of $C_n'$ is of the order $n$ and we find that

$$\left|\int_{C_n'}\frac{(-z)^{s-1}}{e^z-1}\,dz\right| \le An^{\sigma}$$

for some constant $A$.

If $\sigma < 0$, the integral over $C_n'$ will tends to $0$ as $n \to \infty$ and the same is true of the integral over $C_n''$.

$$\therefore \int_{C_n} = \int_{C_n'+C_n''} \to 0.$$

$$\frac{1}{2\pi i}\int_{C_n-C}\frac{(-z)^{s-1}}{e^z-1}\,dz = \frac{1}{2\pi i}\int_{-}\frac{(-z)^{s-1}}{e^z-1}\,dz = -\frac{1}{2\pi i}\int_{C}\frac{(-z)^{s-1}}{e^z-1}\,dz$$

$$\frac{1}{2\pi i}\int_{C_n-C}\frac{(-z)^{s-1}}{e^z-1}\,dz = \frac{\zeta(s)}{\Gamma(1-s)}$$

$$(17.4.1) \Rightarrow \frac{\zeta(s)}{\Gamma(1-s)} = 2\sum_{m=1}^{n}(2m\pi)^{s-1}\sin\frac{\pi s}{2}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}\frac{\zeta(s)}{\Gamma(1-s)} &= 2\sum_{m=1}^{\infty}(2m\pi)^{s-1}\sin\frac{\pi s}{2}\\ &= 2^s\pi^{s-1}\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}m^{s-1}.\end{aligned}$$

For $\sigma \geq 0$, the series $\sum_{m=1}^{\infty} m^{-(1-s)} = \zeta(1-s)$

$$\therefore\ \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

∎

**Note.** Using the identity $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}$,

$$\begin{aligned}\zeta(s) &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\frac{1}{\Gamma(s)}\frac{\pi}{\sin \pi s}\zeta(1-s)\\ &= 2^{s-1}\pi^{s}\frac{1}{\cos\frac{\pi s}{2}}\frac{1}{\Gamma(s)}\zeta(1-s)\end{aligned}$$

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,\zeta(s). \qquad (10.4.2)$$

The content of Theorem 3.3.1 can also be expressed in the following form:

**Corollary 10.4.1.** *The function*

$$\zeta(s) = \frac{1}{2}s(1-s)\,e^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

*is entire and satisfies* $\zeta(s) = \zeta(1-s)$.

**Proof.** Given

$$\zeta(s) = \frac{1}{2}s(1-s)\,e^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Since the factor $1-s$ cancels with the poles of $\zeta(s)$. Also the poles of $\Gamma\left(\frac{s}{2}\right)$ cancel against the trivial zeros of $\zeta(s)$. Hence $\zeta(s)$ is an entire function. By use of (17.4.2) the assertion $\zeta(s) = \zeta(1-s)$ translates to

$$\frac{1}{2}s(1-s)\,e^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}(1-s)(1-(1-s))\,e^{-\frac{(1-s)}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}\,2^{1-s}\,\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

$$\therefore\ \Gamma\left(\frac{1-s}{2}\right)\Gamma(s)\cos\left(\frac{\pi s}{2}\right) = 2^{s-1}\pi^{\frac{1}{2}}\Gamma\left(\frac{s}{2}\right).$$

Because of the relation

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)=\frac{\pi}{\cos\frac{\pi s}{2}},$$

the last equation is equivalent to

$$\pi^{\frac{1}{2}}\Gamma(s)=2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)$$

and this equation is called Legendre's duplication formula. The corollary is proved. ■

**Result.** Prove that the order of $\zeta(s)$ is one.

**Proof.** Since $\zeta(s)=\zeta(1-s)$, it is sufficient to estimate $|\zeta(s)|$ for $\sigma\geq\frac{1}{2}$. As a consequence of Stirling's formula, we have

$$\log\left|\Gamma\left(\frac{s}{2}\right)\right|\leq A\,|s|\log|s|$$

for some constant $A$ and large $|s|$, and this estimate precise for real values of $s$. So to prove that the order is equal to one, we can show that $\zeta(s)$ relatively small when $\sigma\geq\frac{1}{2}$.

Let $[x]$ denote the largest integer $\leq x$. Assume first that $\sigma>1$. Then we have

$$\begin{aligned}\int_N^\infty [x]\,x^{-s-1}\,dx &= \sum_{n=N}^{\infty} n\int_n^{n+1} x^{-s-1}dx\\ &= s^{-1}\sum_{n=N}^{\infty} n\left(n^{-s}-(n+1)^{-s}\right)\\ &= s^{-1}\left(N^{-s+1}+\sum_{n=N+1}^{\infty} n^{-s}\right)\end{aligned}$$

It follows that

$$\zeta(s)=\sum_{n=1}^{N} n^{-s}+\frac{1}{s-1}N^{1-s}-s\int_N^\infty (x-[x])x^{-s-1}dx \tag{10.4.3}$$

For $\sigma>1$ where the integral on the right hand side converges and the equality will therefore remain valid for $\sigma>0$. Incidently, (17.4.3) exhibits the pole at $s=1$ with residue $1$.

For $\sigma\geq\dfrac{1}{2}$, (17.4.3) yields an estimate of the form

$$|\zeta(s)|\leq N+A\,|N|^{-\frac{1}{2}}\,|s|$$

valid for large $|s|$ with $A$ independent of $s$ and $N$. By choosing $N$ as the integer closest to $|s|^{\frac{2}{3}}$, we find that $|\zeta(s)|$ is bounded by a constant times $|s|^{\frac{2}{3}}$. Therefore this factor does not influence the order. ■

## 10.5 The Zeros of the Zeta Function

We know that the product development of $\zeta(s)$ is

$$\zeta(s) = \frac{1}{\prod_{n=1}^{\infty}(1 - p_n^{-s})}$$

for $\sigma = Res > 1$. It follows from this product development that $\zeta(s)$ has no zeros in the half plane $\sigma > 1$. With this information the functional equation implies that the only zeros in the half plane $\sigma < 0$ are the trivial ones.

In other words, all nontrivial zeros lie in the so-called *critical strip* $0 \leq \sigma \leq 1$. The famous Riemann conjecture asserts that all nontrivial zeros lie on the *critical line* $\sigma = \frac{1}{2}$. There are no zeros on $\sigma = 1$ and $\sigma = 0$. Let $N(T)$ be the number of zeros with $0 \leq t \leq T$. For the information of the reader we state without proof that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

# BLOCK-III

# UNIT 11

# Normal Families

**Objectives**

Upon completion of this Unit, students will be able to

x understand the concept of normality.

x prove Arzela's theorem.

x identify the families of analytic functions.

## 11.1 Introduction

A function can be regarded as a point in a space and as such there is no difference between a set of points and of functions. In order to make a clear distinction we shall nevertheless prefer to speak of *families* of functions, and usually we assume that all functions in a family are defined on the same set. We are primarily interested in families of analytic functions, defined in a fixed region. The aim is to study the convergence properties of within such families.

## 11.2 Equicontinuity

Let F denote a family of functions defined in a fixed region $\Omega$ of the complex plane and with values in a metric space $(S, d)$ where $d$ is the distance function in $S$. Let us review the definition of continuous function $f$ with values in a metric space.

**Definition 11.2.1.** A function $f$ is continuous at $z_0$ if for every $s > 0$ there exists a $\delta > 0$ such that

$$d(f(z),\ f(z_0)) < s, \quad \textit{whenever} \quad |z - z_0| < \delta.$$

$f$ is said to be uniformly continuous if we can choose $\delta$ independent of $z_0$.

**Definition 11.2.2.** The function in a family $F$ are said to be equicontinuous on a set $E \subset \Omega$ if and only if, if for each $s > 0$, there exists a $\delta > 0$ such that

$$d(f(z), f(z_0)) < s, \quad whenever \quad |z - z_0| < \delta,$$

and $z_0, z \in E$, simultaneously for all functions $f \in F$.

**Note.** In the above definition, we observe that each $f$ is an equicontinuous family is itself uniformly continuous on $E$.

**Definition 11.2.3.** A family $F$ is said to be *normal* in $\Omega$ if every sequence $\{f_n\}$ of functions $f_n \in F$ contains a subsequence which converges uniformly on every compact subset of $\Omega$.

**Note.** This definition does not require the limit functions of the convergent subsequences to be members of $F$.

## 11.3 Normality and Compactness

We shall prove that in the family of functions there exists the convergence with respect to the distance $\rho$ is equivalent to the uniform convergence on compact sets.
i.e., convergence with respect to $\rho$ if and only if uniform convergence on compact set.

**Proof.** To prove this, we need the following observations;

(i) An exhaustion of $\Omega$ by an increasing sequence of compact sets $E_k \subset \omega$. This means that every compact subset $E$ of $\Omega$ shall be contained in an $E_k$. The construction is possible in many ways: To be specific, let $E_k$ consist of all points in $\Omega$ at distance $\leq k$ rom the origin, and at distance $\geq \frac{1}{k}$ from the boundary $\partial\Omega$. It is clear that each $E_k$ is bounded and closed, and hence compact. Any compact set $E \subset \Omega$ is bounded and at positive distance from $\partial\Omega$; therefore it is contained in an $E_k$.

(ii) Let $f$ and $g$ be any two functions on $\Omega$ with values in $S$. We shall define a distance $\rho(f, g)$ between these functions, not to be confused with the distances $d(f(z)\ g(s))$ between their values. To do this, we first replace $d$ by the distance function

$$\delta(a, b) = \frac{d(a, b)}{1 + d(a, b)}$$

which also satisfies the triangle inequality and all distances lie under a fixed bound and is bounded and so $S$ is bounded.

(iii) Next, we set

$$\delta_k(f, g) = \sup_{z \in E_k} \delta(f(z), g(z))$$

which describes the distance between $f$ and $g$ on $E_k$.

(iv) Finally we define

$$\rho(f, g) = \sum_{k=1}^{n} \delta_k(f, g)\, 2^{-k} \tag{11.3.1}$$

$\rho(f, g)$ is finite and satisfies all the conditions for a distance function.

Now, we prove the result. Suppose that $f_n \to f$ in the sense of $\rho$ distance.

$$\rho(f_n, f) < \varepsilon, \quad for\ sufficiently\ large\ n$$

$$(11.3.1) \quad \sum_{k=1}^{\infty} \delta_k(f_n, f)\, 2^{-k} < \varepsilon$$

$$\delta_k(f_n, f) < 2^k \varepsilon$$

$$\sup_{z \in E_k} \{\delta(f_n(z), f(z))\} < 2^k \varepsilon$$

$$\delta(f_n(z), f(z)) < 2^k \varepsilon$$

$$\frac{d(f_n(z), f(z))}{1 + d(f_n(z), f(z))} < 2^k \varepsilon$$

$$d(f_n(z), f(z)) < \frac{2^k \varepsilon}{1 - 2^k \varepsilon}$$

This implies that $f_n(z) \to f(z)$ uniformly on $E_k$ with respect to $\delta$ metric, but hence also with respect to the $d-$ metric. Since every compact $E$ is contained in an $E_k$ it follows that the convergence is uniform on $E$.

Conversely, suppose that $f_n$ converges uniformly to every compact set.

$$d(f_n(z),\ f(z)) < s\ \ \forall z\ \in E \subset \Omega.$$

Since $\Omega = \cup E_n$ there exists an $n_0$ such that $E \subset E_n$ for every $n \geq n_0$.

$$\begin{aligned} i.e,\ \ d(f_n(z),\ f(z)) &\to 0\ \ \forall z \in E_k,\ \ n \geq n_0 \\ \delta(f_n(z),\ f(z)) &\to 0\ \ \forall z \in E_k,\ \ n \geq n_0 \\ \delta_k(f_n,\ f) &\to 0\ \ \forall k \\ \sum_{k=1} \delta_k(f_n,\ f)2^{-k} &\to 0. \end{aligned}$$

Hence $\rho(f_n,\ f) \to 0$. That is the convergence with respect to the distance $\rho$. Hence we proved that the convergence with respect to the distance $\rho$ is equivalent to the uniform convergence on compact sets. ■

**Recall: Bolzano - Weierstrass Theorem** *A metric space is compact if and only if every sequence has a convergent subsequence.*

**Theorem 11.3.1.** *A family* $F$ *is compact if and only if its closure* $\overline{F}$ *with respect to the distance function* (11.3.1) *is compact.*

**Proof.** Let $F$ be normal. We have to prove that $\overline{F}$ is compact.

Let $\{f_n\}$ be a sequence of functions in $F$ then $f_n \in \overline{F}$ for every $n$. This implies that $f_n$ is a sequence of function in $F$. Since $F$ is normal, every sequence $\{f_n\}$ of functions in $F$ will have a subsequence $\{f_{n_k}\}$ which converges uniformly on every compact subset of $\omega$.

Therefore by Bolzano - Weierstrass theorem, $\overline{F}$ s compact.

Conversely, assume that $\overline{F}$ is compact.

Consider a sequence $\{f_n\}$ of function in $F$. Then $\{f_n\}$ is a sequence of functions in $\overline{F}$. Since $\overline{F}$ is compact, by Bolzano - Weierstrass theorem, $\{f_n\}$ has a convergent subsequence $\{f_{n_k}\}$ with respect to the distance function $\rho$.

$\therefore$ the convergence is uniform, since the convergence with respect to $\rho$ if and only if uniform convergence on compact set.

Hence $\mathcal{F}$ is normal. ■

**Definition 11.3.1.** $\mathcal{F}$ is relatively compact if $\bar{\mathcal{F}}$ is compact.

**Note.** Normal and relatively compact are same.

**Remark.** If $S$ is complete then $\mathcal{F}$ is normal if and only if it is totally bounded.

**Definition 11.3.2.** A set $E$ is totally bounded if for every $\varepsilon > 0$ $E$ can be covered by finitely many balls of radius $\varepsilon$

The following theorem serves to state the condition of total boundedness in terms of the original metric on $S$ rather than in terms of the auxiliary metric $\rho$.

**Theorem 11.3.2.** *The family $\mathcal{F}$ is totally bounded if and only if to every compact set $E \subset \Omega$ and every $\varepsilon > 0$ it is possible to find $f_1, f_2, \cdots f_n \in \mathcal{F}$ such that $f \in \mathcal{F}$ satisfies $d(f, f_j) < \varepsilon$ on $E$ for some $f_j$.*

**Proof.** Assume that $\mathcal{F}$ is totally bounded, then for any $\varepsilon > 0$ there exists $f_1, f_2, \cdots, f_n$ such that for any $f \in \mathcal{F}$,

$$\rho(f, f_j) < \varepsilon \ for \ \ some \ \ f_j$$

But, we have

$$\begin{aligned}
\rho(f, f_j) &= \sum_{k=1}^{\infty} \delta_k(f, f_j) 2^{-k} \\
\therefore \sum_{k=1}^{\infty} \delta_k(f, f_j) 2^{-k} &< \epsilon \\
\delta_k(f, f_j) 2^{-k} &< \epsilon \ \ for \ \ each \ \ k \\
\sup \delta(f(z), f_j(z)) &< 2^k \epsilon \\
\delta(f(z), f_j(z)) &< 2^k \epsilon \\
\frac{d(f(z), f_j(z))}{1 + d(f(z), f_j(z))} &< 2^k \epsilon \\
d(f(z), f_j(z)) &< \frac{2^k \epsilon}{1 - 2^k \epsilon} = \epsilon \\
d(f(z), f_j(z)) &< \epsilon, \ \forall z \in E \ \ and \ \ for \ \ some \ \ f_j \\
d(f, f_j) &< \epsilon \ \ on \ \ for \ \ some \ \ f_j.
\end{aligned}$$

Assume that to every compact set $E$ and every $\epsilon > 0$ it is possible to find $f_1, f_2, \cdots, f_n \in \mathcal{F}$ such that every $f \in \mathcal{F}$ satisfies

$$d(f, f_j) < \epsilon \ \ on \ \ E \ \ for \ \ some \ \ f_j.$$

To prove that $\mathcal{F}$ is totally bounded. That is to prove that $\rho(f, f_j) < \epsilon$ for some $f_j$. We choose $k_0$ such that

$$2^{-k_0} < \frac{\epsilon}{2}.$$

By assumption, we can find $f_1, f_2, \cdots, f_n \in \mathcal{F}$ such that any $f \in \mathcal{F}$ satisfies one of the inequalities

$$\delta(f, f_j) = \frac{d(f, f_j)}{1 + d(f, f_j)} \leq d(f, f_j) < \frac{\epsilon}{2k_0} \ \ on \ \ E_{k_0}$$

Hence it follows that

$$\delta_k(f, f_j) < \frac{\epsilon}{2k_0} \ \ for \ k \geq k_0.$$

But we also know, for $k > k_0$, $\delta_k(f, f_j) < 1$

$$\begin{aligned}
\therefore \rho(f, f_j) &= \sum_{k=1}^{k_0} \delta_k(f, f_j)2^{-k} + \sum_{k=k_0+1}^{\infty} \delta_k(f, f_j)2^{-k} \\
&< \sum_{k=1}^{k_0} \delta_k(f, f_j) + \sum_{k=k_0+1}^{\infty} \delta_k(f, f_j)2^{-k} \\
&< \sum_{k=1}^{k_0} \frac{\varepsilon}{2k_0} + \sum_{k=k_0+1}^{\infty} 2^{-k} \\
&= \frac{\varepsilon}{2} + \frac{1}{2^{k_0}}
\end{aligned}$$

*Hence* $\rho(f, f_j) < \varepsilon$ *for some* $f_j$.

∎

## 11.4 Arzela's Theorem

We shall now study the relationship between Definition 4.1.2 and Definition 4.1.3. The connection is established by a famous and extremely useful theorem known as *Arzela's theorem*(or the Arzela - Ascoli theorem)

**Theorem 11.4.1.** *A family $\mathfrak{F}$ of continuous functions with values in a metric space $S$ is normal in the region $\Omega$ of the complex plane if and only if*
*(i) $\mathfrak{F}$ is equicontinuous on every compact set $E \subset \Omega$;*
*(ii) for any $z \in \Omega$ the values $f(z)$, $f \in \mathfrak{F}$, lie in a compact subset of $S$.*

**Proof.** Necessary part:

Assume that $\mathfrak{F}$ is normal.

To prove (i):

Since $\mathfrak{F}$ is normal, its closure $\overline{\mathfrak{F}}$ is compact. Therefore $\overline{\mathfrak{F}}$ is totally bounded. This implies that $\mathfrak{F}$ is totally bounded. To every compact subset $E \subset \Omega$ and every $\varepsilon > 0$ it is possible to find

$f_1, f_2, \cdots, f_n \in \mathcal{F}$ such that every $f \in \mathcal{F}$ satisfies

$$d(f, f_j) < \epsilon \ \ on \ \ E \ \ for \ \ some \ \ f_j.$$

Since each $f_j$ is continuous on a compact set $E$ and hence each $f_j$ is uniformly continuous on $E$. Hence we can find $\delta > 0$ such that

$$d(f_j(z), f_j(z_0)) < \epsilon \ \ for \ \ z, z_0 \in E \ \ and \ \ |z - z_0| < \delta,$$

$j = 1, 2, \cdots, n$. Therefore for any given $f \in \mathcal{F}$ and corresponding $f_j$, we obtain

$$\begin{aligned} d(f(z), f(z_0)) &\leq d(f(z), f_j(z)) + d(f_j(z), f_j(z_0)) + d(f_j(z_0), f(z_0)) \\ &< \epsilon + \epsilon + \epsilon \\ d(f(z), f(z_0)) &< 3\epsilon, \ \ whenever \ \ |z - z_0| < \delta. \end{aligned}$$

Therefore $\mathcal{F}$ is equicontinuous on $E \subset \Omega$.

To prove (ii):

To prove $\{f(z) : f \in \mathcal{F}\}$ $\forall z \in \Omega$ lies in a compact set.

i.e., we have to show that the closure of the set formed by the values $f(z)$, $f \in \mathcal{F}$ is compact.

Let $\{w_n\}$ be a sequence in this closure. To each $w_n$ we can find $f_n \in \mathcal{F}$ so that

$$d(f_n(z), w_n) < \frac{1}{n}$$

for positive integer $n$. Since $\{f_n\} \in \mathcal{F}$, by the definition of normality, $\{f_n(z)\}$ has a subsequence $\{f_{n_k}\}$ which converges uniformly on every compact subset of $\Omega$.

$\therefore$ the corresponding subsequence $\{w_{n_k}\}$ converges to the same value. Hence an infinite sequence $w_n$ converges. The closure of the image set is compact. Therefore the image set is lies in a compact set.

Sufficiency part:

The sufficiency of (i) together with (ii) is proved by Cantor's famous diagonal process. Let condition (i) and (ii) be true simultaneously. To prove that $\mathcal{F}$ is normal. We shall prove that every sequence $\{f_n\}$ of functions $f_n \in \mathcal{F}$ contains a subsequence $\{f_{n_k}\} \in \mathcal{F}$ which converges uniformly on every compact subset of $\Omega$.

By condition (ii), $\{f_n(\zeta_k)\}$ lies in a compact set. By the definition of compactness, this infinite sequence $\{f_n(\zeta_k)\}$ has a convergent subsequence $\{f_{n_k}(\zeta(k))\}$ lies in a compact set.

By the repeated application of this process for all points of $\zeta_k$ we can obtain an array of subscripts

$$\begin{array}{l} n_{11} < n_{12} < \cdots < n_{1j} < \cdots \\ n_{21} < n_{22} < \cdots < n_{2j} < \cdots \\ \quad . \\ \quad . \\ n_{k1} < n_{k2} < \cdots < n_{kj} < \cdots \\ \ldots \quad . \quad \ldots. \end{array}$$

such that each row is contained in the preceding one, and such that the diagonal sequence $n_{jj}$ is a strictly increasing sequence and thus it forms a subsequence of each row of the above. In other words, the sequence of functions

$$\begin{array}{l} f_{n_{11}}, f_{n12} \cdots \ \ converges\ \ at\ \ \zeta_1 \\ f_{n21}, f_{n22} \cdots \ \ converges\ \ at\ \ \zeta_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{nk1}, f_{nk2} \cdots \ \ converges\ \ at\ \ \zeta_k \end{array}$$

Hence

$$\lim_{j\to\infty} f_{nkj} \ \ exists \ \ \forall k.$$

The diagonal sequence $\{f_{nk\,j}\} = \{f_{n11}, f_{n22} \cdot \cdot \cdot\}$ converges at all points of $\zeta_k$. That is the subsequence $\{f_{n\,j\,j}\}$ of $\{f_n\}$ converges at all points $\zeta_k$. For convenience, we can replace $n_{j\,j}$ by $n_j$. Therefore the subsequence $\{f_{nj}\}$ of $\{f_n\}$ converges at all points of $\zeta_k$. It remains to show that $\{f_{nj}\}$ is uniformly converges on $E$.

By hypothesis (i), the sequence $\{f_{nj}\}$ is equicontinuous on $E$ as $\{f_{nj}\} \in \mathsf{F}$ is equicontinuous.

Therefore for any given $\varepsilon > 0$ we can find $\delta > 0$ such that for every $z_1, z \in E$ and $f_{nj} \in \mathfrak{F}$ we have

$$d(f_{nj}(z), f_{nj}(z_1)) < \frac{\varepsilon}{3} \textit{ whenever } |z - z_0| < \delta.$$

Since $E$ is compact it can be covered by a finite number of $\frac{\delta}{2}$ neighborhoods. We select a point $\zeta_k$ from each of these $\frac{\delta}{2}$ neighborhood so that we can find an $i_0$ such that $i, j > i_0$. This implies that

$$d(f_{ni}(\zeta_k), f_{nj}(\zeta_k)) < \frac{\varepsilon}{3} \ \forall\ \zeta_k$$

For each $z \in E$ one of the $\zeta_k$ is within the distance $\delta$ from $z$. Hence

$$d(f_{ni}(z), f_{ni}(\zeta_k)) < \frac{\varepsilon}{3}$$

$$d(f_{nj}(z), f_{nj}(\zeta_k)) < \frac{\varepsilon}{3} \ \forall\ i, j > i_0$$

$$\begin{aligned} d(f_{ni}(z), f_{nj}(z)) &< d(f_{ni}(z), f_{ni}(\zeta_k)) + d(f_{ni}(\zeta_k), f_{nj}(\zeta_k)) + d(f_{nj}(\zeta_k), f_{nj}(z)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ d(f_{ni}(z), f_{nj}(z)) &< \varepsilon. \end{aligned}$$

Therefore all values $f(z)$ belong to a compact set and consequently a complete subset of $S$, it follows that the sequence $\{f_{nj}\}$ is uniformly convergent on $E$. This implies that $\mathfrak{F}$ is normal. ■

## 11.5 Families of Analytic Functions

Analytic functions have their values in $\mathbb{C}$ the finite complex plane. In order to apply the preceding considerations to families of analytic functions it is therefore natural to choose $S = \mathbb{C}$ with the euclidean distance.

**Theorem 11.5.1.** *A family $\mathfrak{F}$ of analytic functions is normal with respect to $\mathbb{C}$ if and only if the functions in $\mathfrak{F}$ are uniformly bounded on every compact set.*

**Proof.** Necessary Part:
Assume that the family $\mathfrak{F}$ of analytic functions is normal with respect to $\mathbb{C}$.

To prove that the functions in $\mathfrak{F}$ are uniformly bounded on every compact set.

That is to prove that if $E$ is any compact subset of $\Omega$ then $|f(z)| \le M$ for every $z \in E$ and for every $f \in \mathfrak{F}$. Therefore the family $\mathfrak{F}$ satisfies the following conditions:
(i) $\mathfrak{F}$ is equicontinuous on every compact subset $E$ of $\Omega$.

(ii) For any $z \in \Omega$ the values $f(z)$, $f \in \mathfrak{F}$ lies in a compact subset of $\mathbb{C}$.

The condition (ii) implies that these values are bounded for each $z \in \Omega$ and these bounds may depend upon $z$.

Let $E$ be any compact subset of $\Omega$. In $\mathbb{C}$, closed bounded set is a compact set. Therefore for given $z_0 \in \Omega$, we can determine $\rho$ so that the closed disk $|z - z_0| \le \rho$ is contained in $\Omega$. Since this disk is compact, by condition (i), $\mathfrak{F}$ is equicontinuous on this disk, therefore for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad for \quad |z - z_0| < \delta < \rho,$$

$$z, z_0 \in |z - z_0| \le \rho \ \ \forall \ f \in \mathfrak{F}.$$

Consider the $\delta-$ neighborhood for all points in $E$. These form an open covering of $E$. Since $E$ is compact, it has finite subcover. Therefore finite number of these $\delta-$ neighborhood cover $E$.

Let $z_1, z_2 \cdots, z_n$ be the centre of this finite collections of these neighborhoods.
Consider the set $\{|f(z)| : f \in \mathfrak{F}\}$ $i = 1, 2, \cdots, n$ By condition (i) they belong to compact subset $E$ and hence bounded.

That is there exists constant $M_1, M_2, \cdots, M_n$ such that

$$|f(z_i)| \le M_i \ (i = 1, 2, \cdots, n)$$

Let $M = \max\{M_1, M_2, \cdots, M_n\}$. Then

$$|f(z_i)| \le M \ \forall i = 1, 2, \cdots n \ \ \forall f \in \mathfrak{F}.$$

Consider any $z \in E$ then $z$ lies in some $\delta-$ neighborhood of some $z_i$.

$$\therefore |f(z) - f(z_0)| < \varepsilon \ \forall f \in \mathcal{F}.$$

Now consider,

$$\begin{aligned} |f(z)| &= |f(z) - f(z_i) + f(z_i)| \\ &\leq |f(z) - f(z_i)| + |f(z_i)| \\ &< \varepsilon + M \\ |f(z)| &\leq M' \ \forall f \in \mathcal{F}, \ \ \forall z \in E \end{aligned}$$

Hence $\mathcal{F}$ is uniformly bounded. Sufficient Part:

Assume that $\mathcal{F}$ is uniformly bounded on every compact set.

To prove that $\mathcal{F}$ is normal with respect to $\mathbb{C}$, it is enough to prove that the two conditions of Arzela - Ascolis theorem is satisfied.
That is we have to prove that
(i) $\mathcal{F}$ is equicontinuous on every compact subset of $\Omega$.
(ii) For any $z \in \Omega$, the values $f(z)$, $f \in \mathcal{F}$ lies in a compact subset of $\mathbb{C}$.

To prove $\mathcal{F}$ is equicontinuous.
Let $C$ be the boundary of the closed disk in $\Omega$ of radius $r$. Since $f$ is analytic, if $z, z_0$ are points inside $C$, by Cauchy's integral formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \ \ and \\ f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta \\ f(z) - f(z_0) &= \frac{1}{2\pi i}\int_C \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right] f(\zeta) d\zeta. \end{aligned}$$

Since $\mathcal{F}$ is uniformly bounded on $C$, we have $|f(z| \leq M$ on $C$ and if we restrict $z, z_0$ to the smaller circular disk of radius $\frac{r}{2}$, it follows that

$$|f(z) - f(z_0)| \leq \frac{|z - z_0|}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z||\zeta - z_0|} d\zeta$$

$$\leq \frac{M}{2\pi} \frac{2\pi r}{\frac{r}{2}\frac{r}{2}} |z - z_0|$$

$$|f(z) - f(z_0)| \leq \varepsilon \ \ if \ \ |z - z_0| < \delta \ \ and \ \ \delta < \frac{\varepsilon r}{4M} \ \ \forall f \in \mathcal{F}.$$

Thus,

$$|f(z) - f(z_0)| < \varepsilon \ \ if \ \ |z - z_0| < \delta$$

and this proves equicontinuity on the smaller disk. The open disk of radius $\frac{r}{4}$ form an open covering of $E$. Since $E$ is compact, there exists a finite subcover. We select a finite subcovering and denote the corresponding centres, radii and bounded by $\zeta_k$ $r_k$ and $M_k$ respectively.

Let $r$ be the smallest of $r_k'$s and $M$ be the largest of the $M_k'$s.

$$i.e., r = \min\{r_1, r_2, \cdots, r_k, \cdots\}$$

and

$$M = \max\{M_1, M_2, \cdots, M_k, \cdots\}$$

For a given $\varepsilon > 0$, Let

$$\delta = \min\left\{\frac{r}{4}, \frac{\varepsilon r}{4M}\right\}.$$

Let $z, z_0 \in E$ with $|z - z_0| < \delta$. We have to show that

$$|f(z) - f(z_0)| < \varepsilon, \ \forall f \in \mathcal{F}.$$

Since $z_0 \in E$, $z_0$ will belong to one of the balls $B\left(\zeta_k, \frac{r_k}{4}\right)$ for some $k = 1, 2, \cdots, n$. Then

$$|z - \zeta_k| < \frac{r_k}{4}.$$

Also

$$|z - \zeta_k| \leq |z - z_0| + |z_0 - \zeta_k| < \delta + \frac{r_k}{4} \leq \frac{r_k}{4}$$

Hence

$$|f(z) - f(z_0)| \leq |z - z_0| \frac{4M}{r}$$

is applicable. We find

$$|f(z) - f(z_0)| \leq \frac{4M_k}{r_k}|z - z_0|$$
$$|f(z) - f(z_0)| \leq \frac{4M\delta}{r} < \varepsilon$$

Thus we have,

$$|f(z) - f(z_0)| < \varepsilon, \ \forall z, z_0 \in E \ \ \forall \ f \in \mathcal{F}.$$

Therefore $\mathcal{F}$ is continuous on every compact subset of $E$.
Condition (ii): follows immediately, since $\mathcal{F}$ is uniformly bounded on every compact set for any $z \in \Omega$, the values $f(z)$, $f \in \mathcal{F}$ lies in a compact subset of $C$. Hence the family $\mathcal{F}$ is normal.

**Remark.** If a family has the property of the above theorem, we say that it is locally bounded. Indeed, if the family is bounded in a neighborhood of each point, then it is obviously bounded on every compact set.
Therefore the above theorem can be stated as " every sequence has a subsequence which converges uniformly on compact sets if and only if it is locally bounded." ■

**Theorem 11.5.2.** *A locally bounded family of analytic functions has locally bounded derivatives.*

**Proof.** Let $\mathcal{F}$ be a family of locally bounded analytic functions. Take any $f \in \mathcal{F}$ and a point $z_0 \in \omega$. By the property of local boundedness, there exists a neighborhood $|z - z_0| < r$ in which

$$|f(z)| \leq M, \ \ \forall z \in B(z, r) \ \ \text{and} \ \ f \in \mathcal{F}.$$

By the Cauchy's representation of the derivative, we have if $C$ is the boundary of a closed disk in $\Omega$ of radius $r$, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$$
$$|f'(z)| \leq \frac{1}{2\pi}\int_C \frac{|f(\zeta)|}{|\zeta - z|^2}d\zeta$$
$$\leq \frac{1}{2\pi}\frac{M2\pi r}{(\frac{r^2}{4})}$$
$$< M \ \ \forall \ z \in |z - z_0| < \frac{r}{2} \ \ \text{and} \ \ \forall f \in \mathcal{F}$$

Hence

$$f'(z) < M, \quad \forall\ z \in |z - z_0| < \frac{r}{2} \quad \forall\ f \in \mathfrak{F}$$

This implies that $f'$ is locally bounded. We know that what is true for first derivative is also true for higher derivatives.

$$\therefore\ f'', f''' \cdots$$

are locally bounded. Thus a locally bounded family of analytic function has locally bounded derivatives. ■

## 11.6 The Classical Definition

If a sequence tends to $\infty$ there is no great scattering of values, and it may well be argued that for the purposes of normal families such a sequence should be regarded as convergent. This is the classical point of view, and we shall restyle our definition to conform with traditional usage.

**Definition 11.6.1.** A family of analytic functions in a region $\Omega$ is said to be normal if every sequence contains either a subsequence that converges uniformly on every compact set $E \subset \Omega$, or a subsequence that tends uniformly to $\infty$ on every compact set.

**Lemma 11.6.1.** *If a sequence of meromorphic functions converges in the sense of spherical distance, uniformly on every compact set, then the limit function is meromorphic or identically equal to* $\infty$.
*If a sequence of analytic functions converges in the same sense, then the limit function is either analytic or identically equal to* $\infty$.

**Proof.** Let $\{f_n(z)\}$ be a sequence of analytic functions which converges to a limit function $f(z)$ in the sense of spherical distance, uniformly on every compact set. Since the limit function of a uniformly convergent sequence of continuous functions in continuous, $f(z)$ is continuous in the spherical metric.

**Case. (i** If $f(z_0) \neq \infty$ then $f(z)$ is bounded in a neighborhood of $z_0$ and for large $n$, the functions $f_n$ are not equal to $\infty$ in the same neighborhood. Therefore by ordinary form of Weierstrass theorem, $f(z)$ is analytic in a neighborhood of $z_0$.

**Case. (ii)** If $f(z_0) = \infty$, we consider the reciprocal $\frac{1}{f(z)}$ which is the limit of $\frac{1}{f_n(z)}$ in the spherical sense. Therefore by Weierstrass theorem, $\frac{1}{f(z)}$ is analytic near $z_0$. Also $\frac{1}{f(z_0)} = 0$. Thus by Hurwitz's theorem, $\frac{1}{f}$ must be identically zero. Therefore $f$ is identically equal to $\infty$. ■

**Theorem 11.6.1.** ***F. Marty Theorem*** *A family of analytic functions or meromorphic functions f is normal in the classical sense if and only if the expressions*

$$\rho(f) = \frac{2\,|f'(z)|}{1 + |f(z)|^2} \tag{11.6.1}$$

*are locally bounded.*

**Proof. Sufficient Part:**
Suppose the expression (11.6.1) are locally bounded.
By using formula,

$$d(z, z') = \frac{2\,|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

we write

$$d(f(z_1), f(z_2)) = \frac{2\,|f(z_1) - f(z_2)|}{\sqrt{(1 + |f(z_1)|^2)(1 + |f(z_2)|^2)}}$$

Thus $f$ followed by the stereographic projection maps an arc $\gamma$ on an image with length

$$\int_\gamma \rho(f(z))\,|dz|\,.$$

If $\rho(f) \le M$ on the line segment between $z_1$ and $z_2$ we conclude that

$$d(f(z_1), f(z_2)) \le M\,|z - z_1|$$

. This immediately proves the equicontinuity when $\rho(f)$ is locally bounded. Therefore the family $\mathcal{F}$ is normal in the classical sense.

**Necessity Part:**

First let us prove that $\rho(f) = \rho\left(\frac{1}{f}\right)$.

Consider,

$$\rho\left(\frac{1}{f}\right) = \frac{2\left|\left(\frac{1}{f}\right)'(z)\right|}{1+\left|\frac{1}{f(z)}\right|^2}$$

$$= \frac{2|f'(z)|}{1+|f'(z)|^2}$$

$$\therefore \rho\left(\frac{1}{f}\right) = \rho(f).$$

Assume that the family $\mathcal{F}$ of meromorphic functions is normal but $\rho(f)$ fails to be bounded on a compact set $E$. Consider the sequence $\{f_n\}$ in $\mathcal{F}$ such that the maximum of $\rho(f_n)$ on $E$ tends to $\infty$. Let $f$ denote the limit function of a convergent subsequence $\{f_{n_k}\}$. Around each point of $E$ we can find a small closed disk, contained in $\Omega$ on which either $f$ or $\frac{1}{f}$ is analytic.

If $f$ is analytic, it is bounded on the closed disk and by the spherical convergence it follows that $\{f_{n_k}\}$ has no poles in the disk as soon as $k$ is sufficiently large, then by Weierstrass theorem,

$$\rho(f_{n_k}) \to \rho(f)$$

uniformly on a slightly smaller disk. Since $\rho(f)$ is continuous, $\rho(f_{n_k})$ is bounded on the smaller disk.

If $\frac{1}{f}$ is analytic the same proof applies to $\rho\left(\frac{1}{f_{n_k}}\right)$ which is same as $\rho(f_{n_k})$. Since $E$ is compact, it can be covered by a finite number of the smaller disk. Therefore $\rho(f_{n_k})$ are bounded on $E$. This contradicts the hypothesis. Hence $\rho(f)$ is locally bounded. ■

# BLOCK-IV

# UNIT 12

# The Riemann Mapping Theorem

| **Objectives** |
| --- |
| After completion of this Unit, students will be able to |
| x identify that the unit disk can be mapped conformally onto any simply connected region in the plane, other than the plane itself. |
| x understand the concept of boundary behavior and the use of reflection principle. |

## 12.1 Introduction

We shall prove that the unit disk can be mapped conformally onto any simply connected region in the plane, other than the plane itself. This will imply that any two such regions can be mapped conformally onto each other, for we can use the unit disk as an intermediary step. The theorem is applied to polygon regions, and in this case an explicit form for the mapping function is derived.

## 12.2 Statement and Proof

**Definition 12.2.1. Univalent Function.** An analytic function $g$ is defined on a region $\Omega$ is *univalent* if $g(z_1) = g(z_2)$ only for $z_1 = z_2$. In other words, if the mapping by $g$ is one - one.

**Theorem 12.2.1.** ***Riemann Mapping Theorem.*** *Given any simply connected region $\Omega$ which is not the whole plane, and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in $\Omega$, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that $f(z)$ defines a one - to - one mapping of $\Omega$ onto the disk $|\omega| < 1$.*

**Proof.** Suppose that $\Omega$ is any simply connected region which is not the whole plane and $z_0 \in \Omega$.

**Existence of $f(z)$ :**

Let us consider the family $\mathsf{F}$ formed by all functions $g$ with the following properties:
(i) $g$ is analytic and univalent in $\Omega$.
(ii) $|g(z)| \le 1$ in $\Omega$.
(iii) $g(z_0) = 0$ and $g'(z_0) > 0$

The proof will consists of three parts: That is, we have to show that
(i) $\mathsf{F}$ is non - empty;
(ii) $f \in \mathsf{F}$ with maximal derivative;
(iii) $f$ has the desired properties.

To prove that $\mathsf{F}$ is non - empty:

Since by hypothesis $\Omega$ is not the whole plane, there exists at least one point $a \neq \infty$ which is not in $\Omega$, then $z - a$ has no zero in $\Omega$. Also $z - a$ is analytic in $\Omega$. Since $\Omega$ is simply connected, it is possible to define a single-valued analytic branch of $\sqrt{z-a}$ in $\Omega$, we denote it by $h(z)$. That is

$$h(z) = \sqrt{z-a}.$$

This function does not take the same value twice, nor does it take opposite values.
For, if

$$\begin{aligned} h(z_1) &= h(z_2) \\ \sqrt{z_1 - a} &= \sqrt{z_2 - a} \\ z_1 &= z_2 \end{aligned}$$

This implies that $h$ is univalent in $\Omega$.

Also if $h(z_1) = b$ and $h(z_2) = -b$, $z_1 \neq z_2$

$$h^2(z_1) = h^2(z_2) \Rightarrow z_1 = z_2,$$

which is a contradiction. Therefore $h$ will not take opposite values. This implies that $h$ is a constant analytic function. But by open mapping theorem, we have that "A non - constant analytic function maps open sets onto open sets". Therefore the image of $\Omega$ under $h$ is an open set. That is $h(\Omega)$ is open. Since $h(\Omega)$ is open there exists a real number $\rho > 0$ such that the neighborhood

$|\omega - h(z_0)| < \rho$ is contained in $h(\Omega)$.

Since opposite values are not taken by $h$ in $\Omega$. There is no point $z_0 \in \Omega$, for which $h(z)$ takes opposite values $h(z_0)$, $-h(z_0)$. Therefore $h(\Omega)$ does not meet the disk $|\omega + h(z_0)| < \rho$. The distance between all points of $h(\Omega)$ and $h(z_0)$ must be $\geq \rho$. That is to say for every $h(z) \in h(\Omega)$, we have

$$|h(z) - (-h(z_0))| \geq \rho$$

In other words,

$$|h(z) + h(z_0)| \geq \rho.$$

When $z = z_0$,

$$2\,|h(z_0)| \geq \rho \Rightarrow |h(z_0)| \geq \frac{\rho}{2} \Rightarrow \frac{1}{|h(z_0)|} \leq \frac{2}{\rho}$$

Now consider the function $g_0(z)$ defined as

$$g_0(z) = \frac{\rho}{4}\frac{|h'(z_0)|}{|h(z_0)|^2}\frac{h(z_0)}{h'(z_0)}\frac{(h(z) - h(z_0))}{(h(z) + h(z_0))} \tag{12.2.1}$$

Since $h(z)$ is analytic and univalent in $\Omega$, and $g_0(z)$ is linear transformation in $h(z)$, $g_0(z)$ is also analytic and univalent in $\Omega$. Also $g_0(z_0) = 0$ Differentiating with respect to $z$, we get

$$g_0'(z) = \frac{\rho}{2}\frac{|h'(z_0)|}{|h(z_0)|^2}\frac{h(z_0)}{h'(z_0)}\frac{h(z_0)h'(z)}{[h(z) + h(z_0)]^2}.$$

Now

$$g'_0(z_0) = \frac{\rho}{8}\frac{|h'(z_0)|}{|h(z_0)|^2} > 0$$

Finally for all $z \in \Omega,$ we have

$$
\begin{aligned}
|g_0(z)| &= \frac{\rho}{4}\frac{|h'(z_0)|}{|h(z_0)|}\,\frac{|h(z_0)|}{|h'(z_0)|}\,\frac{|h(z)-h(z_0)|}{|h(z)+h(z_0)|}\\
&= \frac{\rho}{4}\frac{1}{|h(z_0)|}\,\frac{|h(z)+h(z_0)-2h(z_0)|}{|h(z)+h(z_0)|}\\
&= \frac{\rho}{4}\left|\frac{1}{h(z_0)}-\frac{2}{h(z)+h(z_0)}\right|\\
&\le \frac{\rho}{4}\left(\frac{1}{|h(z_0)|}+\frac{2}{|h(z)+h(z_0)|}\right)\\
&\le \frac{\rho}{4}\left(\frac{2}{\rho}+\frac{2}{\rho}\right)
\end{aligned}
$$

$$\therefore |g_0(z)| \le 1 \ in\ \Omega$$

$$\therefore g_0(z) \in \mathsf{F}.$$

This implies that $\mathsf{F}$ is non - empty.

(2) To prove that $f \in \mathsf{F}$ with a maximal derivative:
Let

$$B = \sup\{g(z) : g \in \mathsf{F}\}$$

which may be finite or infinite. There is a sequence of functions $g_n \in \mathsf{F}$ such that $g'_n(z_0) \to B.$ Since $|g(z)| < 1,\ \forall g \in \mathsf{F}$ and $\forall z \in \Omega.$ The family $\mathsf{F}$ is normal.(since $\mathsf{F}$ is totally bounded)

Since $\mathsf{F}$ is normal, there exists a subsequence $\{g_{n_k}\}$ which is uniformly convergent to an analytic function $f(z)$ on a compact set.

$$\therefore \lim_{k\to\infty} g'_{n_k}(z_0) = f'(z_0)$$

Since $|g_n(z)| < 1$ in $\Omega,$ $|f(z)| \le 1$ in $\Omega.$ Also

$$\lim_{n\to\infty} g_n(z_0) = f(z_0)$$

Therefore $f(z_0) = 0,$ since $g(z_0) = 0,\ g \in \mathsf{F}$

$$\lim_{k\to 0} g'_{n_k}(z_0) = B \Rightarrow f'(z_0) = B \Rightarrow B\ \ is\ \ finite.$$

Next, we prove that $f$ is univalent. Since $f'(z_0) = B > 0,$ $f$ is not a constant. Choose a point

$z \in \Omega$ and consider the function

$$g_1(z) = g(z) - g(z_1), \quad g \in \mathfrak{F}.$$

They are all not equal to zero in the region $\Omega' = \Omega - \{z_1\}$. By Hurwitz's theorem, every limit function is either identically zero or never zero. But $f(z) - f(z_1)$ is a limit function and it is not identically zero. Hence

$$f(z) \neq -f(z_1), \ \forall z \neq z_1.$$

Since $z_1$ is arbitrary, $f$ is univalent in $\Omega$. Therefore $f \in \mathfrak{F}$ and $f$ has maximal derivative $B$ at $z_0$.

(3) To show that $f$ takes every value $\omega$ with $|\omega| < 1$.

Suppose it were true that $f(z) \neq \omega_0$, for some $\omega_0$, $|\omega_0| < 1$. Since $\Omega$ is simply connected, it is possible to define a single valued branch of

$$F(z) = \sqrt{\frac{f(z) - \omega_0}{1 - \omega_0 f(z)}} \tag{12.2.2}$$

It is clear that $F$ is univalent and that $|F| \leq 1$. To normalize it we form

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$$

Clearly $G$ is univalent and $G(z_0) = 0$.

$$G'(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F'(z) - F(z_0)\overline{F(z_0)}F'(z)}{[1 - \overline{F(z_0)}F(z)]^2}.$$

Now

$$\begin{aligned} G'(z_0) &= \frac{|F'(z_0)|}{F'(z_0)} F'(z_0) \frac{1 - F(z_0)\overline{F(z_0)}}{(1 - \overline{F(z_0)}F(z_0))^2} \\ &= \frac{|F'(z_0)|}{(1 - |F(z_0)|^2)} \\ \textit{Also } F(z) &= \sqrt{\frac{f(z_0) - \omega_0}{1 - \omega_0 f(z_0)}} \end{aligned}$$

Since $f(z_0) = 0$, $F(z_0) = \sqrt{-\omega_0}$

$$\therefore\ |F(z_0)| = |(-\omega_0)|^{\frac{1}{2}} = |\omega_0|^{\frac{1}{2}} < 1$$

$$\log F(z) = \frac{1}{2}\log(f(z) - \omega_0) - \frac{1}{2}\log(1 - \bar{\omega}_0 f(z))$$

On Differentiation,

$$\frac{F'(z)}{F(z)} = \frac{1}{2}\frac{f'(z)}{(f(z) - \omega_0)} + \frac{\bar{\omega}_0 f'(z)}{2(1 - \bar{\omega}_0 f(z))}$$

$$\begin{aligned}
At\ \ z = z_0,\quad \frac{F'(z_0)}{F(z_0)} &= \frac{1}{2}\frac{f'(z_0)}{(f(z_0) - \omega_0)} + \frac{\omega_0 f'(z_0)}{2(1 - \bar{\omega}_0 f(z))} \\
&= \frac{1}{2}\cdot\frac{-B}{\omega_0} + \frac{1}{2}\bar{\omega}_0 B \\
&= -B\,\frac{1 - |\omega_0|^2}{2\omega_0} \\
|F'(z_0)| &= \frac{B}{2|\omega_0|}(1 - |\omega_0|^2)\,|F(z_0)| \\
&= \frac{B}{2|\omega_0|}(1 - |\omega_0|^2)\,|\omega_0|^{\frac{1}{2}} \\
|F'(z_0)| &= \frac{B}{2|\omega_0|^{\frac{1}{2}}}(1 - |\omega_0|^2)
\end{aligned}$$

$$\begin{aligned}
\therefore\ G'(z_0) &= \frac{B(1 - |\omega_0|^2)}{2|\omega_0|^{\frac{1}{2}}(1 - |\omega_0|)} \\
G'(z_0) &= \frac{B}{2}\,\frac{1 + |\omega_0|}{|\omega_0|^{\frac{1}{2}}} \\
&> B \\
G'(z_0) &\geq f'(z_0)
\end{aligned}$$

Thus $G(z)$ is analytic and univalent in $\Omega$. $|G(z)| < 1$, $G(z_0) = 0$ and $G'(z_0) > 0$. Therefore $G(z) \in \mathcal{F}$ and $G'(z_0) > f^{-1}(z_0)$. This is a contradiction. (since $f'(z_0) = B$ and $f'(z_0)$ is the only maximum). Therefore assumes every value $\omega$ with $|\omega| < 1$.

To prove Uniqueness:

Suppose that $f_1(z)$ and $f_2(z)$ are two functions which map $\Omega$ onto $|\omega| < 1$. Then $f_1[f_2^{-1}(\omega)]$

is a one - to - one mapping of $|\omega| < 1$ onto itself. Also the mapping is conformal. We know that such a mapping is given by a linear transformation S. Also

$$S(\omega)^{\cdot} f_1 \cdot f_2^{-1}(0) = f_1^{\cdot} f_2^{-1}(0)^{\cdot} = f_1(z_0) = 0$$

and

$$S'(0) = {}^{\cdot}(f_1 \cdot f_2^{-1})(0)^{\cdot\prime} > 0,$$

because both $f_1'(z_0)$ and $f_2'(z_0)$ are greater than 0. This conditions implies that $S(\omega) = \omega$. Hence S is the identity transformation $I$.

$$i.e., f_1 f_2^{-1} = I \Rightarrow f_1 = f_2.$$

This completes the proof. ■

## 12.3 Boundary Behavior

**Definition 12.3.1.** Let $\Omega$ be a simply connected region. Consider a sequence $\{z_n\}$ of points in $\Omega$ or consider an arc $z(t)$, $0 \leq t \leq 1$ such that all $z(t)$ are in $\Omega$. We say that the sequence or the arc tends to the boundary if the points $z_n$ or $z(t)$ will ultimately stay away from any point in $\Omega$. In other words, if $z \in \Omega$, there shall exists an $\varepsilon > 0$ and an $n_0$ or a $t_0$ such that $|z_n - z| \geq \varepsilon$ for $n > n_0$ or such that $|z(t) - z| \geq \varepsilon$ for all $t > t_0$.

**Note.** The disks of centre $z$ and radius $\varepsilon$ (which may depend on $z$) form an open covering of $\Omega$.

Hence any compact subset $K \subset \Omega$ is covered by the finite number of these disks.

**Result.** A sequence of points or an arc in a simply connected region $\Omega$ tends to the boundary of $\Omega$ if and only if for every compact $K \subset \Omega$ there exists a tail end of the sequence or of the arc which does not meet $K$.

**Theorem 12.3.1.** *Let $f$ be a topological mapping of a region $\Omega$ onto a region $\Omega'$. If $\{z_n\}$ or $z(t)$ tends to the boundary of $\Omega$, then $\{f(z_n)\}$ or $\{f(z(t))\}$ tends to the boundary of $\Omega'$.*

**Proof.** Given that (i) $f$ is a topological mapping of a region $\Omega$ onto the region $\Omega'$. Then $f$ is one - to - one and onto, $f$ and $f^{-1}$ are continuous

(ii) $\{z_n\}$ or $z(t)$ tends to the boundary of $\Omega$.

To prove that $\{f(z_n)\}$ or $\{f(z(t))\}$ tends to the boundary of $\Omega'$. It is enough to prove that $\{f(z_n)\}$ or $f(z(t))$ stay away from every compact subset $K$ in $\Omega'$.

Let $K$ be a compact subset of $\Omega'$. Since $f$ is a topological mapping of $\Omega$ onto $\Omega'$, $f^{-1}(k)$ is compact in $\Omega$. Since $\{z_n\}$ or $z(t)$ tends to the boundary of $\Omega$. We have that the end of the sequence or of the arc does not meet $f^{-1}(K)$. That is there exists an integer $n_0$ or a real number $t_0$ such that for every $n > n_0$, $\{z_n\}$ is not contained in $f^{-1}(K)$ or for every $t > t_0$, $z(t)$ is not in $f^{-1}(K)$.

Since $f$ is a topological mapping, $f(z_n)$ not in $K$ if $n > n_0$ or $f(z(t))$ not in $K$ if $t > t_0$. This implies that $\{f(z_n)\}$ or $f(z(t))$ does not meet $K$ ultimately. That is $\{f(z_n)\}$ or $f(z(t))$ stay away from every compact subset $K$ in $\Omega'$. Hence $\{f(z_n)\}$ or $f(z(t))$ tends to the boundary of $\Omega'$. ■

## 12.4 Use of Reflection Principle

**Definition 12.4.1. Free Boundary Arc.** Let $\Omega$ be a simply connected region and let $\partial\Omega$ be its boundary containing a segment $\gamma$ of a straight line which is the real axis, let it be the interval $a < x < b$. Then $\gamma$ is said to be free boundary arc, if to each point of $\gamma$ there exists a neighborhood whose intersection with the boundary $\partial\Omega$ is the same as its intersection with $\gamma$. In other words, $\gamma$ is a free boundary arc if to each point $x_0 \in \gamma$ there exists a neighborhood $\Delta$ of $x_0$ such that $\Delta \cap \partial\Omega = \Delta \cap \gamma$ = the real diameter of the disk $\Delta$ along the real axis.

**Definition 12.4.2.** Let $\gamma$ be a free boundary arc of the region $\Omega$ then $\Delta \cap \partial\Omega = \Delta \cap \gamma$ = the real diameter of the disk $\Delta$ where $\Delta$ is the neighborhood of a point in $\gamma$. Then it is clear that each of the half disks determined by this diameter are entirely in or entirely outside of $\Omega$ and at least one must be inside. If any one is inside, we call the point a one sided boundary point. If both are inside, it is a two sided boundary point.

**Theorem 12.4.1.** *Suppose that the boundary of a simply connected region $\Omega$ contains a line segment $\gamma$ as a one - sided free boundary arc. Then the function $f(z)$ which maps $\Omega$ onto the unit disk can be extended to a function which is analytic and one to one on $\Omega \cup \gamma$. The image of $\gamma$ is an arc $\gamma'$ on the unit circle.*

**Proof.** Suppose that the boundary of a simply connected region $\Omega$ contains a line segment $\gamma$ as a one - sided free boundary arc. To prove that the function $f(z)$ which maps $\Omega$ onto the unit disk can be extended to a function which is analytic and one - to - one on $\Omega \cup \gamma$ and the image of $\gamma$ is an arc $\gamma$ on the unit circle. Since $\Omega$ is a simply connected region which is not the whole plane by Riemann mapping theorem, we can find a unique univalent function and analytic function $f : \Omega \to |\omega| < 1$ such that

$$f(z_0) = 0 \; and \; f'(z_0) > 0 \;\; for \;\; some \;\; z_0 \in \Omega.$$

Since $\gamma$ is a free boundary arc, every point of $\gamma$ has a neighborhood $\Delta$ whose intersection with the boundary $\partial\Omega$ containing a segment of a straight line is same as the intersection with $\gamma$, which is also equal to the real diameter of the disk $\Delta$. Since $\gamma$ is a one sided free boundary arc, one of the half disks determined by this diameter is entirely inside $\Omega$.

Consider the disk $\Delta$ around $x_0 \in \gamma$ which is so small that the half disk in the region $\Omega$ will not contain a point $z_0$ with $f(z_0) = 0$. That is for every $z$ in this half disk, $f(z_0) \neq 0$. This implies that $\log f(z)$ has a single valued branch in the half disk $\Delta$ and the real part by $\log f(z) \to 0$ as $z$ approaches the diameter. Because as $z \to$ the boundary of $\gamma$ in $\Omega \cup \gamma$, $f(z)$ approaches the boundary $|\omega| < 1$. So that $|f(z)| \to 1$.

Therefore $\log |f(z)| \to 0$ as $z \to \gamma$. Hence by the reflection principle, $\log f(z)$ has an analytic extension to the whole disk. Therefore $\log f(z)$ and consequently $f(z)$ is analytic at $z_0$. The extension to overlapping disks must coincide and define a function which is analytic on $\Omega \cup \gamma$. Since $f(z)$ is analytic at $x_0$, $f'(z) \neq 0$ on $\gamma$.

For, suppose $f'(z) = 0$ for $z = x_0$. That is $f'(x_0) = 0$. This implies that $f(x_0)$ where a multiple value so that the two subarcs of $\gamma$ meeting at $x_0$ would be mapped on arcs forming an angle $\frac{\pi}{n}$ with $n \geq 2$; this is clearly impossible. This is a contradiction. Therefore $f'(x_0) \neq 0$.

Hence considering the upper half disks lying in $\Omega$,

$$\frac{\partial}{\partial y} \log |f(z)| = -\frac{\partial}{\partial x} \arg(f(z)) < 0 \;\; on \;\; \gamma.$$

Hence by the reflection principle, $\arg(f(z))$ moves constantly in the same direction. ■

# BLOCK-IV

# UNIT 13

# Conformal Mapping of Polygons

**Objectives**

After completion of this Unit, students will be able to

- find the mapping function.
- identify the Schwarz - Christoffel formula.
- understand the concept of mapping on rectangle

## 13.1 Introduction

When $\Omega$ is a polygon, the mapping problem has an almost explicit solution. Indeed, we shall find that the mapping function can be expressed through a formula in which only certain parameters have values that depend on the specific shape of the polygon.

## 13.2 The Behavior at an Angle.

Assume that $\Omega$ is bounded simply connected region whose boundary is closed polygon line without self-intersections. Let the consecutive vertices be $z_1, z_2, \cdots, z_n$ in positive cyclic order. We set $z_{n+1} = z_1$. Let the angle at $z_k$ be $\alpha_k \pi$.

$$\therefore \alpha_k \pi = \arg\left(\frac{z_{k-1} - z_k}{z_{k+1} - z_k}\right) = \theta, \quad where\ 0 < \theta < 2\pi.$$

When $\theta = 0$, $\alpha_k \pi = 0 \Rightarrow \alpha_k = 0$.

When $\theta = 2\pi$, $\alpha_k\pi = 2\pi \Rightarrow \alpha_k = 2$. Therefore $0 < \alpha_k < 2$.

Also the outer angle at $z_k = \pi - \theta = \pi - \alpha_k\pi = \beta_k\pi$, where $\beta_k = 1 - \alpha_k$. Then as $0 < \alpha_k < 2$, $-1 < \beta_k < 1$. Also we have $\sum \beta_k = 2$. Since the sum of the interior angles of apolygon of $n-$ sides is $(n-2)\pi$ and hence

$$\sum_{k=1}^{n} \alpha_k\pi = (n-2)\pi$$

$$\begin{aligned}
\sum_{k=1}^{n} \alpha_k &= n-2\\
\sum_{k=1}^{n} (1-\beta_k) &= n-2\\
\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \beta_k &= n-2\\
n - \sum_{k=1}^{n} \beta_k &= n-2\\
\sum_{k=1}^{n} \beta_k &= 2.
\end{aligned}$$

The polygon is convex if and only if all $\beta_k > 0$.

## 13.3 The Schwarz- Christoffel Formula

The formula we are looking for refers not to the function $f$, but to its inverse function, which we shall denote by $F$.

**Theorem 13.3.1.** *The functions $z = F(\omega)$ which map $|\omega| < 1$ conformally onto polygons with angles $\alpha_k\pi$ $(k = 1, 2, \cdots, n)$ are of the form*

$$F(\omega) = C\int_0^{\infty} \prod_{k=1}^{n} (\omega - \omega_k)^{-\beta_k} d\omega + C' \qquad (13.3.1)$$

*where* $\beta_k = 1 - \alpha_k$, *the* $\omega_k$ *are points on the unit circle, and* $C$, $C'$ *are complex constants.*

**Proof.** Let $\Omega$ is bounded simply connected region whose boundary is closed polygon line without self-intersections. Let the consecutive vertices be $z_1, z_2, \cdots, z_n$ in positive cyclic order. We set $z_{n+1} = z_1$. Let the angle at $z_k$ be $\alpha_k\pi$.

$$\therefore \alpha_k\pi = \arg\left(\frac{z_{k-1} - z_k}{z_{k+1} - z_k}\right) = \theta, \quad where\ 0 < \theta < 2\pi.$$

When $\theta = 0$, $\alpha_k\pi = 0 \Rightarrow \alpha_k = 0$.

When $\theta = 2\pi$, $\alpha_k\pi = 2\pi \Rightarrow \alpha_k = 2$. Therefore $0 < \alpha_k < 2$.

Also the outer angle at $z_k = \pi - \theta = \pi - \alpha_k\pi = \beta_k\pi$, where $\beta_k = 1 - \alpha_k$. Then as $0 < \alpha_k < 2, -1 < \beta_k < 1$. Also we have $\sum \beta_k = 2$.

The mapping function $f(z)$ can be extended by continuity to any side of the polygon and that each side is mapped in one - one way onto an arc of the unit circle. Consider a circular sector $S_k$ which is the intersection of $\Omega$ with a sufficiently small disk about $z_k$. A single valued branch of $\zeta = (z - z_k)^{\frac{1}{\alpha_k}}$ maps $S_k$ onto a half disk $S'_k$.

A suitable branch of $z_k + \zeta^{\alpha_k}$ has its values in $\Omega$ and we may consider the function

$$g(\zeta) = f(z_k + \zeta^{\alpha_k})$$

in $S'_k$. Then as $\zeta$ approaches the diameter $|g(\zeta)|$. Therefore by the reflection principle, we conclude, $g(\zeta)$ has an analytic continuation to the whole disk.

Since $g(\zeta) = f(z_k + \zeta^{\alpha_k})$ is analytic at the origin, it has the Taylor's series development,

$$g(\zeta) = f(z_k + \zeta^{\alpha-k}) = f(z_k) + \frac{f'(z_k)}{1!}(z_k + \zeta^{\alpha_k})^2 + \cdots$$

$$f(z_k + \zeta^{\alpha_k}) = \omega_k^m + \sum_{m=1}^{\infty} a_m\zeta \tag{13.3.2}$$

where $a_1 \neq 0$ for otherwise the image of the half disk $S'_k$ could not be contained in the unit disk.

Therefore the series can be innverted.

Set

$$\omega = f(z_k + \zeta^{\alpha_k}) \tag{13.3.3}$$

$$\omega = \omega_k + \sum_{m=1}^{\infty} a_m \zeta^m \Rightarrow \omega - \omega_k = \sum_{m=1}^{\infty} a_m \zeta^m.$$

After inversion we have

$$\zeta = \sum_{m=1}^{\infty} b_m (\omega - \omega_k)^m \ with \ b_1 = 0,$$

the development being valid in a neighborhood of $\omega_k$.

$$\therefore \zeta^{\alpha_k} = \left( \sum_{m=1}^{\infty} b_m (\omega - \omega_k)^m \right)^{\alpha_k}$$

$$\zeta^{\alpha_k} = (\omega - \omega_k)^{\alpha_k} G_k(\omega) \tag{13.3.4}$$

where $G_k(\omega)$ is analytic and $\neq 0$ near $\omega_k$.

$$(16.3.3) \Rightarrow f^{-1}(\omega) = z_k + \zeta^{\alpha_k} = F(\omega) \Rightarrow \zeta^{\alpha_k} = F(\omega) - z_k.$$

$$(16.3.4) \Rightarrow F(\omega) - z_k = (\omega - \omega_k)^{\alpha_k} G_k(\omega).$$

Differentiating with respect to $\omega$, we get

$$\begin{aligned}
F'(\omega) &= \alpha_k(\omega - \omega_k)^{\alpha_k - 1} G_k(\omega) + (\omega - \omega_k)^{\alpha_k} G'_k(\omega) \\
\frac{F'(\omega)}{(\omega - \omega_k)^{\alpha_k - 1}} &= \alpha_k G_k(\omega) + (\omega - \omega_k) G'_k(\omega) \\
F'(\omega)(\omega - \omega_k)^{\beta_k} &= \alpha_k G_k(\omega) + (\omega - \omega_k) G'_k(\omega).
\end{aligned}$$

Since $G_k(\omega)$ is analytic and not equal to zero in the neighborhood of $\omega_k$, $(\omega - \omega_k)^{\beta_k}$, $F'(\omega)$ is analytic and not equal to zero at $\omega_k$. Consider the product,

$$H(\omega) = F'(\omega) \prod_{k=1}^{n} (\omega - \omega_k)^{\beta_k}$$

which is analytic and not equal to zero in the closed unit disk $|\omega| < 1$. We shall complete the proof by showing that $H(\omega)$ is constant.

For this purpose, we examine its argument when $\omega = e^{i\theta}$ lies on the unit circle between $\omega_k = e^{i\theta_k}$ and $\omega_{k+1} = e^{i\theta_{k+1}}$. Consider $F(\omega) = F(e^{i\theta})$

$$F'(\omega) = \frac{dF}{d\omega} = F'(e^{i\theta})$$

$$\arg F' = \arg dF - \arg d\omega$$

where $\arg dF$ denotes the angle of the tangent to the unit circle at $\omega = e^{i\theta}$ and $\arg d\omega$ denotes the angle of tangent to its image $F(\omega) = F(e^{i\theta})$.

Since $F$ describes a straight line, $\arg dF$ is a constant and we have

$$\arg d\omega = \theta + \frac{\pi}{2}.$$

Hence

$$\arg F' = -\theta - \frac{\pi}{2} + constant$$

Also

$$\begin{aligned}
\omega - \omega_k &= e^{i\theta} - e^{i\theta_k} \\
&= 2\sin\left(\frac{\theta+\theta_k}{2}\right)\sin\left(\frac{\theta_k-\theta}{2}\right) + i\,2\cos\left(\frac{\theta+\theta_k}{2}\right)\sin\left(\frac{\theta-\theta_k}{2}\right) \\
&= 2i\sin\left(\frac{\theta-\theta_k}{2}\right)\left[\cos\left(\frac{\theta+\theta_k}{2}\right) + i\sin\left(\frac{\theta+\theta_k}{2}\right)\right] \\
\omega - \omega_k &= 2i\sin\left(\frac{\theta-\theta_k}{2}\right) e^{i\frac{\theta+\theta_k}{2}} \\
\Rightarrow \arg(\omega - \omega_k) &= \frac{\theta}{2} + constant.
\end{aligned}$$

Also

$$
\begin{aligned}
\arg(H(\omega)) &= \arg F'(\omega) + \arg \left( \prod_{k=1}^{n} (\omega - \omega_k)^{\beta_k} \right) \\
&= \arg F'(\omega) + \sum_{k=1}^{n} \beta_k \arg(\omega - \omega_k) \\
&= \arg F'(\omega) + \beta_1 \arg(\omega - \omega_1) + \cdots + \beta_n \arg(\omega - \omega_n) \\
&= -\theta - \frac{\pi}{2} + constant + \beta_1 \left( \frac{\theta}{2} + constant \right) + \cdots + \beta_n \left( \frac{\theta}{2} + constant \right) \\
&= -\theta - \frac{\pi}{2} + \frac{\theta}{2}(\beta_1 + \beta_2 + \cdots + \beta_n) + constant \\
&= -\theta - \frac{\pi}{2} + \frac{\theta}{2}(2) + constant \\
\arg(H(\omega)) &= -\frac{\pi}{2} + constant
\end{aligned}
$$

Thus we conclude that, $H(\omega)$ is a constant between $\omega_k$ and $\omega_{k+1}$ and since it is continuous, it must be constant on the whole unit circle. Therefore by the maximum principle,

$$\arg(H(\omega) = Img(\log(H(\omega)) = constant\ inside\ the\ unit\ disk$$

This implies that

$$H(\omega) = constant = c(say)$$

Also

$$
\begin{aligned}
H(\omega) &= F'(\omega) \prod_{k=1}^{n} (\omega - \omega_k)^{\beta_k} \\
F'(\omega) &= H(\omega) \prod_{k=1}^{n} (\omega - \omega_k)^{-\beta_k}
\end{aligned}
$$

$$F'(\omega) = c \prod_{k=1}^{n} (\omega - \omega_k)^{-\beta_k}$$

On integrating from 0 to $\omega$, we get

$$\begin{aligned}
\int_0^{\omega} F'(\omega)d\omega &= c\int_0^{\omega}\prod_{k=1}^{n}(\omega-\omega_k)^{-\beta_k}d\omega + constant\\
F(\omega) &= c\int_0^{\omega}\prod_{k=1}^{n}(\omega-\omega_k)^{-\beta_k}d\omega + constant + F(0)\\
F(\omega) &= c\int_0^{\omega}\prod_{k=1}^{n}(\omega-\omega_k)^{-\beta_k}d\omega + c'
\end{aligned}$$

■

## 13.4 Mapping on a Rectangle

$$F(\omega) = c\int_0^{\omega}\prod_{k=1}^{n}(\omega-\omega_k)^{-\beta_k}d\omega + c'$$

$F(\omega)$ maps the unit disk $|\omega| < 1$ conformally onto polygons with angles $\alpha_k\pi$ (interior) and $\beta_k\pi = (1-\alpha_k)\pi$ (exterior) and $\omega_k$ are the points on the unit circle and $c, c'$ are complex constants and $\Omega$ is a bounded simply connected region whose boundary is the above said closed polygonal lines.

If $\Omega$ becomes a rectangle then

$$\begin{aligned}
\sum_{k=1}^{4}\beta_k\pi &= 2\pi\\
\beta_1+\cdots+\beta_4 &= 2\\
\therefore\ \beta_1\pi = \beta_2\pi = \beta_3\pi = \beta_4\pi &= \frac{\pi}{2}
\end{aligned}$$

in a rectangle. We have

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{2}$$

Choosing the three vertices as $x_1 = 0$, $x_2 = 1$, $x_3 = \rho < 1$ and for $c = 1$, $c' = 0$ the above mapping function will be given by

$$F(\omega) = \int_0^{\omega} (\omega - 0)^{-\frac{1}{2}}(\omega - 1)^{-\frac{1}{2}}(\omega - \rho)^{-\frac{1}{2}} d\omega$$
$$F(\omega) = \int_0^{\omega} \frac{d\omega}{\sqrt{\omega(\omega - 1)(\omega - \rho)}}$$

This equation is called an elliptic integral. To avoid ambiguity, let $\sqrt{\omega}$, $\sqrt{\omega - 1}$, $\sqrt{\omega - \rho}$ lies in the first quadrant. Consider the mapping $F(\omega)$ as $\omega$ traces the real axis. When $\omega$ is real each of the square root is either positive or purely imaginary with a positive imaginary part.

As $0 < \omega < 1$ and $\rho > 1$, there are one real and two imaginary square roots, this means $F(\omega)$ decrease from $0$ to a value $-K$ where

$$K = \int_0^1 \frac{dt}{\sqrt{t(1 - t)(\rho - t)}} \tag{13.4.1}$$

For $1 < \omega < \rho$ there is only one imaginary square root and the integral

$$\int_1^{\omega} \frac{d\omega}{\sqrt{\omega(\omega - 1)(\omega - \rho)}}$$

is purely imaginary with a positive imaginary part. Thus $F(\omega)$ follows a vertical segment from $-K$ to $-K - iK'$ where $K'$ is given by

$$\int_1^{\rho} \frac{dt}{\sqrt{t(1 - t)(\rho - t)}}$$

Therefore for $\omega > \rho$, the integral is positive and $F(\omega)$ will trace a horizontal segment in the positive direction. Since the image is to be a rectangle, it terminates at $-iK'$ and the lengths of the segment is given by

$$\int_{\rho}^{\infty} \frac{dt}{\sqrt{t(t - 1)(t - }$$

Let

$$t = \frac{\rho - u}{1 - u} \Rightarrow u = \frac{\rho - t}{1 - t}$$

$$dt = \frac{\rho - 1}{(1-u)^2}\, du$$

$$\int_\rho^\infty \frac{dt}{\sqrt{t(t-1)(t-\rho)}} = \int_0^1 \frac{\frac{\rho-1}{(1-u)^2}\,du}{\sqrt{\left(\frac{\rho-u}{1-u}\right)\left(\frac{\rho-1}{1-u}\right)\left(\frac{u(\rho-1)}{1-u}\right)}}$$

$$\int_\rho^\infty \frac{dt}{\sqrt{t(t-1)(t-\rho)}} = \int_0^1 \frac{du}{\sqrt{(1-u)(\rho-u)u}} = K$$

By Cauchy's theorem, $\frac{1}{\sqrt{t(t-1)(t-\rho)}}$ is analytic within a semicircle with radius $R$ and as $R \to \infty$,

$$\int_{-}^{R} \frac{dt}{\sqrt{t(t-1)(t-}} = \int_{-}^{\infty} \frac{dt}{\sqrt{t(t-1)(t-}} \to 0$$

This implies that $K \to 0$. That is the real part, become zero. Therefore we can claim that the horizontal segment are equal.

Similarly, when the imaginary part is zero, $-\infty < \omega < 0$ is mapped onto the segment $-iK'$ to $0$. Therefore $-K$ means $(-K, 0)$. $-K - iK'$ implies $(-K, -K')$ and $1 - iK'$ implies that $(0, -K')$. These points form a rectangle.

**Note.** If we consider the vertices as $\pm\frac{1}{K}$, $0 < K < 1$,

$$F(\omega) = \int_0^\omega \frac{d\omega}{\sqrt{(1-\omega^2)(1-K^2\omega^2)}}$$

as $\omega^2$, $K^2\omega^2$ are positive $\sqrt{1-\omega^2}$, $\sqrt{1-K^2\omega^2}$ have the positive real part so that the vertices of the rectangle become $-\frac{K}{2}$, $\frac{K}{2}$, $\frac{K}{2} + iK'$, $-\frac{K}{2} + iK'$, where

$$K = \int_{-1}^{1} \frac{dt}{\sqrt{(1-t)^2(1-K^2t^2)}} \quad and \quad K' = \int_1^{\frac{1}{K}} \frac{dt}{\sqrt{(t^2-1)(1-K^2t^2)}}$$

and the corresponding rectangle is given by $ABCD$.

# BLOCK-IV

# UNIT 14

# A Closer Look at Harmonic Functions

**Objectives**

Upon completion of this Unit, students will be able to

- understand the concept of Mean Value Property.
- identify the Harnack's Principle.

## 14.1 Introduction

We have already discussed the basic properties of harmonic functions. At that time it was expedient to use a rather crude definition, namely one that requires all second-order derivatives to be continuous. This was sufficient to prove the mean-value property from which we could in turn derive the Poisson representation and the reflection principle. We shall now show that a more satisfactory theory is obtained if we make the mean-value property rather than the Laplace equation our starting point.

In this connection we shall also derive an important theorem on monotone sequence of harmonic functions, usually referred to as *Harnack's Principle.*

## 14.2 Functions with the Mean-Value Property

Let $u(z)$ be a real valued continuous function in a region $\Omega$. We say that $u$ satisfies the mean value property if

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \tag{14.2.1}$$

when the disk $|z - z_0| \leq r$ is contained in $\Omega$.

**Theorem 14.2.1.** *A continuous function* $u(z)$ *which satisfies condition* (14.2.1) *is necessarily harmonic.*

**Proof.** Let $u(z)$ be a real valued continuous function in a region $\Omega$. Suppose that the disk $|z - z_0| \leq \rho$ is contained in $\Omega$ and let $u(z)$ satisfies the mean-value property.

$$i.e., u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

By the use of Poisson's formula, we can construct a function $v(z)$ which is harmonic for $|z - z_0| \leq \rho$ continuous and equal to $u(z)$ on $|z - z_0| = \rho$. Since $v(z)$ is harmonic, it satisfies both maximum and minimum principle. This implies that $u(z) - v(z)$ satisfies maximum principle on the boundary of the disk $|z - z_0| \leq \rho$ contained in $\Omega$. Therefore $u \leq v$.

Similarly, it can be proved using minimum principle that $u \geq v$. Thus, we have $u = v$ in the whole disk. Since $v(z)$ is already harmonic, $u(z)$ is harmonic. Thus the function $u(z$ satisfies the mean-value property is necessarily harmonic. ■

The implication of Theorem 3.1.1 is that we may, if we choose, define a harmonic function to be a continuous function with the mean-value property. Such a function has automatically continuous derivatives of all orders, and it satisfies Laplace's equation.

**Result.** Suppose that $u(z)$ is continuous and that the derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exists and satisfy $\Delta u = 0$ then $u$ is harmonic.

**Proof.** By use of Poisson's formula, we can construct a function $v(z)$ which is harmonic for $|z - z_0| < \rho$. Also $u = v$ on the boundary $|z - z_0| = \rho$.

Let

$$V = u - v + s(x - x_0)^2, \quad s > 0$$

If $V$ had a maximum the rules of the calculus would yield $\dfrac{\partial^2 V}{\partial x^2} \le 0$, $\dfrac{\partial^2 V}{\partial y^2} \le 0$, and hence $\Delta V \le 0$ at that point. On the other hand,

$$\Delta V = \Delta u - \Delta v + 2s = 2s > 0.$$

The contradiction shows that $V$ has a maximum on the boundary. That is $V \le s\rho^2$ in the disk $|z - z_0| \le \rho$. Since on the boundary $u = v$ and

$$x - x_0 = Re(z - z_0) \le |z - z_0| \le \rho.$$

Again by minimum principle $V \ge s\rho^2$. Therefore $V$ reduce to a constant. Hence

$$s\rho^2 = u - v + s(x - x_0)^2.$$

Let $s \to 0$, we find that $u = v$. Therefore $u$ is harmonic. ■

## 14.3 Harnack's Principle

**Theorem 14.3.1.** ***Harnack's inequality.*** *Let $u(z)$ be a positive harmonic function defined in the disk $|z - 0| = |z| \le \rho$ contained in $\Omega$. Then for any $z$ such that $|z - 0| = r\rho$, we have*

$$\frac{\rho - r}{\rho + r} u(0) \le u(z) \le \frac{\rho + r}{\rho - r} u(0) \tag{14.3.1}$$

**Proof.** Let us consider a circle $|z| = r < \rho$. By Poisson's formula, the harmonic function $u(z)$ can be expressed throughout its value on the circle and it is given by

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta \tag{14.3.2}$$

where $u$ is assumed to be harmonic in $|z| \le \rho$. (on harmonic for $|z| < \rho$, continuous for $|z| \le \rho$)
We know that

$$|\rho e^{i\theta} - z| \le |\rho e^{i\theta}| + |z| = \rho + r.$$

and

$$\left|\rho e^{i\theta} - z\right| \geq \left|\rho e^{i\theta}\right| - |z| = \rho - r.$$

Therefore

$$\rho - r \leq \left|\rho e^{i\theta} - z\right| \leq \rho + r$$

$$\frac{1}{\rho - r} \geq \frac{1}{|\rho e^{i\theta} - z|} \geq \frac{1}{\rho + r}$$

$$\frac{\rho^2 - r^2}{(\rho + r)^2} \leq \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho^2 - r^2}{(\rho - r)^2}$$

$$\therefore \frac{\rho - r}{\rho + r} \leq \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + r}{\rho - r}$$

$$(14.3.2) \Rightarrow |u(z)| = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2}\right| \left|u(\rho e^{i\theta})\right| d\theta$$

Since $u(z)$ is a positive harmonic function, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho - r}{\rho + r} u(\rho e^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho + r}{\rho - r} u(\rho e^{i\theta})d\theta$$

$$\frac{1}{2\pi}\frac{\rho - r}{\rho + r}\int_0^{2\pi} u d\theta \leq u(z) \leq \frac{1}{2\pi}\frac{\rho + r}{\rho - r}\int_0^{2\pi} u d\theta, \quad u(\rho e^{i\theta}) = u$$

But the arithmetic mean of $u(\rho e^{i\theta})$ equals $u(0)$. Therefore, we have

$$\left(\frac{\rho - r}{\rho + r}\right) u(0) \leq u(z) \leq \left(\frac{\rho + r}{\rho - r}\right) u(0)$$

■

The main application of (14.3.1) is to series with positive terms or, equivalently, increasing sequence of harmonic functions. It leads to a powerful and simple theorem known as Harnack's principle.

**Theorem 14.3.2.** ***Harnack's Principle.*** *Consider a sequence of functions $u_n(z)$, each defined and harmonic in a certain region $\Omega_n$. Let $\Omega$ be a region such that every point in $\Omega$ has a neighborhood contained in all but a finite number of the $\Omega_n$, and assume moreover that in this neighborhood $u_n(z) \leq u_{n+1}(z)$ as soon as n is suflciently large. Then there are only two possibilities: either $u_n(z)$ tends uniformly to $+\infty$ on every compact subset of $\Omega$, or $u_n(z)$ tends* *to a harmonic limit function* $u(z)$ *in* $\Omega$, *uniformly on compact sets.*

**Proof.** Given that,
(i) the sequence of functions $u_n(z)$ is defined and harmonic in a certain region $\Omega_n$.
(ii) $\Omega$ is a region such that every point in $\Omega$ has a neighborhood contained in all but finite number of $\Omega_n$.
(iii) In the above neighborhood, $u_n(z) \leq u_{n+1}(z)$ as soon as $n$ is sufficiently large.
To prove that, either $u_n(z)$ tends uniformly to $\infty$ on every compact subset of $\Omega$ or $u_n(z)$ tends to a harmonic limit function $u(z)$ in $\Omega$ uniformly on compact sets.

Suppose that

$$\lim_{n\to\infty} u_n(z) = u(z).$$

It is enough if we prove that, either $u(z)$ tends to $\infty$ or $u(z)$ is harmonic, and in both the cases the convergence is uniform in all compact subset of $\Omega$.

**Case.(i)** Let $z_0 \in \Omega$ be such that $u(z_0) = \infty$.

$$i.e., \lim_{n\to\infty} u_n(z_0) = \infty,$$

then by hypothesis of the theorem (i), (ii), (iii) there exists $r$ and $m$ such that the function $u_n(z)$ are harmonic and form a non-decreasing sequence for $|z - z_0| < r$ and $n \geq m$.

$$u_n(z) \geq u_m(z) \ for \ \ n \geq m.$$

$$\therefore u_n(z) - u_m(z)$$

is a positive harmonic function. Hence from the left hand Harnack's inequality applied to $|z - z_0| \leq \frac{r}{2} < r$, we have

$$\begin{aligned} \frac{r - \frac{r}{2}}{r + \frac{r}{2}}(u_n(z_0) - u_m(z_0)) &\leq u_n(z) - u_m(z), \quad \forall n \geq m \\ \frac{1}{3}(u_n(z_0) - u_m(z_0)) &\leq u_n(z) - u_m(z), \quad \forall n \geq m \\ \lim_{n\to\infty} u_n(z) &= \infty \end{aligned}$$

in the disk $|z - z_0| \leq \frac{r}{2}$. Since

$$\lim_{n\to\infty} u_n(z) = \infty \ at \ z_0.$$

That is $u_n(z_0) = \infty$. Therefore $u_n(z) \to \infty$ on every compact subset of $\Omega$.

**Case.(ii)** Suppose that $\lim_{n\to\infty} u_n(z_0) < \infty$. Then by the same argument, from right handed Harnack's inequality applied to $|z - z_0| \leq \frac{r}{2} < r$.

$$u_n(z) - u_m(z) \leq \frac{r + \frac{r}{2}}{r - \frac{r}{2}} (u_n(z_0) - u_m(z_0))$$

$$u_n(z) - u_m(z) \leq 3(u_n(z_0) - u_m(z_0))$$

It shows that $u_n(z)$ is bounded on the disk $|z - z_0| \leq \frac{r}{2}$. As $\{u_n\}$ is monotonic and bounded it converges to the function $u(z)$. Therefore the sets on which $\lim u_n(z)$ is respectively finite or infinite are both open, and since $\Omega$ is connected, one of the sets must be empty. Suppose that the limit is infinite at a single point say $z_0$, it is hence identically infinite.The uniformity follows by the usual compactness argument.

Suppose that the limit function $u(z)$ is finite everywhere with the same notations as above. By using Harnack's inequality

$$u_{n+p}(z) - u_n(z) \leq 3(u_{n+p}(z) - u_n(z_0))$$

for $|z - z_0| < \frac{r}{2}$ and $n + p \geq n \geq m$. Hence convergence at $z_0$ implies uniform convergence in the neighborhood of $z_0$ and use of the Heine Borel property shows that the convergence is uniform on every compact set. The harmonicity of the limit function can be inferred from the fact that $u(z)$ can be represented by Poisson's formula.

∎

# BLOCK-V

# UNIT 15

# Simply Periodic Functions

**Objectives**

Upon completion of this Unit, students will be able to

- know the concept of simply periodic function.
- understand the concept of Fourier development and functions of finite order.

## 15.1 Introduction

**Definition 15.1.1.** A function $f(z)$ is said to be periodic with period $\omega(\neq 0)$ if $f(z+\omega) = f(z)$, $\forall z$.

**Example 15.1.1.** Let $f(z) = e^z$.

$$\begin{aligned} f(z+2\pi i) &= e^{z+2\pi i} \\ &= e^z(\cos 2\pi + i\sin 2\pi) \\ &= e^z \\ f(z+2\pi i) &= f(z). \end{aligned}$$

Hence $e^z$ is a periodic function with period $2\pi i$.

**Example 15.1.2.** $\sin z$ and $\cos z$ have the period $2\pi$, since $\sin(2\pi + z) = \sin z$ and $\cos(2\pi + z) = \cos z$.

**Note.(i)** If $\omega$ is a period, so are all integrals multiples $n\omega$. That is, if $\omega$ is a period then the integrals multiples $n\omega$ are also the periods.

**Note.(ii)** If $\omega_1$ and $\omega_2$ are the periods of $f(z)$ then a linear combination of $\omega_1$ and $\omega_2$, $n_1\omega_1 + n_2\omega_2$ is a period.

## 15.2 Representation by Exponentials

***(i)*** The simplest function with period $\omega$ is the exponential $e^{\frac{2\pi iz}{\omega}}$. It is a fundamental fact that any function with period $\omega$ can be expressed in terms of this particular function.

**(ii)** Let $\Omega$ be a region with the property that $z \in \Omega$ implies $z+\omega \in \Omega$ and $z - \omega \in \Omega$. We define $\Omega'$ in the $\zeta-$ plane to be the image of $\Omega$ under the mapping $\zeta = e^{\frac{2\pi iz}{\omega}}$; it is obviously a region. For instance, if $\Omega$ is the whole plane then $\Omega'$ is the plane punctured at $0$. If $\Omega$ is a parallel strip defined by $a < Im\left(\frac{2\pi z}{\omega}\right) < b$, then $\Omega'$ is the annulus $e^{-b} < |\zeta| < e^{-a}$.

For proving this, let $\zeta = e^{\frac{i2\pi z}{\omega}}$, $z = x + iy$ and $\omega = a + ib$. Then

$$Im\left(\frac{2\pi z}{\omega}\right) = \frac{2\pi}{a^2 + b^2}(ay - bx).$$

Now

$$\begin{aligned}
e^{\frac{2\pi iz}{\omega}} &= e^{\frac{2\pi}{a^2+b^2}((bx-ay)+i(ax+by))} \\
&= e^{\frac{2\pi(bx-ay)}{a^2+b^2}} e^{\frac{i2\pi(ax+by)}{a^2+b^2}} \\
Hence, |\zeta| &= e^{\frac{2\pi(bx-ay)}{a^2+b^2}} \\
|\zeta| &= e^{-\frac{2\pi(ay-bx)}{a^2+b^2}}
\end{aligned}$$

Now consider,

$$
\begin{aligned}
a &< Im\left(\frac{2\pi z}{\omega}\right) < b \\
a &< \frac{2\pi}{a^2+b^2}(ay-bx) < b \\
-a &> \frac{-2\pi}{a^2+b^2}(ay-bx) > -b
\end{aligned}
$$

$$
\begin{aligned}
e^{-a} &> |\zeta| > e^{-b} \\
\therefore e^{-b} &< |\zeta| < e^{-a}.
\end{aligned}
$$

**(iii)** Suppose that $f(z)$ is meromorphic in $\Omega$ and has a period $\omega$ then there exists a unique function $F$ in $\Omega'$ such that

$$f(z) = F\left(\frac{2\pi i z}{\omega}\right). \tag{15.2.1}$$

Indeed, to determine $F(\zeta)$ we write $\zeta = e^{\frac{2\pi i z}{\omega}}$; $z$ is unique upto an additive multiple of $\omega$, and this multiple does not influence the value $f(z)$. Since $f(z)$ is meromorphic, $F$ is also meromorphic. Conversely, if $F$ is meromorphic in $\Omega'$ then there exists a function $f$ in $\Omega$ with period $\omega$ which is also meromorphic in $\Omega$ given by (15.2.1).

## 15.3 The Fourier Development

Assume that $\Omega'$ contains an annulus $r_1 < |\zeta| < r_2$ in which $F$ has no poles. In this annulus $F$ has a Laurent development

$$F(\zeta) = \sum_{n=-\infty}^{\infty} C_n \zeta^n,$$

and we obtain

$$f(z) = \sum_{-\infty}^{\infty} C_n \left(e^{\frac{2\pi i z}{\omega}}\right)^n = \sum_{-\infty}^{\infty} C_n e^{\frac{2n\pi i z}{\omega}},$$

since

$$f(z) = F\left(e^{\frac{2\pi iz}{\omega}}\right).$$

This is the complex Fourier development of $f(z)$ valid in the parallel strip that corresponds to the given annulus. The Fourier coefficients are given by

$$C_n = \frac{1}{2\pi i}\int_{|\zeta|=r} F(\zeta)\zeta^{-n-1}d\zeta, \quad (r_1 < r < r_2).$$

Substitute $\zeta = e^{\frac{2\pi iz}{\omega}}$, we get

$$C_n = \frac{1}{\omega}\int_a^{a+\omega} f(z)\, e^{-\frac{2\pi inz}{\omega}}dz$$

Here $a$ is an arbitrary point in the parallel strip and integration is along any path $a$ to $a+\omega$ which remains within the strip. If $f(z)$ is analytic in the whole plane, the same Fourier development is valid everywhere.

## 15.4 Functions of Finite Order

When $\Omega$ is the whole plane, $F(\zeta)$ has isolated singularities at $\zeta = 0$ and $\zeta = \infty$. If both these singularities are in essential that is either removable singularities or poles, then $F$ is a rational function. We say in this case that $f$ has finite order, equal to the order of $F$.

A rational function assumes every complex value including $\infty$, the same number of times, provided that we observe the usual multiplicity convention. If $\omega$ is a period of a simply periodic function and if there is no distinction between $z$ and $z + \omega$, we obtain a same result for simply periodic functions.

For convenient terminology, we say that $z + n\omega$ is equivalent to $z$. If $f$ is of order $m$, we find that every complex value $c \neq F(0)$ and $F(\infty)$ is assumed at $m$ inequivalent points with due count of multiplicities. We observe further that,

$f(z) \to F(0)$ when $Im \frac{z}{\omega} \to -\infty$ and

$f(z) \to F(\infty)$ when $Im \frac{z}{\omega} \to \infty$. If we are willing to agree that these values are also assumed, we can maintain that all complex values are exactly $m$ times.

For another interpretation we may consider the period strip, defined by $0 \leq Im \frac{z}{\omega} \leq 2\pi$. Since this strip contains only one representation from each equivalence class we find that $f(z)$ assumes each complex value $m-$ times in the period strip, except that the values $F(0)$ and $F(\infty)$ require a special convention.

# BLOCK-V

# UNIT 16

# Doubly Periodic Functions

| **Objectives** |
| --- |
| Upon completion of this Unit, students will be able to |
| x know the concept of doubly periodic function. |
| x identify periodic modules. |
| x understand the concept of canonical basis. |
| x prove the properties of elliptic functions. |

## 16.1 Introduction

The terms *elliptic function* and *doubly periodic function* are interchangeable; we have already met examples of such functions in connection with the conformal mapping of rectangles and certain triangles. Elliptic functions have been the object of very extensive study, partly because of their function theoretic properties and partly because of their importance in algebra and number theory. Our introduction to the topic covers only the most elementary aspects.

**Definition 16.1.1.** An analytic function $f(z)$ is said to be doubly periodic function with period $\omega_1$ and $\omega_2$ if

(i) $f(z+\omega_1) = f(z)$

(ii) $f(z+\omega_2) = f(z)$

(iii) $\dfrac{\omega_2}{\omega_1}$ is non-real.

**Definition 16.1.2.** A doubly periodic meromorphic function defined in the whole complex plane is called an elliptic function.

**Example 16.1.1.** Consider

$$f(z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z-(m+in)]^3}$$

(i)

$$\begin{aligned}
f(z+1) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z+1-(m+in)]^3}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z-(m-1)-in]^3}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z-(m+in)]^3}\\
f(z+1) &= f(z).
\end{aligned}$$

Hence $f(z)$ is a periodic function with period 1.

(ii)

$$\begin{aligned}
f(z+i) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z+i-(m+in)]^3}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z-(m+n-1)i]^3}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{[z-(m+in)]^3}\\
f(z+i) &= f(z).
\end{aligned}$$

Therefore $f(z)$ is a periodic function with period $i$. Hence $f(z)$ is doubly periodic function with period 1 and $i$. Also $f(z)$ has got a pole at $z = m + in$ of order 3. Therefore $f(z)$ is an elliptic function.

## 16.2 The period Module

**Definition 16.2.1. Periodic Module.** Let $f(z)$ be meromorphic in the whole plane. Let $M$ be the set of all its periods. If $\omega$ is a period, so are all integral multiples $n\omega$, and if $\omega_1$ and $\omega_2$ belong to $M$ so does $\omega_1 + \omega_2$, as a consequence all linear combinations $n_1\omega_1 + n_2\omega_2$ are in $M$. A set with these properties is called a module and we shall call $M$ the period module.(This module can be called more precisely as module over the integers.)

**Result.** The points of a period module $M$ are isolated.

**Proof.** Let $f$ be a meromorphic function. If $\omega$ is a period of $f$, we have

$$f(\omega) = f(0), \forall \omega \in M, \quad since\, f(\omega) = f(0+\omega) = f(0).$$

$$i.e., f(\omega) - f(0) = 0.$$

Therefore $\omega$ is a zero of $f(z) - f(0)$. Since the zeros of a meromorphic function are isolated, it follows that the periods are isolated. ■

**Definition 16.2.2.** A module with isolated points is said to be discrete.

Our first step is to determine all discrete modules.

**Theorem 16.2.1.** *A discrete module consists either of zero alone, of the integral multiples* $n\omega$ *of a single complex number* $\omega = 0$, *or of all linear combinations* $n_1\omega_1 + n_2\omega_2$ *with integral coeflcients of two numbers* $\omega_1$, $\omega_2$ *with non-real ratio* $\dfrac{\omega_2}{\omega_1}$.

**Proof.** Let $M$ be a discrete module. Then $M$ is a module with isolated points. If $M$ consists of a number $\omega \quad 0$ then $n\omega \in M \ \forall n \in Z$. Also it contains one number call it $\omega_1$, whose absolute value is a minimum. That is $\omega_1 \in M$ such that $|\omega_1| < |\omega|$, $\forall \omega \in M$.

Consider a disk $|z| \leq r$ for sufficiently large $r$ so that the disk contains at least one non - zero integral multiple of $\omega$. That is for large $r$, the disk $|z| \leq r$ contains a point from $M$, other than zero. Since $M$ is a discrete module, its points are isolated. Because the points are isolated there are only a finite number of such points, and we choose $\omega_1$ to be one closest the origin. Since $\omega_1 \in M$, the multiples $n\omega_1$ are in $M$, $\forall n \in Z$. Thus $M$ may be just the set of all integral multiples $n\omega_1$, $\omega_1 /= 0$.

Suppose now there exists an $\omega \in M$ which is not an integral multiple of $\omega_1$. Among all such there is one $\omega_2$ whose absolute value is smallest.

Now, let us claim that $\dfrac{\omega_2}{\omega_1}$ is not real. If $\dfrac{\omega_2}{\omega_1}$ is real then we can find an integer $n$, such that $n < \dfrac{\omega_2}{\omega_1} < n+1$. This implies that

$$0 < \frac{\omega_2}{\omega_1} - n < 1 \Rightarrow 0 < \left|\frac{\omega_2}{\omega_1} - n\right| < 1 \Rightarrow 0 < |\omega_2 - n\omega_1| < |\omega_1|.$$

Hence $n\omega_1 - \omega_2 \in M$ and $|n\omega_1 - \omega_2| < |\omega_1|$. This is a contradiction to our choice of $\omega_1$. Hence our assumption that $\dfrac{\omega_2}{\omega_1}$ is real is wrong. Therefore $\dfrac{\omega_2}{\omega_1}$ is non-real.

Now let us assume that any complex number $\omega$ can be written in the form $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ with real $\lambda_1$ and $\lambda_2$. Consider the equations

$$\begin{aligned} \omega &= \lambda_1\omega_1 + \lambda_2\omega_2 \\ \overline{\omega} &= \lambda_1\overline{\omega_1} + \lambda_2\overline{\omega_2} \end{aligned}$$

Since the determinant $\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}$ is $0$ the system has a unique solution $(\lambda_1, \lambda_2)$. Now,

$$\begin{aligned} \overline{\omega} &= \overline{\lambda_1}\overline{\omega_1} + \overline{\lambda_2}\overline{\omega_2} \\ \omega &= \overline{\lambda_1}\omega_1 + \overline{\lambda_2}\omega_2 \end{aligned}$$

These two equations also have unique solution $(\overline{\lambda_1}, \overline{\lambda_2})$. Therefore the equations

$$x\omega_1 + y\omega_2 = \omega \text{ and } x\overline{\omega_1} + y\overline{\omega_2} = \overline{\omega}$$

are both satisfied by $(\lambda_1, \lambda_2)$, and $(\overline{\lambda_1}, \overline{\lambda_2})$. Since the solutions are unique, we must have

$$\overline{\lambda_1} = \lambda_1 \text{ and } \overline{\lambda_2} = \lambda_2.$$

Hence $\lambda_1$ and $\lambda_2$ are real. Hence any complex number $\omega$ can be written in the form $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ uniquely, where $\lambda_1$, $\lambda_2$ are real.

Now, let us show that any $\omega \in M$ can be written uniquely in the form $n_1\omega_1 + n_2\omega_2$ where $n_1$ and $n_2$ are integers. Suppose $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$. Let $m_1$ and $m_2$ be integers such that

$$|\lambda_1 - m_1| \leq \frac{1}{2} \text{ and } |\lambda_2 - m_2| \leq \frac{1}{2}.$$

Let $\omega' = \omega - m_1\omega_1 - m_2\omega_2$. If $\omega \in M$, then $\omega' \in M$.

Now

$$\begin{aligned}
|\omega'| &= |\omega - m_1\omega_1 - m_2\omega_2| \\
&= |\lambda_1\omega_1 + \lambda_2\omega_2 - m_1\omega_1 - m_2\omega_2| \\
&\leq |\lambda_1 - m_1|\,|\omega_1| + |\lambda_2 - m_2|\,|\omega_2| \\
&< \frac{1}{2}|\omega_1| + \frac{1}{2}|\omega_2| \\
|\omega'| &\leq |\omega_2|
\end{aligned}$$

where the first inequality is strict because $\omega_2$ is not a real multiple of $\omega_1$. (Since $\dfrac{\omega_2}{\omega_1}$ is not real)

Equality holds if

$$|\lambda_1 - m_1| = |\lambda_2 - m_2|$$

and

$$\begin{aligned}
|\omega_1| + |\omega_2| &= |\omega_1 + \omega_2| \\
\left|1 + \frac{\omega_2}{\omega_1}\right| &= 1 + \frac{|\omega_2|}{|\omega_1|} \\
\left|1 + \frac{\omega_2}{\omega_1}\right|^2 &= \left(1 + \frac{|\omega_2|}{|\omega_1|}\right)^2 \\
\left(1 + \frac{\omega_2}{\omega_1}\right)\left(1 + \frac{\overline{\omega_2}}{\overline{\omega_1}}\right) &= 1 + 2\frac{|\omega_2|}{|\omega_1|} + \frac{|\omega_2|^2}{|\omega_1|^2} \\
\frac{\omega_2}{\omega_1} + \frac{\overline{\omega_2}}{\overline{\omega_1}} &= 2\frac{|\omega_2|}{|\omega_1|} \\
2\,Re\left(\frac{\omega_2}{\omega_1}\right) &= 2\frac{|\omega_2|}{|\omega_1|}
\end{aligned}$$

This implies that $\dfrac{\omega_2}{\omega_1}$ is real, which is a contradiction. Therefore, by the way $\omega_2$ was chosen, it follows that $\omega'$ is an integral multiple of $\omega_1$ and hence $\omega$ has the asserted form. ■

## 16.3 Unimodular Transformations

**Definition 16.3.1.** Any pair $(\omega_1, \omega_2)$ is called a basis of $M$, if any $\omega \in M$ has a unique representation of the form $\omega = n_1\omega_1 + n_2\omega_2$ where $n_1$, $n_2$ are integers.

**Definition 16.3.2.** A linear transformation of the form

$$\omega'_2 = a\omega_2 + b\omega_1 \;\; and \;\; \omega'_1 = c\omega_2 + d\omega_1$$

where $(\omega_1, \omega_2)$ and $(\omega'_1, \omega'_2)$ are two bases and $a, b, c, d$ are all integers is such that the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \pm 1$$

then this transformation is called an unimodular transformation.

**Result.** Any two bases of the same module are connected by a unimodular transformation.

**Proof.** Let $(\omega_1, \omega_2)$ and $(\omega'_1, \omega'_2)$ be two bases for the period module $M$. Since $(\omega_1, \omega_2)$ is a basis, there exists an integers $a, b, c, d$ such that

$$\begin{aligned} \omega'_2 &= a\omega_2 + b\omega_1 \\ \omega'_1 &= c\omega_2 + d\omega_1 \end{aligned} \tag{16.3.1}$$

This can be put in matrix form

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \tag{16.3.2}$$

Taking conjugate of (16.3.1) we get

$$\begin{aligned} \overline{\omega'_2} &= a\overline{\omega_2} + b\overline{\omega_1} \\ \overline{\omega'_1} &= c\overline{\omega_2} + d\overline{\omega_1} \end{aligned}$$

The matrix form of the equation is

$$\begin{pmatrix} \overline{\omega'_2} \\ \overline{\omega'_1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{\omega_2} \\ \overline{\omega_1} \end{pmatrix} \tag{16.3.3}$$

Thus we have

$$\begin{pmatrix} \omega'_2 & \overline{\omega'_2} \\ \omega'_1 & \overline{\omega'_1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix} \tag{16.3.4}$$

Since $(\omega_1', \omega_2')$ is also a basis, we have

$$\begin{aligned}
\omega_2 &= a'\omega_2' + b'\omega_1' \\
\omega_1 &= c'\omega_2' + d'\omega_1'
\end{aligned}$$

$$\begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_2' & \overline{\omega_2'} \\ \omega_1' & \overline{\omega_1'} \end{pmatrix} \tag{16.3.5}$$

where $a', b', c', d'$ are integers. From (16.3.4) and (16.3.5), we have

$$\begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix} \tag{16.3.6}$$

Hence

$$\begin{vmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{vmatrix} = \omega_2\overline{\omega_1} - \omega_1\overline{\omega_2} \neq 0.$$

A matrix with determinant $\neq 0$ has an inverse matrix and if we multiply (16.3.6) by the inverse of the matrix

$$\begin{pmatrix} \omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that the matrices

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and are inverse to each other. In particular, their determinant must satisfy

$$\begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since both the determinant are integer valued, we have

$$\begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1.$$

Hence the transformation is an unimodular transformation. Thus any two bases of the same module are connected by a unimodular transformation. ■

**Note. (i)** The set of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integral entries and with determinant $\pm 1$ form a group under multiplication. This group is called as modular group.

(i.e.,) the unimodular matrices, or the corresponding linear transformation, forms a group, the *modular group*

**(ii)** We divide the complex plane geometrically into parallelogram spanned by the period module $M$, with $(\omega_1, \omega_2)$ as a basis. Also $\dfrac{\omega_2}{\omega_1}$ is non-real. Therefore the whole complex plane is divided into network of congruent parallelogram. The vertices of the parallelogram $n_1\omega_1 + n_2\omega_2$. If $f(z)$ is $\omega-$ periodic with periods $(\omega_1, \omega_2)$ then we see that the values of $f(z)$ are identical in each and every one of the congruent parallelogram. So it is enough to study the properties of $f(z)$ in one parallelogram.

**(iii)** The following figure shows two bases of the same module. Observe that the parallelogram have equal area.

## 16.4 The Canonical basis

Among all bases of $M$ it is possible to single out one, almost uniquely, to be called the canonical basis. If we call the ratio $\dfrac{\omega_2}{\omega_1}$, the following theorem shows that there exists a basis $(\omega_1, \omega_2)$ with special requirement on $\tau$, such a basis is called a canonical basis.

**Theorem 16.4.1.** *There exists a basis* $(\omega_1, \omega_2)$ *such that the ratio* $\tau = \dfrac{\omega_2}{\omega_1}$ *satisfies the following conditions: (i)* $Im\,\tau > 0$, *(ii)* $-\dfrac{1}{2} < Re\,\tau \le \dfrac{1}{2}$, *(iii)* $|\tau| \ge 1$, *(iv)* $Re\,\tau \ge 0$ *if* $|\tau| = 1$. *The ratio* $\tau$ *is uniquely determined by these conditions, and there is a choice of two, four, or six corresponding bases.*

**Proof.** Let $M$ be the period module. We choose $\omega_1 \in M$ to be the one closer to the origin. There are always possible two, four, or six closest points. Next, we select $\omega_1$ and $\omega_2$ such that $\omega_1$ is having smallest absolute value. Also

$$\begin{aligned} |\omega_1| &\le |\omega_2| \\ |\omega_2| &\le |\omega_1| + |\omega_2| \\ |\omega_2| &\le |\omega_1 - \omega_2| \end{aligned}$$

Let $\tau = \dfrac{\omega_2}{\omega_1}$. Since

$$|\omega_1| \le |\omega_2| \Rightarrow 1 \le \frac{|\omega_2|}{|\omega_1|} \Rightarrow |\tau| \ge 1$$

This proves (iii).

$$\begin{aligned} Since\quad |\omega_2| &\le |\omega_1 + \omega_2| \\ \frac{|\omega_2|}{|\omega_1|} &\le \left|\frac{\omega_1 + \omega_2}{\omega_1}\right| \\ |\tau| &\le |1 + \tau| \end{aligned}$$

$$|\tau|^2 \le |1 + \tau|^2 \tag{16.4.1}$$

Again

$$\omega_2 \le |\omega_1 - \omega_2|$$

$$\Rightarrow |\tau|^2 \le |1 - \tau|^2 \tag{16.4.2}$$

From (16.4.1), we have

$$
\begin{aligned}
\tau\bar{\tau} &\leq (1+\tau)\overline{(1+\tau)}\\
0 &\leq 1+\tau+\bar{\tau}\\
-1 &\leq \tau+\bar{\tau}\\
-1 &\leq 2\,Re\,\tau\\
Re\,\tau &\geq -\frac{1}{2}
\end{aligned}
$$

From (16.4.2), we have

$$
\begin{aligned}
\tau\bar{\tau} &\leq (1-\tau)\overline{(1-\tau)}\\
0 &\leq 1-\tau-\bar{\tau}\\
1 &\geq 2\,Re\,\tau\\
Re\,\tau &\leq \frac{1}{2}\\
\therefore -\frac{1}{2} &\leq Re\,\tau \leq \frac{1}{2}.
\end{aligned}
$$

This proves (ii).

Next we prove that $Im\,\tau > 0$. If $Im\,\tau < 0$, we replace the basis $(\omega_1, \omega_2)$ by $(-\omega_1, \omega_2)$. This makes $Im > 0$ without changing the condition on $Re\,\tau$. If $Re\,\tau = -\frac{1}{2}$, we replace $(\omega_1, \omega_2)$ by $(\omega_1, \omega_1+\omega_2)$, and if $|\tau| = 1$, $Re\,\tau < 0$ we replace it by $(-\omega_2, \omega_1)$. After these minor changes all the conditions are satisfied.

Now, we prove that the four conditions fix $\tau$ uniquely. Suppose there exists another basis $(\omega_1', \omega_2')$ satisfying four conditions. Then these two bases are connected by a modular transformation.

$$
\begin{aligned}
\omega_2' &= a\omega_2 + b\omega_1\\
\omega_1' &= c\omega_2 + d\omega_1
\end{aligned}
$$

$$
\frac{\omega_2'}{\omega_1'} = \tau' = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1} = \frac{a\cdot\frac{\omega_2}{\omega_1} + b}{c\cdot\frac{\omega_2}{\omega_1} + d}
$$

$$
\Rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad with \ \ ad - bc = \pm 1
$$

$$Now,\ \tau' = \frac{(a\tau + b)(c\bar{\tau} + d)}{|c\tau + d|^2}$$

$$\tau' = \frac{ac\,|\tau|^2 + bd + (ad\tau + bc\bar{\tau})}{|c\tau + d|^2}.$$

Hence

$$Im\,\tau' = (ad - bc)\frac{Im\,\tau}{|c\tau + d|^2}$$

First as $Im\,\tau$ and $Im\,\tau'$ are both $> 0$ and $ad - bc = 1$.

$$Im\,\tau' = \frac{Im\,\tau}{|c\tau + d|^2} \tag{16.4.3}$$

Without loss of generality we may assume that $Im\,\tau' \geq Im\,\tau$ then $|c\tau + d| \leq 1$. As $c$ and $d$ are integers there are very few possibilities for this inequality to hold.

**Case.(i)** $c = 0,\ d = \pm 1$, the relation $ad - bc = 1$ reduces to either $a = d = 1$ or $a = d = -1$. Hence $\tau' = \tau \pm b$. That is $\tau' - \tau$ is real. Since $-\frac{1}{2} < Re\,\tau,\ Re\,\tau' < \frac{1}{2}$. $|b| = |Re\,\tau' - Re\,\tau| < 1$. As $b$ is an integer, $b = 0$. Hence $\tau' = \tau$.

**Case.(ii)** $d = 0$, then $ad - bc = 1 \Rightarrow bc = -1$. Therefore $b = 1, c = -1$ or $b = -1, c = 1$. Further $|c\tau + d| \leq 1$ becomes $\tau \leq 1$. Since $\tau \geq 1$ by assumption. $\tau = 1$. Then

$$\tau' = \pm a - \frac{1}{\tau} = \pm a - \bar{\tau}.$$

Thus $\pm a = \tau' + \bar{\tau}$ and hence

$$\begin{aligned} |a| &= Re\,(\tau' + \bar{\tau}) \\ &= Re\,\tau' + Re\,\tau \\ |a| &\leq \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

If $Re\,\tau' + Re\,\bar{\tau} = 1$ then $Re\,\tau' = Re\,\tau = \frac{1}{2}$ as $|\tau| = 1$. It follows that

$$\tau = \tau' = e^{\frac{i\pi}{3}},$$

$$Re\,\tau' + Re\,\bar{\tau} < 1 \Rightarrow |a| < 1 \Rightarrow a = 0.$$

Therefore

$$\tau' = -\frac{1}{\tau} = -\bar{\tau}.$$

But since $|\tau| = 1$, $Re\,\tau > 0$.

$$\therefore\ Re\,\tau' = -Re\,\bar{\tau} = -Re\,\tau < 0.$$

But $Re\tau' \geq 0$, so $Re\,\tau = 0 = Re\,\tau'$. As $|\tau| = 1 = |\tau'|$. This implies that $\tau = \tau' = 1$.

**Case.(iii)** $c \neq 0$, $d \neq 0$ then $|cd| \geq 1$ as $c$ and $d$ are integers. Now,

$$\begin{aligned} |c\tau + d|^2 &= c^2\,|\tau|^2 + d^2 + 2cd\,Re\,\tau \\ &> c^2 + d^2 + 2cd(-\frac{1}{2}) \end{aligned}$$

Add and subtract $|cd|$, wehave

$$|c\tau + d|^2 = (|c| - |d|)^2 + |cd| \geq 1.$$

Therefore $|c\tau + d| > 1$, a contradiction. Hence this case cannot arise. Thus $\tau = \tau'$ and the uniqueness of $\tau$ satisfying conditions (i) to (iv) above is established.

Geometrically, the conditions (i) to (iv) means that the point $\tau$ lies in the part of the complex plane shown in the diagram below. It is bounded by the circle $|\tau| = 1$ and the vertical lines $Re\,\tau = \pm\frac{1}{2}$, but only part of the boundary is included. Although the set is not open, it is referred to as the *fundamental region* of the unimodular group. ■

## 16.5 General Properties of Elliptic Functions

Let $f(z)$ be a meromorphic function which admits all numbers in the module $M$ with basis $(\omega_1, \omega_2)$ as periods. We shall not assume that the basis is canonical, and it will not be required that $M$ comprise all the periods.

We say that $z_1$ is congruent to $z_2$, $z_1 = z_2(\ (\text{mod}\ M))$, if the difference $z_1 - z_2 \in M$, i.e., $z_1 - z_2 = n_1\omega_1 + n_2\omega_2$.

The function $f$ takes the same values at the congruent points, and may thus be regarded as a function on the congruent classes. Let $P_a$ denote the parallelogram with vertices at $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ where $a$ is any complex number. By including part of the boundary we may represent each congruence class by exactly one point in $P_a$, and then $f$ is completely determined by its values on $P_a$. The choice of $a$ is irrelevant, and we leave it free in order to attain, for instance, that $f$ has no poles on the boundary of $P_a$.

**Theorem 16.5.1.** *An elliptic function without poles is a constant.*

**Proof.** Let $f(z)$ be an elliptic function without poles. Let $P_a$ denote the parallelogram vertices at $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ where $a$ is any complex number. Since $f(z)$ has no poles, $f(z)$ is either within and on $P_a$. It follows that $f(z)$ is continuous. Therefore $f(z)$ is bounded on the closure of $P_a$. By double periodicity, $f(z)$ is analytic. Thus, $f(z)$ is analytic and bounded in the whole complex plane. Therefore by Liouville's theorem, $f(z)$ must reduce to a constant. Hence an elliptic function without poles is a constant. ■

**Theorem 16.5.2.** *The sum of the residues of an elliptic function is zero.*

**Proof.** Let $P_a$ denote the parallelogram vertices at $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$. Let us choose the complex number $a$, so that none of the poles fall on the boundary of $P_a$. If the boundary $\partial P_a$ is traced in the positive sense, the sum of the residues at the poles in $P_a$ is given by

$$\frac{1}{2\pi i}\int_{\partial P_a} f(z)dz.$$

$$i.e., \text{the sum of the residues} = \frac{1}{2\pi i}\int_{\partial P_a} f(z)dz.$$

Consider

$$\begin{aligned}\int_{\partial P_a} f(z)dz &= \int_a^{a+\omega_1} f(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)dz + \int_{a+\omega_1+\omega_2}^{a+\omega_2} f(z)dz \\ \int_{\partial P_a} f(z)dz &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

In $I_3$, Put $z = u + \omega_2$. Then $dz = du$ and $u = z - \omega_2$

$$\begin{aligned} I_1 + I_3 &= \int_a^{a+\omega_1} f(z)dz + \int_{a+\omega_1}^{a} f(u+\omega_2)du \\ I_1 + I_3 &= 0. \end{aligned}$$

Similarly, in $I_2$, put $z = u + \omega_1$, we get

$$I_2 + I_4 = \int_a^{a+\omega_2} f(u+\omega_1) + \int_{a+\omega_2}^{a} f(z)dz = 0.$$

Hence

$$\int_{\partial P_a} f(z)dz = 0.$$

That is the sum of the residues of $f(z)$ at its poles in $P_a$ is zero. ■

**Note.** From the above theorem, it is clear that every elliptic function should have at least two simple poles or a simple of order two. That is there does not exists an elliptic function with a single simple pole.

**Theorem 16.5.3.** *A non-constant elliptic function has equally many poles as it has zeros.*

**Proof.** Let $P_a$ denote the parallelogram vertices at $a$, $a+\omega_1$, $a+\omega_2$, $a+\omega_1+\omega_2$. Let $N$ and $P$ denote the number of zeros and poles of an elliptic function $f(z)$ with $P_a$, each zero and pole being counted according to its multiplicity. From the calculus of residues, we have

$$\frac{1}{2\pi i}\int_{\partial P_a} \frac{f'(z)}{f(z)}dz = N - P.$$

Since $f(z)$ is an analytic function, $f'(z)$ is also an elliptic function. Hence $\frac{f'(z)}{f(z)}$ is also an elliptic function.

$$\therefore \frac{1}{2\pi i}\int_{\partial P_a} \frac{f'(z)}{f(z)}dz = sum\ of\ residues\ of\ \frac{f'(z)}{f(z)} = 0.$$

$$N = P \Rightarrow Number\ of\ zeros\ of\ f(z) = Number\ of\ poles\ of\ f(z).$$

Hence a non - constant elliptic function has same number of poles and zeros. ■

**Note.** If $c$ is any constant, $f(z) - c$ has the same poles as $f(z)$. Therefore all values are assumed equally many times. The number of incongruent roots of the equations $f(z) = c$ is called the order of the elliptic function.

**Theorem 16.5.4.** *The zeros $a_1, \cdots, a_n$ and poles $b_1, \cdots, b_n$ of an elliptic function satisfy* $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{M}$.

**Proof.** Let $f(z)$ be an elliptic function defined in a period parallelogram $P_a$. Let $a_1, \cdots, a_n$ and $b_1 \cdots, b_n$ be the zeros and poles of $f(z)$ respectively. Choose $a$ such that none of the zeros and poles lie on the boundary of $P_a$.

$$\frac{1}{2\pi i}\int_{\partial P_a}\frac{zf'(z)}{f(z)}dz = (a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n) \tag{16.5.1}$$

$$\therefore \frac{1}{2\pi i}\int_{\gamma}\frac{g(z)f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j)g(a_j) - \sum_k n(\gamma, b_k)g(b_k),$$

from the argument principle. Hence from (16.5.1), $g(z) = z$, $g(a_j) = a_j$ and $g(b_k) = b_k$. Now consider,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial P_a}\frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{zf'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{a+\omega_1}^{a+\omega_1+\omega_2}\frac{zf'(z)}{f(z)}dz\\ &+ \frac{1}{2\pi i}\int_{a+\omega_1+\omega_2}^{a+\omega_2}\frac{zf'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{a+\omega_2}^{a}\frac{zf'(z)}{f(z)}dz\end{aligned}$$

$$\frac{1}{2\pi i}\int_{\partial P_a}\frac{zf'(z)}{f(z)}dz = I_1 + I_2 + I_3 + I_4 \tag{16.5.2}$$

Consider $I_3$, put $z = u + \omega_2$, we have

$$\begin{aligned}I_1 + I_3 &= \frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{zf'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{a+\omega_1}^{a}(u+\omega_2)\frac{f'(u+\omega_2)}{f(u+\omega_2)}du\\ &= -\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(u)}{f(u)}du\\ I_1 + I_3 &= -\omega_2(-n_2),\end{aligned}$$

where $n_2$ represents the winding number around the origin of the closed curve denoted by $f(z)$ where $z$ varies from $a$ to $a + \omega_1$ and consequently it is an integer. Thus

$$I_1 + I_3 = n_2\omega_2$$

Similarly,

$$I_2 + I_4 = n_1\omega_1.$$

$$\therefore (16.5.2) \Rightarrow \frac{1}{2\pi i}\int_{\partial P_a} \frac{z f'(z)}{f(z)} dz = n_1\omega_1 + n_2\omega_2 \qquad (16.5.3)$$

From (16.5.1) and (16.5.3), we have

$$\begin{aligned}(a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n) &= n_1\omega_1 + n_2\omega_2\\ \Rightarrow (a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n) &\in M\\ \Rightarrow (a_1 + a_2 + \cdots + a_n) &\equiv (b_1 + b_2 + \cdots + b_n) \pmod M.\end{aligned}$$

Hence the theorem is proved. ■

# BLOCK-V

# UNIT 17

# The Weierstrass Theory

**Objectives**

After completion of this Unit, students will be able to

- identify the Weierstrass $\wp$ function.
- prove the differential equation satisfied by $\wp(z)$.
- solve problems in Weierstrass function.

## 17.1 Introduction

The simplest elliptic functions are of order 2, and such functions have either a double pole with residue zero, or two simple poles with opposite residues. We shall follow the classical example of Weierstrass, who chose a function with a double pole as the starting point of a systematic theory.

## 17.2 The Weierstrass $\wp$ Function

Weierstrass considered an elliptic function $f(z)$ with double pole at the origin.

$$\therefore\ f(z) = \frac{1}{z^2} + \textit{regular part} \Rightarrow f(-z) = \frac{1}{z^2} + \textit{regular part}$$

$$\therefore\ f(z) - f(-z) = k.$$

Put $z = \frac{\omega_1}{2}$ then $f\left(\frac{\omega_1}{2}\right) - f\left(\frac{-\omega_1}{2}\right) = k$

$$f\left(\frac{\omega_1}{2}\right) - f\left(-\omega_1 + \frac{\omega_1}{2}\right) = k$$

Since $\omega_1$ is a period of $f(z)$, $-\omega_1$ is also a period. Therefore

$$f\left(-\omega_1 + \frac{\omega_1}{2}\right) = f\left(\frac{\omega_1}{2}\right) \Rightarrow k = 0$$

$$\therefore\ f(z) = f(-z)$$

This implies that $f(z)$ is an even function. Hence

$$f(z) = \frac{1}{z^2} + a_1 z^2 + a_2 z^4 + \cdots.$$

Thus $f(z)$ is uniquely determined and it is denoted by a special typographical symbol $\wp(z)$.

$$\wp(z) = \frac{1}{z^2} + a_1 z^2 + a_2 z^4 + \cdots$$

is an elliptic function which is even and has a double pole at the origin and points of the form $n_1\omega_1 + n_2\omega_2$. Our next result shows that,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

where the sum ranges over all $\omega = n_1\omega_1 + n_2\omega_2$ except $0$.

**Result.** The Weierstrass ℘ function has the following properties,

(i) $\wp(z)$ is an even function.

(ii) $\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$ is well defined, where the sum ranges over all $\omega = n_1\omega_1 + n_2\omega_2$ except $0$.

(iii) $\wp(z)$ is meromorphic with double poles at the origin and all the points $\omega = n_1\omega_1 + n_2\omega_2$

(iv) $\wp(z)$ is doubly periodic with periods $\omega_1$ and $\omega_2$.

**Proof.** To prove (i):
From the definition

$$\wp(z) = \frac{1}{z^2} + a_1 z^2 + a_2 z^4 + \cdots \Rightarrow \wp(-z) = \wp(z)$$

$\therefore\ \wp(z)$ is an even function.To prove (ii):

To prove that the series

$$\sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)$$

converges. Consider the singular part $\dfrac{1}{(z-\omega)^2}$. Choose $z$, such that $|\omega| > 2|z|$.

$$\begin{aligned}
\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2} &= \frac{2z\omega - z^2}{\omega^2(z-\omega)^2}\\
&= \frac{z\left(2-\frac{z}{\omega}\right)}{\omega^3\left(1-\frac{z}{\omega}\right)^2}
\end{aligned}$$

$$\begin{aligned}
\left|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right| &\leq \frac{|z|\left(2+\frac{|z|}{|\omega|}\right)}{|\omega|^3\left(1-\frac{|z|}{|\omega|}\right)^2}\\
&= \frac{|z|\left(2+\frac{1}{2}\right)}{|\omega|^3\left(1-\frac{1}{2}\right)^2}\\
\left|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right| &\leq \frac{10\,|z|}{|\omega|^3}
\end{aligned}$$

Hence the series is uniformly convergent on every compact set, if the series

$$\sum_{\omega\neq 0}\frac{1}{|\omega|^3} < \infty.$$

Now,

$$\sum_{\omega\neq 0}\frac{1}{|\omega|^3} = \sum_{\omega\neq 0}\frac{1}{|n_1\omega_1 + n_2\omega_2|^3}.$$

$\dfrac{n_1\omega_1 + n_2\omega_2}{|n_1| + |n_2|}$ is the arithmetic mean of $(|n_1|+|n_2|)$ quantities. Since $\dfrac{\omega_2}{\omega_1}$ is nonreal, the arithmetic mean is zero. Hence there exists $k$ such that

$$\frac{|n_1\omega_1 + n_2\omega_2|}{|n_1| + |n_2|} \geq k, \quad for \ real \ pairs \ (n_1, n_2)$$

$$\frac{1}{|n_1\omega_1 + n_2\omega_2|} \leq \frac{1}{k(|n_1| + |n_2|)}$$

$$\therefore \sum \frac{1}{|n_1\omega_1 + n_2\omega_2|^3} \leq \sum \frac{1}{k^3(|n_1| + |n_2|)^3}$$

If we consider only integers there are $4n$ pairs $(n_1, n_2)$ with $|n_1| + |n_2| = n, \quad n = 1, 2, \cdots$

$$\sum \frac{1}{k^3(|n_1| + |n_2|)^3} = \frac{1}{k^3} \sum_{1}^{\infty} \frac{4n}{n^3}$$

$$\sum \frac{1}{k^3(|n_1| + |n_2|)^3} < \infty$$

$$\therefore \sum \frac{1}{|\omega|^3} < \infty.$$

Hence the series

$$\sum_{\omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is convergent.

$$\therefore \wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is well defined.

To prove (iii):

Since $\wp(z)$ is a meromorphic function with double pole at $z = 0$, and at all convergent points $z = \omega = n_1\omega_1 + n_2\omega_2$.

To prove (iv):

First to prove that, $\wp(z + \omega_1) = \wp(z), \quad \forall z$.

Consider

$$\therefore \wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Since it is convergent, it is differentiable term by term.

$$\therefore \wp'(z) = \sum_{\omega=-\infty}^{\infty} \frac{-2}{(z-\omega)^3}$$

Putting $z = z + \omega_1$, we get

$$\begin{aligned}\wp'(z+\omega_1) &= \sum_{\omega=-\infty}^{\infty} \frac{-2}{[z-(\omega-\omega_1)]^3}\\ &= \sum_{\omega=-\infty}^{\infty} \frac{-2}{(z-\omega)^3}\end{aligned}$$

Hence $\wp'(z+\omega_1) = \wp'(z)$. On integrating, we get

$$\wp(z+\omega_1) = \wp(z) + c.$$

Take $z = -\frac{\omega_1}{2}$. Then

$$\wp\left(-\frac{\omega_1}{2} + \omega_1\right) = \wp\left(-\frac{\omega_1}{2}\right) + c = \wp\left(\frac{\omega_1}{2}\right) + c.$$

$$c = \wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega_1}{2}\right) = 0.$$

Hence $\wp(z+\omega_1) = \wp(z)$. Similarly, we can prove that $\wp(z+\omega_2) = \wp(z)$. Therefore $\wp(z)$ is doubly periodic with periods $\omega_1$ and $\omega_2$. ■

**Note.** For convenient reference we display the important formula

$$\wp'(z) = -2\sum_{\omega=-\infty}^{\infty} \frac{1}{(z-\omega)^3}.$$

## 17.3 The Functions $\zeta(z)$ and $\sigma(z)$

~~Because $\wp(z)$ has zero residues, it is the derivative of a single - valued function. It is traditional~~

to denote the antiderivative of $\wp(z)$ by $-\zeta(z)$, and to normalize it so that it is odd.

**Weierstrass zeta function:**

We obtain Weierstrass zeta function $\zeta(z)$ using the relation,

$$\wp(z) = -\zeta'(z),$$

provided that

$$\lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0.$$

To prove that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

Consider, $\zeta'(z) = -\wp(z)$.

$$\zeta'(z) = -\frac{1}{z^2} - \sum_{\omega \neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

$$\zeta'(z) + \frac{1}{z^2} = -\sum_{\omega \neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

Integrating along the path $0$ to $z$, we have

$$\int_0^z \left(\zeta'(z) + \frac{1}{z^2}\right) dz = -\int_0^z \sum_{\omega \neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) dz$$

$$\left[\zeta(z) - \frac{1}{z}\right]_0^z = \sum_{\omega \neq 0}\left[\frac{1}{z-\omega} + \frac{z}{\omega^2}\right]_0^z$$

Since

$$\lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0,$$

we get

$$\zeta(z) - \frac{1}{z} = \sum_{\omega \neq 0}\left[\left(\frac{1}{z-\omega} + \frac{z}{\omega^2}\right) - \left(-\frac{1}{\omega} + 0\right)\right]$$

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0}\left(\frac{1}{z-\omega} + \frac{z}{\omega^2} + \frac{1}{\omega}\right).$$

where $\zeta(z)$ is an odd function and also it has a simple pole at the origin with residue $1$.

**Legendre's Relation.**

The constants $\eta_1$ and $\eta_2$ are connected with $\omega_1$ and $\omega_2$ by the relation, $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$.

**Proof.** Consider Weierstrass zeta function $\zeta(z)$. Let us choose a periodic parallelogram $P_a$ having vertices at $a$, $a+\omega_1$, $a+\omega_1+\omega_2$, $a+\omega_2$. Let $a$, be chosen, so that the origin is the only pole of $\zeta(z)$ lying inside $P_a$.
By Cauchy's residue theorem,

$$\int_{\partial P_a} \zeta(z)dz = 2\pi i[Residue\ of\ \zeta(z)\ at\ z=0].$$

But, $\zeta(z)$ has a simple pole at $z=0$ with residue $1$. Hence, we have

$$\int_{\partial P_a} \zeta(z)dz = 2\pi i(1) = 2\pi i.$$

Consider,

$$\begin{aligned}\int_{\partial P_a} \zeta(z)dz &= \int_a^{a+\omega_1} \zeta(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} \zeta(z)dz + \int_{a+\omega_1+\omega_2}^{a+\omega_2} \zeta(z)dz + \int_{a+\omega_2}^{a} \zeta(z)dz \\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

Consider,

$$I_3 = \int_{a+\omega_1+\omega_2}^{a+\omega_2} \zeta(z)dz$$

Put $z = u+\omega_2 \Rightarrow u = z-\omega_2 \Rightarrow du = dz$.

$$\therefore I_3 = \int_{a+\omega_1}^{a} [\zeta(u)+\zeta(\omega_2)]du = \int_{a+\omega_1}^{a} (\zeta(u)+\eta_2)du,$$

where $\eta_2 = \zeta(\omega_2)$. Now

$$\begin{aligned} I_1 + I_3 &= \int_a^{a+\omega_1} \zeta(z)dz - \int_a^{a+\omega_1} (\zeta(u)+\eta_2)du \\ &= -\eta_2 \int_a^{a+\omega_1} du \\ I_1 + I_3 &= -\eta_2\omega_1 \end{aligned}$$

Now consider,

$$I_2 = \int_{a+\omega_1}^{a+\omega_1+\omega_2} \zeta(z)dz$$

Put $z = u + \omega_1 \Rightarrow u = z - \omega_1 \Rightarrow dz = du$.

$$\therefore I_2 = \int_a^{a+\omega_2} \zeta(u + \omega_1)du = \int_a^{a+\omega_2} (\zeta(u) + \eta_1)du$$

$$\begin{aligned} I_2 + I_4 &= \int_a^{a+\omega_2} (\zeta(u) + \eta_1)du - \int_a^{a+\omega_2} \zeta(z)dz \\ &= \eta_1 \int_a^{a+\omega_2} d\omega \\ I_2 + I_4 &= \eta_1\omega_2 \end{aligned}$$

$$\therefore \int_{\partial P_a} \zeta(z)dz = \eta_1\omega_2 - \eta_2\omega_1$$

Hence $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. ■

**Weierstrass $\sigma$ Function.** The canonical product representation of Weierstrass sigma function $\sigma(z)$ is given by

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

where the product ranges over $\omega = \eta_1\omega_1 + \eta_2\omega_2$ except $0$.

**Proof.** Consider the Weierstrass zeta function

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \left(\frac{1}{z - \omega} + \frac{z}{\omega^2} + \frac{1}{\omega}\right).$$

This is analytic at the origin. Hence the series,

$$\sum_{\omega \neq 0} \left(\frac{1}{z - \omega} + \frac{z}{\omega^2} + \frac{1}{\omega}\right)$$

converges absolutely and uniformly about the origin. Hence we can integrate the series termwise along any path starting from the origin and not passing through the point $z = \omega$.

$$\text{Consider} \quad \int_0^z \left(\zeta(z) - \frac{1}{z}\right) dz = \int_0^z \sum_{\omega \neq 0} \left(\frac{1}{z-\omega} + \frac{z}{\omega^2} + \frac{1}{\omega}\right) dz$$

$$= \sum_{\omega \neq 0} \left(\log\left(\frac{z-\omega}{-\omega}\right) + \frac{z^2}{2\omega^2} + \frac{z}{\omega}\right)$$

$$= \sum_{\omega \neq 0} \left(\log\left(1 - \frac{z}{\omega}\right) + \log e^{\frac{z^2}{2\omega^2}} + \log e^{\frac{z}{\omega}}\right)$$

$$= \sum_{\omega \neq 0} \log\left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

$$e^{\int_0^z \left(\zeta(z) - \frac{1}{z}\right) dz} = \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

The $\sigma$ function is defined by

$$\sigma(z) = z e^{\int_0^z \left(\zeta(z) - \frac{1}{z}\right) dz}$$

This implies that

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}.$$

where the product converges and represents an entire function. This is the canonical product representation of $\sigma(z)$. ■

**Properties of $\sigma$ function.**

(i) $\dfrac{\sigma'(z)}{\sigma(z)} = \zeta(z)$.

(ii) $\sigma(z)$ is an odd function.

(iii) When $z$ is changed to $z + \omega_1$, $\sigma(z)$ is multiplied by an exponential function.

$$\sigma(z + \omega_1) = -\sigma(z)\, e^{\eta_1\left(z + \frac{\omega_1}{2}\right)}$$

$$\sigma(z + \omega_2) = -\sigma(z)\, e^{\eta_2\left(z + \frac{\omega_2}{2}\right)}$$

**Proof.** To prove (i):

Consider

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}.$$

Taking logarithms on both sides, we get

$$\log(\sigma(z)) = \log z + \sum_{\omega \neq 0} \left[\log\left(1 - \frac{z}{\omega}\right) + \frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right].$$

On differentiating, we get

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \neq 0}\left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right] = \zeta(z).$$

Hence

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z).$$

To prove (ii):

Consider

$$\sigma(-z) = -z e^{\int_0^{-z} \left(\zeta(-z) + \frac{1}{z}\right) d(-z)}$$

Consider

$$\int_0^{-z} \left[\zeta(-z) + \frac{1}{z}\right] d(-z) = \int_0^{-z} -\left[\zeta(z) - \frac{1}{z}\right](-dz)$$

Since $\zeta$ is an odd function, $\zeta(-z) = -\zeta(z)$. Therefore

$$\int_0^{-z} \left[\zeta(-z) + \frac{1}{z}\right] d(-z) = \int_0^{-z} \left[\zeta(z) - \frac{1}{z}\right] dz$$

Put $-z = t$, we have

$$\int_0^{-z} \left[\zeta(z) - \frac{1}{z}\right] dz = \int_0^{t} \left[\zeta(t) - \frac{1}{t}\right] dt = \int_0^{z} \left[\zeta(z) - \frac{1}{z}\right] dz$$

Therefore

$$\sigma(-z) = -z e^{\int_0^z \zeta(z) - \frac{1}{z} dz} = -\sigma(z)$$

This implies that $\sigma(z)$ is an odd function.

To prove (iii):

We know that $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$. When $z = z + \omega_1$, we have

$$\begin{aligned}
\frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)} &= \zeta(z+\omega_1) \\
&= \zeta(z) + \eta_1 \\
\frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)} &= \frac{\sigma'(z)}{\sigma(z)} + \eta_1
\end{aligned}$$

Integrating with respect to $z$, we get

$$\begin{aligned}
\log\sigma(z+\omega_1) &= \log(\sigma(z)) + \eta_1 z + C_1 \\
&= \log\sigma(z) + \log e^{\eta_1 z + C_1} \\
\log\sigma(z+\omega_1) - \log\sigma(z) &= \log e^{\eta_1 z + C_1} \\
\frac{\sigma(z+\omega_1)}{\sigma(z)} &= e^{\eta_1 z + C_1}
\end{aligned}$$

To find $C_1$:
Put $z = -\frac{\omega_1}{z}$ in the above equation, we have

$$\begin{aligned}
\frac{\sigma\left(\frac{\omega_1}{2}\right)}{\sigma\left(-\frac{\omega_1}{2}\right)} &= e^{-\eta_1\frac{\omega_1}{2}+C_1} \\
\frac{\sigma\left(\frac{\omega_1}{2}\right)}{-\sigma\left(\frac{\omega_1}{2}\right)} &= e^{-\eta_1\frac{\omega_1}{2}+C_1} \\
-1 &= e^{-\eta_1\frac{\omega_1}{2}+C_1} \\
C_1 &= \log(-1) + \frac{\eta_1\omega_1}{2} \\
\therefore \frac{\sigma(z+\omega_1)}{\sigma(z)} &= e^{\eta_1 z + \frac{\eta_1\omega_1}{2} + \log(-1)} \\
&= -e^{\eta_1\left(z+\frac{\omega_1}{2}\right)} \\
\sigma(z+\omega_1) &= -\sigma(z)e^{\eta_1\left(z+\frac{\omega_1}{2}\right)}
\end{aligned}$$

Similarly, we can prove

$$\sigma(z+\omega_2) = -\sigma(z)e^{\eta_2\left(z+\frac{\omega_2}{2}\right)}$$

∎

**Note. (i)** $\sigma(z)$ is not an elliptic function and it has a single simple pole at the origin with residue 1 and $\omega_1$, $\omega_2$ are not periods.

**Note. (ii)** The functions $\zeta(z)$ and $\sigma(z)$ are called pseudo periodic functions of Weierstrass.

## 17.4 The Differential Equations

Prove that the differential equation satisfied by $\wp(z)$ is

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

**Proof.** Consider the function $\zeta(z)$,

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \left( \frac{1}{z-\omega} + \frac{z}{\omega^2} + \frac{1}{\omega} \right).$$

The Laurent's expansion of $\zeta(z)$ around the origin can be obtained. Consider

$$\begin{aligned}
\zeta(z) &= \frac{1}{z} + \sum_{\omega \neq 0} \left( -\frac{1}{\omega}\frac{1}{1-\frac{z}{\omega}} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \\
&= \frac{1}{z} + \sum_{\omega \neq 0} \left( -\frac{1}{\omega}\left(1 + \frac{z}{\omega} + \frac{z^2}{\omega^2} + \cdots\right) + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \\
\zeta(z) - \frac{1}{z} &= \sum_{\omega \neq 0} \left( -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots \right) \\
\zeta(z) - \frac{1}{z} &= -z^2 \sum_{\omega \neq 0} \frac{1}{\omega^3} - z^3 \sum_{\omega \neq 0} \frac{1}{\omega^4} - \cdots
\end{aligned}$$

Since $\zeta(z)$ is an odd function only odd powers of $z$ occurs in its expression (regular part). Therefore

$$\zeta(z) - \frac{1}{z} = -z^3 \sum_{\omega \neq 0} \frac{1}{\omega^4} - z^5 \sum_{\omega \neq 0} \frac{1}{\omega^6} \cdots \qquad (17.4.1)$$

Let

$$G_2 = \sum_{\omega \neq 0} \frac{1}{\omega^4}, \quad G_3 = \sum_{\omega \neq 0} \frac{1}{\omega^6}, \ ect$$

In general,

$$G_k = \sum_{\omega \neq 0} \frac{1}{\omega^{2k}}$$

$$(17.4.1) \Rightarrow \zeta(z) - \frac{1}{z} = -G_2 z^3 - G_3 z^5 - \cdots = -\sum_{k=2}^{\infty} G_k z^{2k-1}$$

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k z^{2k-1}$$

Differentiating with respect to $z$, we get

$$\begin{aligned}
\zeta'(z) &= -\frac{1}{z^2} - \sum_{k=2}^{\infty} G_k(2k-1)z^{2k-2} \\
-\wp'(z) &= -\frac{1}{z^2} - \sum_{k=2}^{\infty} (2k-1)G_k z^{2k-2} \\
\wp(z) &= \frac{1}{z^2} + 3G_2 z^2 + 5G_4 z^4 + \cdots \\
\wp'(z) &= -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3 + \cdots \\
(\wp'(z))^2 &= \frac{4}{z^6} - 24\frac{G_2 z}{z^3} - 80\frac{G_3 z^3}{z^3} + \cdots
\end{aligned}$$

Now

$$\begin{aligned}
\wp^3(z) &= \frac{1}{z^6} + \frac{9G_2 z^2}{z^4} + \frac{15G_3 z^4}{z^4} + \cdots \\
4\wp^3(z) &= \frac{4}{z^6} + \frac{36G_2}{z^2} + 60G_3 + \cdots
\end{aligned}$$

$$60G_2\wp(z) = \frac{60G_2}{z^2} + 180G_2 z^2 + \cdots$$

*Consider* $(\wp'(z))^2 - 4\wp^3(z) + 60G_2\wp(z) = \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 - \frac{4}{z^6} - \frac{36G_2}{z^2} - 60G_3 + \frac{60G_2}{z^2} + \cdots$

$$= -140G_3 + \cdots$$

Here left hand side is doubly periodic function and right hand side has no poles. Also right hand side is an analytic function in the whole complex plane. Therefore right hand side is an elliptic function without poles. Hence it must reduce to a constant. Let the constant be $k = -140G_3$.

Hence we have

$$(\wp'(z))^2 - 4\wp^3(z) + 60G_2\wp(z) = -140G_3$$
$$(\wp'(z))^2 = 4\wp^3(z) - 60G_2\wp(z) - 140G_3$$

Take $g_2 = 60G_2$ and $g_3 = 140G_3$ then we have

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

■

**Note.** $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$ is a first order differential equation for $\omega = \wp(z).\omega(z)$

$$= \wp(z)$$
$$\frac{d\omega(z)}{dz} = \wp'(z)$$
$$\text{we have } \left(\frac{d\omega}{dz}\right)^2 = 4\omega^3 - g_2\omega - g_3$$
$$\frac{d\omega}{dz} = \sqrt{4\omega^3 - g_2\omega - g_3}$$
$$z = \int_0^{\omega} \frac{1}{\sqrt{4\omega^3 - g_2\omega - g_3}}\, d\omega$$

which shows that $\wp(z)$ is the inverse of an elliptic integral. Moreover,

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{d\omega}{\sqrt{4\omega^3 - g_2\omega - g_3}}$$

where the path of integration is the image under $\wp$ of a path from $z_0$ to $z$ that avoids the zeros and poles of $\wp'(z)$ and where the sign of the square root must be chosen so that it actually equals $\wp'(z)$.

**Problems.**

**Problem 17.4.1.** Show that $\wp(z) - \wp(u) = -\dfrac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$

**Solution.** Let $f(z) = \wp(z) - \wp(u)$. Then $f(z)$ is an elliptic function with zeros at $z = u$ and

$z = -u$ double pole at the origin. Therefore $f(z)$ can be written as

$$
\begin{aligned}
f(z) &= c\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z-0)\sigma(z-0)} \quad c = constant \\
f(z) &= c\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2} \\
\wp(z) - \wp(u) &= c\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2} \\
\left(\frac{1}{z^2} + a_1 z^2 + a_2 z^4 + \cdots\right) - \wp(u) &= c\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2} \\
(1 + a_1 z^4 + a_2 z^6 + \cdots) - z^2\wp(u) &= cz^2\frac{\sigma(z-u)\sigma(z+u)}{\sigma(} \\
&= cz^2\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2 \prod_{\omega \neq 0}\left(1-\frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}}
\end{aligned}
$$

Taking $\lim z \to 0$ we get

$$
\begin{aligned}
1 &= \lim_{z\to 0}\frac{c\sigma(z-u)\sigma(z+u)}{\prod_{\omega \neq 0}\left(1-\frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}} \\
1 &= c\sigma(-u)\sigma(u) \\
c &= -\frac{1}{\sigma(u)^2}
\end{aligned}
$$

Hence

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

**Problem 17.4.2.** Prove that $\dfrac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$.

**Solution.**

$$Since\ \wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

Taking logarithmic derivatives, we get

$$-\frac{\wp'(z)}{\wp(u)-\wp(z)} = \frac{\sigma'(z-u)}{\sigma(z-u)} + \frac{\sigma'(z+u)}{\sigma(z+u)} - 2\frac{\sigma'(z)}{\sigma(z)}$$

$$-\frac{\wp'(z)}{\wp(u)-\wp(z)} = \zeta(z-u)+\zeta(z+u)-2\zeta(z)$$
$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u)+\zeta(z+u)-2\zeta(z)$$

**Problem 17.4.3.** Prove that $\zeta(z+u)=\zeta(z)+\zeta(u)+\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$

**Solution.** We have

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u)+\zeta(z+u)-2\zeta(z)$$

Also,

$$\frac{\wp'(u)}{\wp(u)-\wp(z)} = \zeta(u-z)+\zeta(u+z)-2\zeta(u)$$
$$-\frac{\wp'(u)}{\wp(z)-\wp(u)} = -\zeta(z-u)+\zeta(z+u)-2\zeta(u)$$

$$\begin{aligned}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)} &= 2\zeta(z+u)+\zeta(z-u)-\zeta(z-u)-2\zeta(z)-2\zeta(u)\\ \Rightarrow \zeta(z+u) &= \zeta(z)+\zeta(u)+\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\end{aligned}$$

**Problem 17.4.4.** The addition theorem for the $\wp-$ function $\wp(z+u) = -\wp(z)-\wp(u)+\frac{1}{4}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2$.

**Solution.** We have

$$\zeta(z+u)=\zeta(z)+\zeta(u)+\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$$

Differentiating with respect to $z$, we have

$$\zeta'(z+u)=\zeta'(z)+\frac{1}{2(\wp(z)-\wp(u))^2}\left((-\wp(u)+\wp(z))\wp''(z)-(\wp'(z)-\wp'(u))\wp'(z)\right) \tag{17.4.2}$$

Differentiating with respect to $u$

$$\zeta'(z+u)=\zeta'(u)+\frac{1}{2(\wp(z)-\wp(u))^2}\left((\wp(z)-\wp(u))\wp''(u)-(\wp'(z)-\wp'(u))\wp'(u)\right) \tag{17.4.3}$$

Adding (17.4.2) and (17.4.3), we get

$$-2\wp(z+u)=-\wp(z)-\wp(u)+\frac{\wp''(z)-\wp''(u)}{2(\wp(z)-\wp(u))}-\frac{1}{2}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2$$

Consider the differential equation

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

Differentiating we get

$$\begin{aligned} 2\wp'(z)\wp''(z) &= 12\wp^2(z)\wp'(z) - g_2\wp'(z) \\ \wp''(z) &= 6\wp^2(z) - \frac{1}{2}g_2 \\ Similarly,\ \wp''(u) &= 6\wp^2(u) - \frac{1}{2}g_2 \\ \wp''(z) - \wp''(u) &= 6(\wp^2(z) - \wp^2(u)) \end{aligned}$$

$$\begin{aligned} \therefore -2\wp(z+u) &= -(\wp(z)+\wp(u)) + \frac{6(\wp^2(z) - \wp^2(u))}{2(\wp(z) - \wp(u))} - \frac{1}{2}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2 \\ \wp(z+u) &= -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2 \end{aligned}$$

**Problem 17.4.5.** Prove that $\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$

**Solution.** In addition theorem, put $u = z + h$, then we get

$$\wp(2z+h) = -\wp(z) - \wp(z+h) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(z+h)}{\wp(z) - \wp(z+h)}\right)^2$$

Taking limit $h \to 0$, we get

$$\wp(2z) = \frac{1}{4}\frac{\wp'(z)}{\wp'(z)} - 2\wp(z).$$

**Problem 17.4.6.** Prove $\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}$

**Solution.** Consider

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

Put $u = z + h$

$$\therefore \wp(z) - \wp(z+h) = -\frac{\sigma(-h)\sigma(2z+h)}{\sigma(z)^2\sigma(z+h)^2}$$

$$= -\frac{-\frac{h}{\omega_0}\left(1+\frac{h}{\omega}\right)e^{-\frac{h}{\omega}+\frac{1}{2}\left(\frac{h}{\omega}\right)^2}\sigma(2z+h)}{\sigma(z)^2\sigma(z+h)^2}$$

$$\frac{\wp(z) - \wp(z+h)}{h} = \frac{\frac{1}{\omega_0}\left(1+\frac{h}{\omega}\right)e^{-\frac{h}{\omega}+\frac{1}{2}\left(\frac{h}{\omega}\right)^2}\sigma(2z+h)}{\sigma(z)^2\sigma(z+h)^2}$$

$$\lim_{h\to 0}\frac{\wp(z) - \wp(z+h)}{h} = \lim_{h\to 0}\frac{\frac{1}{\omega_0}\left(1+\frac{h}{\omega}\right)e^{-\frac{h}{\omega}+\frac{1}{2}\left(\frac{h}{\omega}\right)^2}\sigma(2z+h)}{\sigma(z)^2\sigma(z+h)^2}$$

$$-\wp'(z) = \frac{\sigma(2z)}{\sigma(z)^2\sigma(z)^2}$$

$$\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}$$

**Problem 17.4.7.** Prove that

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0$$

**Solution.** Since $\wp(z)$ and $\wp(z+u)$ are both elliptic function of same periods there exists an algebraic relation between the two functions. To determine this relation we proceed as follows: Let

$$f(z) = \wp'(z)A\wp(z) + B \tag{17.4.4}$$

where $A$ and $B$ are constant. Since $\wp(z)$ is an elliptic function of order $2$ with a double pole at the origin and its congruent points. $\wp'(z)$ is a pole of order $3$ at $z = 0$ and its congruent points. Since $\wp(z) = \frac{1}{z^2} + \cdots \Rightarrow \wp'(z) = -\frac{2}{z^3} + \cdots$

The incongruent points of $f(z)$ are $0$, $0$ and $0$. Hence $f(z)$ has $3$ zeros and their sum is zero in the fundamental parallelogram. Let $\alpha$, $z$ and $u$ be the zeros of $f(z)$. We have

$$
\begin{aligned}
sum\ of\ zeros &= sum\ of\ poles \\
z + u + \alpha &= 0 + 0 + 0 \\
\alpha &= -(z+u)
\end{aligned}
$$

$\therefore$ the incongruent zeros of $f(z)$ are $z,\ u,\ -(z+u)$.

$$f(z) = 0 \Rightarrow \wp'(z) + A\wp(z) + B = 0 \tag{17.4.5}$$

$$f(u) = 0 \Rightarrow \wp'(u) + A\wp(u) + B = 0 \tag{17.4.6}$$

$$f(-(z+u)) = 0 \Rightarrow \wp'(-(z+u)) + A\wp(-(z+u)) + B = 0$$

$$-\wp'(z+u) + A\wp(z+u) + B = 0 \tag{17.4.7}$$

Eliminating $A$ and $B$ from (17.4.5), (17.4.6) and (17.4.7) we get

$$\begin{vmatrix} \wp'(z) & \wp(z) & 1 \\ \wp'(u) & \wp(u) & 1 \\ -\wp'(z+u) & \wp(z+u) & 1 \end{vmatrix} = \begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(z+u) & -\wp'(z+u) & 1 \end{vmatrix} = 0$$

## 17.5 The Modular Function $\lambda(\tau)$

The Weierstrss $\wp-$ function satisfies the differential equation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$

Let $e_1,\ e_2$ and $e_3$ be the roots of the polynomial

$$4\wp(z)^3 - g_2\wp(z) - g_3.$$

Therefore

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{17.5.1}$$

To find the value of $e_k$, we determine the zeros of $\wp'(z)$. Since $\wp(z)$ is periodic and even.

$$\wp(\omega_1 - z) = \wp(z - \omega_1) = \wp(z)$$

Differentiating with respect to $z$, we get

$$\wp'(\omega_1 - z) = -\wp'(z)$$

Substitute $z = \frac{\omega_1}{2}$.

$$\wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(\frac{\omega_1}{2}\right) \Rightarrow \wp'\left(\frac{\omega_1}{2}\right) = 0.$$

$\therefore \frac{\omega_1}{2}$ is a zero of $\wp'(z)$. Similarly, we can show that $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ are also zeros of $\wp'(z)$. The numbers $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are the incongruent roots. We note that all the zeros of $\wp'(z)$ are simple and $\wp'(z)$ is of order $3$. Now, we can set,

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), e_2 = \wp\left(\frac{\omega_2}{2}\right) \textit{ and } e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$$

$\wp(z)$ takes each value $e_k$ with multiplicity $2$. If any two roots are equal, then that value will be taken four times which is a contradiction to the fact that $\wp(z)$ is of order $2$. Therefore all the roots are distinct.

Consider the function,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right].$$

Substitute $z = \frac{\omega_1}{2}$.

$$\wp\left(\frac{\omega_1}{2}\right) = \frac{4}{\omega^2} + \sum_{\omega \neq 0} \left[\frac{4}{(\omega_1 - 2\omega)^2} - \frac{1}{\omega^2}\right].$$

$$\therefore e_1 = \frac{4}{\omega_1^2} + \sum_{\omega \neq 0} \left[\frac{4}{(\omega_1 - 2\omega)^2} - \frac{1}{\omega^2}\right].$$

Replace $\omega_1$ by $t\omega_1$, we get

$$\therefore e_1 = \frac{4}{t^2\omega_1^2} + \sum_{\omega \neq 0} \left[\frac{4}{(t\omega_1 - 2\omega)^2} - \frac{1}{\omega^2}\right] = t^{-2}\cdot\varphi(\omega_1).$$

If the period are multiplied by $t$, then $e_k$ are multiplied by $t^{-2}$ (i.e., $e_k$ are homogeneous of order $-2$ in $\omega_1, \omega_2$). Now we consider the quantity,

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} \tag{17.5.2}$$

where $\lambda$ depends only on the ratio $\tau = \frac{\omega_2}{\omega_1}$. Since $e_1 \neq e_2$, $\lambda$ is analytic rather than meromorphic. Again, since $e_3 \neq e_2$, $\lambda$ is never equal to zero. Also $e_3 \neq e_1$, $\lambda$ is never equal to one. Also $\lambda(\tau)$ is a quotient of two analytic function in the upper half plane $Im(\tau) > 0$. Now, we shall study the dependence of $\tau$. If the periods are subjected to unimodular transformation

$$\begin{aligned} \omega'_2 &= a\omega_2 + b\omega_1 \\ \omega'_1 &= c\omega_2 + d\omega_1 \end{aligned} \tag{17.5.3}$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$$

Then the $\wp$ function does not change. Therefore by looking at (17.5.1) the roots $e_k$ can atmost be permuted. If $a \equiv d \equiv 1 \pmod 2$ and $b \equiv c \equiv 0 \pmod 2$ then $\omega'_2 = \omega_2$; $\omega'_1 = \omega_1$. This implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

under this condition the $e_k$ do not change and we have shown that

$$\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

**Congruence subgroup** (mod 2). The transformation which satisfy the congruence relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

form a subgroup of the modular group, known as the *congruence subgroup* (mod 2)

***Automorphic Function.*** *An analytic function or meromorphic function which is invariant under a group of linear transformation is called automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.*

***Result.*** *Show that* $\lambda(\tau)$ *satisfies the functional equation* $\lambda(\tau+1) = \dfrac{\lambda(\tau)}{\lambda(\tau)-1}$ *and* $\lambda\left(-\dfrac{1}{\tau}\right) = 1-\lambda(\tau)$.

**Solution.** Consider the matrices congruent (mod 2)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the first case,

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod 2$$

then

$$\frac{\omega_2'}{2} = \frac{\omega_2 + \omega_1}{2} \equiv \frac{\omega_3}{2}$$

and

$$\frac{\omega_1'}{2} = \frac{\omega_1}{2}.$$

This means that, $e_2$ and $e_3$ are interchanged while $e_1$ remains fixed and hence $\lambda$ goes over into $\dfrac{e_2 - e_3}{e_1 - e_3}$.

Now

$$\frac{\lambda(\tau)}{\lambda(\tau)-1} = \frac{\dfrac{e_3 - e_2}{e_1 - e_2}}{\dfrac{e_3 - e_2}{e_1 - e_2} - 1} = \frac{e_3 - e_2}{e_3 - e_1}$$

$$\therefore \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}.$$

In the second case, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$$

Then

$$\frac{\omega_2'}{2} = \frac{\omega_1}{2} \Rightarrow \frac{\omega_1'}{2} = \frac{\omega_2}{2}.$$

This means that $e_1$ and $e_2$ are interchanged and hence $\lambda$ goes over into $\frac{e_3-e_1}{e_2-e_1}$. Now consider,

$$1 - \lambda(\tau) = 1 - \frac{e_3-e_2}{e_1-e_2} = \frac{e_1-e_3}{e_1-e_2} = \frac{e_3-e_1}{e_2-e_1}.$$

Thus $\lambda\left(\frac{1}{\tau}\right) = 1 - \lambda(\tau)$.

## 17.6 The Conformal Mapping by $\lambda(\tau)$

**Theorem 17.6.1.** *The modular function $\lambda(\tau)$ effects a one - one conformal mapping of the region $\omega$ onto the upper half plane. The mapping extends continuously to the boundary in such way that $\tau = 0, 1\ \infty$ correspond to $\lambda = 1, \infty, 0$.*

**Proof.** Consider the modular function,

$$\lambda(\tau) = \frac{e_3-e_2}{e_1-e_2}$$

where $\tau = \dfrac{\omega_2}{\omega_1}$ is non - real. $\tau$ is normalized by taking $\omega_1 = 1$ and $\omega_2 = \tau$.

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

$$\begin{aligned}
Consider,\ e_1 &= \wp\left(\frac{\omega_1}{2}\right) = \frac{4}{\omega_1^2} + \sum_{\omega \neq 0} \left[ \frac{1}{\left(\frac{\omega_1}{2} - \omega\right)^2} - \frac{1}{\omega^2} \right] \\
&= \frac{4}{1} + \sum_{n_1 + n_2\tau \neq 0} \left[ \frac{1}{\left(\frac{1}{2} - n_1 - n_2\tau\right)^2} - \frac{1}{(n_1 + n_2\tau)^2} \right] \qquad (\omega = n_1\omega_1 + n_2\omega_2 \Rightarrow \omega = n_1 + n_2\tau) \\
&= 4 + \sum_{n_1 + n_2\tau \neq 0} \left[ \frac{1}{\left(n_1 - \frac{1}{2} + n_2\tau\right)^2} - \frac{1}{(n_1 + n_2\tau)^2} \right] \\
e_1 &= \sum_{m,n=-\infty}^{\infty} \left[ \frac{1}{\left(m - \frac{1}{2} + n\tau\right)^2} - \frac{1}{(m + n\tau)^2} \right]
\end{aligned}$$

Similarly,

$$\begin{aligned}
e_2 &= \wp\left(\frac{\omega_2}{2}\right) = \frac{4}{\omega_2^2} + \sum_{\omega \neq 0} \left[ \frac{1}{\left(\frac{\omega_2}{2} - \omega\right)^2} - \frac{1}{\omega^2} \right] \\
&= \frac{4}{\tau^2} + \sum_{n_1 + n_2\tau \neq 0} \left[ \frac{1}{\left(\frac{\tau}{2} - n_1 - n_2\tau\right)^2} - \frac{1}{(n_1 + n_2\tau)^2} \right] \\
e_2 &= \sum_{m,n=-\infty}^{\infty} \left[ \frac{1}{\left(m + \tau\left(n - \frac{1}{2}\right)\right)^2} - \frac{1}{(m + n\tau)^2} \right] \\
e_3 &= \wp\left(\frac{\omega_1 + \omega_2}{2}\right) \\
&= \frac{4}{(\omega_1 + \omega_2)^2} + \sum_{\omega \neq 0} \left[ \frac{1}{\left(\frac{\omega_1 + \omega_2}{2} - \omega\right)^2} - \frac{1}{\omega^2} \right] \\
&= \frac{4}{(1+\tau)^2} + \sum_{n_1 + n_2\tau \neq 0} \left[ \frac{1}{\left(\frac{1+\tau}{2} - n_1 - n_2\tau\right)^2} - \frac{1}{(n_1 + n_2\tau)^2} \right] \\
e_3 &= \sum_{m,n=-\infty}^{\infty} \left[ \frac{1}{\left(m - \frac{1}{2} + \tau\left(n - \frac{1}{2}\right)\right)^2} - \frac{1}{(m + n\tau)^2} \right]
\end{aligned}$$

Consider

$$e_3 - e_2 = \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m - \frac{1}{2} + \tau\left(n - \frac{1}{2}\right)\right)^2} - \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m + \tau\left(n - \frac{1}{2}\right)\right)^2}$$

and

$$e_1 - e_2 = \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m - \frac{1}{2} + n\tau\right)^2} - \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m + \tau\left(n - \frac{1}{2}\right)\right)^2}$$

Consider, when $\tau$ is purely imaginary $(\bar{\tau} = -\tau)$

$$\overline{e_1} = \sum_{m,n=-\infty}^{\infty} \left[ \frac{1}{\left(m - \frac{1}{2} + n(\bar{\tau})\right)^2} - \frac{1}{(m + n(\bar{\tau}))^2} \right]$$

$$\overline{e_1} = e_1 = \sum_{m,n=-\infty}^{\infty} \left[ \frac{1}{\left(m - \frac{1}{2} + n\tau\right)^2} - \frac{1}{(m + n\tau)^2} \right]$$

$e_1$ is purely real when $\tau$ lies on the imaginary axis. This implies that $e_k's$ are real and $\tau$ is imaginary. Therefore $\lambda(\tau)$ is real when $\tau$ lies on the imaginary axis.

Now, we consider a matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

in the congruence subgroup (mod 2), we have $\lambda(\tau + 2) = \lambda(\tau)$. This implies that $\lambda$ is periodic function of period 2. $\lambda(\tau)$ can be written in the form $e^{i\pi\tau}$ because

$$e^{i\pi(\tau+2)} = e^{i\pi\tau}$$

To show that $\lambda(\tau) \to 0$ as $Im(\tau) \to \infty$.

We know that

$$\frac{\pi^2}{\sin \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z - m)^2}$$

But

$$e_3 - e_2 = \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m - \frac{1}{2} + \tau\left(n - \frac{1}{2}\right)\right)^2} - \sum_{m,n=-\infty}^{\infty} \frac{1}{\left(m + \tau\left(n - \frac{1}{2}\right)\right)^2}$$

Keeping $n$ fixed and letting $m$ to vary, we get

$$e_3 - e_2 = \sum_{n=-\infty}^{\infty} \frac{\pi^2}{\sin^2 \pi\left[\frac{1}{2} - \tau\left(n - \frac{1}{2}\right)\right]} - \sum_{n=-\infty}^{\infty} \frac{\pi^2}{\sin^2 \pi\left[-\left(n - \frac{1}{2}\right)\tau\right]}$$

$$= \pi^2 \left[ \sum_{n=-\infty}^{\infty} \frac{1}{\cos^2 \pi\left(n - \frac{1}{2}\right)\tau} - \sum_{n=-\infty}^{\infty} \frac{1}{\sin^2 \pi\left(n - \frac{1}{2}\right)\tau} \right]$$

Similarly,

$$e_1 - e_2 = \sum_{m,n=-\infty}^{\infty} \left[\frac{1}{\left(m - \frac{1}{2} + n\tau\right)^2} - \frac{1}{\left(m + \tau\left(n - \frac{1}{2}\right)\right)^2}\right]$$

Keeping $n$ fixed, we get

$$e_3 - e_2 = \sum_{n=-\infty}^{\infty} \left[\frac{\pi^2}{\sin^2\left(n - \frac{1}{2}\right)\pi\tau} - \frac{\pi^2}{\sin^2 \pi n\tau}\right]$$

$$e_1 - e_2 = \sum_{n=-\infty}^{\infty} \left[\frac{\pi^2}{\cos^2 n\pi\tau} - \frac{\pi^2}{\sin^2\left(n - \frac{1}{2}\right)\pi\tau}\right]$$

Now the series are strongly convergent for both $n \to \infty$ and $n \to -\infty$. Also, $|\cos n\pi\tau|$ and $\sin n\pi\tau$ are comparable to $e^{|n|\pi Im(\tau)}$. The consequence is uniform for $Im(\tau) \geq \delta > 0$. Now, once again consider the function

$$\lambda = \frac{e_3 - e_2}{e_1 - e_2}.$$

Now we take term wise limits. When $n = 0$, $e_3 - e_2 = 0$ and $e_1 - e_2 = \pi^2$. Therefore $\lambda(\tau) \to 0$ as $Im(\tau) \to \infty$ uniformly with respect to the real part of $\tau$. When $\tau \to 0$ along the imaginary axis.

$$i.e., \lim_{\tau\to 0} = \lim_{\tau\to 0} \frac{e_3 - e_2}{e_1 - e_2} = 1,$$

along the imaginary axis. Along the imaginary axis the series $e_3 - e_2$ and $e_1 - e_2$ are the terms corresponds to $n = 0$ and $n = 1$. Therefore the sum of the terms when $n = 0$, and $n = 1$ in $e_3 - e_2$, we have

$$\begin{aligned}
e_3 - e_2 &= \pi^2\left(\frac{1}{\cos^2\frac{\pi\tau}{2}} - \frac{1}{\sin^2\frac{\pi\tau}{2}} + \frac{1}{\cos^2\frac{\pi\tau}{2}} - \frac{1}{\sin^2\frac{\pi\tau}{2}}\right)\\
&= 2\pi^2\left(\frac{1}{\cos^2\frac{\pi\tau}{2}} - \frac{1}{\sin^2\frac{\pi\tau}{2}}\right)\\
&= 2\pi^2\left(\frac{4e^{i\pi\tau}}{(e^{i\pi\tau}+1)^2} + \frac{4e^{i\pi\tau}}{(e^{i\pi\tau}-1)^2}\right)\\
e_3 - e_2 &= \frac{16\pi^2 e^{i\pi\tau}(1 + e^{2\pi i\tau})}{(e^{i2\pi\tau} - 1)^2}
\end{aligned}$$

Similarly,

$$e_1 - e_2 = \pi^2\left(\frac{1}{\cos^2 \pi\tau} + \left(\frac{e^{i\pi\tau}+1}{e^{i\pi\tau} - 1}\right)^2\right)$$

$$\lim_{Im(\tau)\to\infty} \lambda(\tau)e^{-i\pi\tau} = 16.$$

Consider the region $\Omega$ bounded by the imaginary axis, the line $Re(\tau) = 1$ and circle $\left|\tau - \frac{1}{2}\right| = \frac{1}{2}$. The transformation $\tau + 1$ maps the imaginary axis onto $Re(\tau) = 1$ and $1 - \frac{1}{\tau}$ maps $Re(\tau) = 1$ on $\left|\tau - \frac{1}{2}\right| = \frac{1}{2}$. Since $\lambda(\tau)$ is real on the imaginary axis, it follows by virtue of the relations

$$\lambda(\tau + 1) = \frac{\lambda(\tau)}{1 - \lambda(\tau)}$$

and

$$\lambda\left(1 - \frac{1}{\tau}\right) = 1 - \lambda(\tau)$$

that it is real on the whole boundary of $\Omega$. Furthermore, $\lambda(\tau) \to 1$ as $\tau$ tends to *zero* and $\lambda(\tau) \to \infty$ as $\tau$ tends to $1$ inside $\Omega$.

We apply the argument principle to determine the number of times $\lambda(\tau)$ takes a nonreal value $\omega_0$ in $\Omega$. Cut off the corners of $\Omega$ by means of a horizontal line segment $Im(\tau) = t_0$ and its images under the transformations $-\frac{1}{\tau}$ and $1-\frac{1}{\tau}$. For sufficiently large $t_0$ it is clear that $\lambda(\tau) \neq \omega_0$ in the portions that have been cut off.

The circle near $\tau = 1$ is mapped by $\lambda(\tau)$ on a curve $\lambda = \lambda\left(1-\frac{1}{\tau}\right) = 1-\frac{1}{\lambda(\tau)}$; where $\tau = s+it_0$, $0 \le s \le 1$. But $\lambda(\tau)e^{i\pi\tau} \to 16$ as $Im(\tau) \to \infty$. This is approximately a large semicircle in the upper half plane. The image of the contour of the truncated region $\Omega$ has winding number about $\omega_0$, when $Im(\omega_0) > 0$ and winding number 0, when $Im(\omega_0) < 0$. Therefore $\lambda(\tau)$ takes every value in the upper half plane exactly once in $\Omega$ and no value in the lower half plane. ■

**Theorem 17.6.2.** *Every point* $\tau$ *in the upper half plane is equivalent under the congruence subgroup* (mod 2) *to exactly one point in* $\overline{\Omega \cup \Omega'}$.

**Proof.** Consider the linear transformation $\tau,\ -\frac{1}{\tau},\ \tau-1,\ \frac{1}{1-\tau},\ \frac{\tau-1}{\tau},\ \frac{\tau}{1-\tau}$ which are denoted by $S_1, S_2, S_3, S_4, S_5, S_6$ respectively.

(i) Consider the region bounded by $Re(\tau) = 0$, $Re(\tau) = \frac{1}{2}$ $|\tau| \ge 1$. We denote it by $\Delta_1$.

(ii) $\tau' = -\frac{1}{\tau} \Rightarrow \tau = -\frac{1}{\tau'}$.

$Re(\tau) = 0 \Rightarrow \tau + \tau' = 0.$

$$-\frac{1}{\tau'} - \frac{1}{\bar{\tau}'} = 0 \Rightarrow \frac{\bar{\tau}' + \tau'}{\tau'\bar{\tau}'} \Rightarrow \tau' + \bar{\tau}' = 0 \Rightarrow Re(\tau') = 0.$$

$$Re(\tau) = \frac{1}{2} \Rightarrow 2\,Re(\tau) = 1.$$

$$\tau + \bar{\tau} = 1 \Rightarrow -\frac{1}{\tau'} - \frac{1}{\bar{\tau}'} = 1$$

$$1 + \frac{1}{\tau'} = -\frac{1}{\bar{\tau}'} \Rightarrow |\tau' + 1| = 1.$$

Also,

$$|\tau'| = \left|-\frac{1}{\tau}\right| = \frac{1}{|\tau|}.$$

$$|\tau| \geq 1 \Rightarrow |\tau'| \leq 1.$$

Clearly, $\tau'$ lies in the region bounded by $Re(\tau)' = 0$.

$$|\tau' + 1| = 1 \quad and \quad |\tau'| \leq 1.$$

Let it be $\Delta_2$.

(iii) Consider the transformation $\tau - 1$. Put $\tau' = \tau - 1$.

$$Re(\tau) = 0 \Rightarrow Re(\tau') = Re(\tau) - 1 = -1 \Rightarrow Re(\tau') = -1.$$

$$Re(\tau) = \frac{1}{2} \Rightarrow Re(\tau') = \frac{1}{2} - 1 = -\frac{1}{2}.$$

$$\tau \geq 1 \Rightarrow \tau' = \tau - 1 \Rightarrow |\tau' + 1| = |\tau| \geq 1.$$

$\tau'$ lies in the circle centre at $-1$ and radius $1$. $\tau'$ lies in the region bounded by $Re(\tau') = -1$.

$Re(\tau') = -\frac{1}{2}$, $|\tau' + 1| \geq 1$. Let it be $\Delta_3$.

(iv) Put $\tau' = \dfrac{1}{1 - \tau}$

$$\tau = 1 - \frac{1}{\tau'}$$

$$Re(\tau) = 0 \Rightarrow \tau + \bar{\tau} = 0 \Rightarrow \left(1 - \frac{1}{\tau'}\right) + \left(1 - \frac{1}{\bar{\tau}'}\right) = 0 \Rightarrow \frac{2\tau' - 1}{\tau'} = \frac{1}{\bar{\tau}'}$$

$$\left|\frac{2\tau' - 1}{\;}\right| = \left|\frac{1}{\;}\right| \qquad |2\tau' - 1| = 1 \Rightarrow \left|\tau' - \frac{1}{2}\right| = \frac{1}{2}.$$

This is a circle with centre at $\tau' = \frac{1}{2}$ and radius $\frac{1}{2}$.

$$Re(\tau) = \frac{1}{2} \Rightarrow \tau + \bar{\tau} = 1 \Rightarrow 1 - \frac{1}{\tau'} + 1 - \frac{1}{\bar{\tau}'} = 1 \Rightarrow \frac{\tau' - 1}{\tau'} = \frac{1}{\bar{\tau}'}$$

$$\left|\frac{\tau' - 1}{\tau'}\right| = \left|\frac{1}{\bar{\tau}'}\right|$$

$$|\tau' - 1| = 1$$

$$|\tau| \geq 1 \Rightarrow \left|1 - \frac{1}{\tau'}\right| \geq 1 \Rightarrow |\tau' - 1| \geq |\tau'|$$

$$\begin{aligned}
|\tau' - 1|^2 &\geq |\tau'|^2 \\
(\tau' - 1)(\bar{\tau}' - 1) &\geq \tau'\bar{\tau}' \\
\tau' + \bar{\tau}' &\leq 1 \\
2\,Re(\tau') &\leq 1 \\
Re(\tau') &\leq \frac{1}{2}
\end{aligned}$$

Therefore $\tau'$ lies in the region bounded by $Re(\tau') \leq \frac{1}{2}$

$$|\tau' - 1| = 1 \quad and \quad \left|\tau' - \frac{1}{2}\right| = \frac{1}{2}.$$

Let it be $\Delta_4$.

(v) Put $\tau' = \dfrac{\tau - 1}{\tau} = 1 - \dfrac{1}{\tau}$

$$\tau = \frac{1}{1 - \tau'}$$

$$
\begin{aligned}
Re(\tau) &= 0\\
\tau + \tau' &= 0\\
\frac{1}{1-\tau'} + \frac{1}{1-\overline{\tau'}} &= 0\\
\tau' + \overline{\tau'} &= 2\\
2\,Re(\tau') &= 2\\
Re(\tau') &= 1
\end{aligned}
$$

$$
\begin{aligned}
Re(\tau) &= \frac{1}{2}\\
\tau + \tau' &= 1\\
\frac{1}{1-\tau'} + \frac{1}{1-\overline{\tau'}} &= 1\\
1 - \overline{\tau'} + 1 - \tau' &= 1 - \tau' + \overline{\tau'} + |\tau'|^2
\end{aligned}
$$

$$|\tau'|^2 = 1 \Rightarrow |\tau'| = 1.$$

$$|\tau| \geq 1 \Rightarrow \left|\frac{1}{1-\tau'}\right| \geq 1$$

$$|1 - \tau'| \leq 1 \Rightarrow |\tau' - 1| \leq 1.$$

Therefore $\tau'$ lies in the region enclosed by

$$Re(\tau') = 1, \quad |\tau'| = 1, \quad |\tau' - 1| \leq 1.$$

(vi) Put $\tau = \dfrac{\tau}{1-\tau}$.

This implies that $\tau = \dfrac{\tau'}{1+\tau'}$

$$\begin{aligned}
Re(\tau) &= 0\\
\tau + \bar{\tau} &= 0\\
\frac{\tau'}{1+\tau'} + \frac{\bar{\tau}'}{1+\bar{\tau}'} &= 0\\
\frac{1}{\bar{\tau}'} + \frac{1}{\tau'} &= -2\\
\left|2 + \frac{1}{\tau'}\right| &= \left|-\frac{1}{\tau'}\right|\\
\left|\frac{1}{\tau'} + \frac{1}{2}\right| &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
Re(\tau) &= \frac{1}{2}\\
\tau + \bar{\tau} &= 1\\
\frac{\tau'}{1+\tau'} + \frac{\bar{\tau}'}{1+\bar{\tau}'} &= 1\\
\tau' + \tau'\bar{\tau}' + \bar{\tau}' + \tau'\bar{\tau}' &= 1 + \bar{\tau}' + \tau' + \tau'\bar{\tau}'
\end{aligned}$$

$$|\tau'|^2 = 1 \Rightarrow |\tau'| = 1$$

$$\begin{aligned}
|\tau| &\geq 1\\
\left|\frac{\tau'}{1+\tau'}\right| &\geq 1\\
|\tau'|^2 &\geq |1+\tau'|^2\\
|\tau'|^2 &\geq (1+\tau')(1+\bar{\tau}')\\
|\tau'|^2 &\geq 1 + \tau' + \bar{\tau}' + \tau'\bar{\tau}'\\
2\,Re(\tau') &\leq -1\\
Re(\tau') &\leq -\frac{1}{2}.
\end{aligned}$$

Therefore $\tau'$ lies in the region bounded by $Re(\tau') \leq -\frac{1}{2}$, $|\tau'| = 1$, $\left|\tau' + \frac{1}{2}\right| = \frac{1}{2}$. Let it be $\Delta_6$.
Then $\tau$ is mapped on the region by means of the transformation $\tau$, $-\frac{1}{\tau}$, $\tau - 1$, $\frac{1}{1-\tau}$, $\frac{\tau-1}{\tau}$,

$\frac{\tau}{1-\tau}$. Let the transformation be $S_k,\ k = 1, 2, \cdots 6$.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \begin{pmatrix}1&-1\\0&1\end{pmatrix},\ \begin{pmatrix}0&1\\-1&1\end{pmatrix},\ \begin{pmatrix}1&-1\\1&0\end{pmatrix},\ \begin{pmatrix}1&0\\-1&1\end{pmatrix}$$

The matrices of the inverse transformation are

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}0&-1\\1&0\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}1&-1\\1&0\end{pmatrix},\ \begin{pmatrix}0&1\\-1&1\end{pmatrix},\ \begin{pmatrix}1&0\\1&1\end{pmatrix}$$

The corresponding transformation $S_k^{-1},\ k = 1, 2, \cdots 6$ are $\tau,\ -\frac{1}{\tau},\ \tau+1,\ \frac{\tau-1}{\tau},\ \frac{1}{1-\tau},\ \frac{\tau}{\tau+1}$.

We note that every unimodular matrix is congruent (mod 2) to exactly one of them. Therefore these matrices form a complete set of mutually incongruent matrices. Let $\Delta'_k\ (k = 1, 2, \cdots 6)$ denote the regions symmetric to the imaginary axis. Therefore the transformations $S'_k\ (k = 1, 2, \cdots 6)$ corresponding to the regions $\Delta'_k\ (k = 1, 2, \cdots 6)$ are obtained by replacing $\tau$ by $-\tau$.

The transformations are $-\tau, \frac{1}{\tau'},\ -\tau-1,\ \frac{1}{\tau+1},\ \frac{\tau+1}{\tau},\ \frac{-\tau}{\tau+1}$. The region $\Delta'$ is mapped on the unshaded region $\Delta'_1,\ \Delta'_2, \cdots \Delta'_k$ by the means of the linear transformation $S'_k,\ k = 1, 2, \cdots 6$. The matrices obtained by $S'_k\ k = 1, 2, \cdots 6$ are

$$\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}-1&-1\\0&1\end{pmatrix},\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}1&1\\1&0\end{pmatrix},\ \begin{pmatrix}-1&0\\1&1\end{pmatrix}$$

The matrices of the inverse transformations are

$$\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}-1&-1\\0&1\end{pmatrix},\ \begin{pmatrix}-1&1\\1&0\end{pmatrix},\ \begin{pmatrix}0&1\\1&-1\end{pmatrix},\ \begin{pmatrix}-1&0\\1&1\end{pmatrix}$$

The transformation $(S'_k)^{-1}\ (k = 1, 2, \cdots 6)$ are $-\tau,\ \frac{1}{\tau},\ \frac{-\tau-1}{1},\ \frac{-\tau+1}{\tau},\ \frac{1}{\tau-1},\ \frac{-\tau}{\tau+1}$. These matrices form a complete set of mutually incongruent matrices. Clearly the image of $\Delta,\ \Delta'$ are $\Delta_1, \Delta_2, \cdots \Delta_6$ and $\Delta'_1, \Delta'_2, \cdots \Delta'_6$. They cover $\overline{\Omega} \cup \overline{\Omega'}$.

Let τ be any point in the upper half plane. The set $\overline{\Delta} \cup \overline{\Delta'}$ can be identified with the closure of the shaded region. There exists a modular transformations such that $S\tau$ lies in $\overline{\Delta} \cup \overline{\Delta'}$. Suppose that $S\tau \subset \overline{\Delta}$. The matrices of $S$ is congruent $\pmod 2$ to the matrix of an $\overline{S_k'}$. The matrix of $T = S_k S$ is congruent to the identity matrix; in other words, $T$ belongs to the congruence subgroup. Since $S\tau$ lies in $\overline{\Delta}$ we have $T\tau = S_k(S\tau)$ lies in $\overline{\Omega} \cup \overline{\Omega'}$.

Similarly, if we suppose that $S_2 \in \overline{\Delta'}$. The matrix of $T' = S_k' S$ is congruent $\pmod 2$ to the matrix of an $(S_k')^{-1}$. Therefore the matrix of $T' = S_k' S$ is congruent to the identity matrix.

Therefore $\overline{T}$ belongs to the congruent subgroup. Since $S\tau$ lies in $\overline{\Delta'}$ we have

$$T'_\tau = S_k'(S\tau)$$

lies in $\overline{\Omega} \cup \overline{\Omega'}$. Therefore there is always a map $T\tau$ in $\overline{\Omega} \cup \overline{\Omega'}$. Trivially which can be choosen in $\overline{\Omega} \cup \overline{\Omega'}$. Since $S_k$ and $S_k'$ are mutually incongruent. $T\tau$ is unique. ∎

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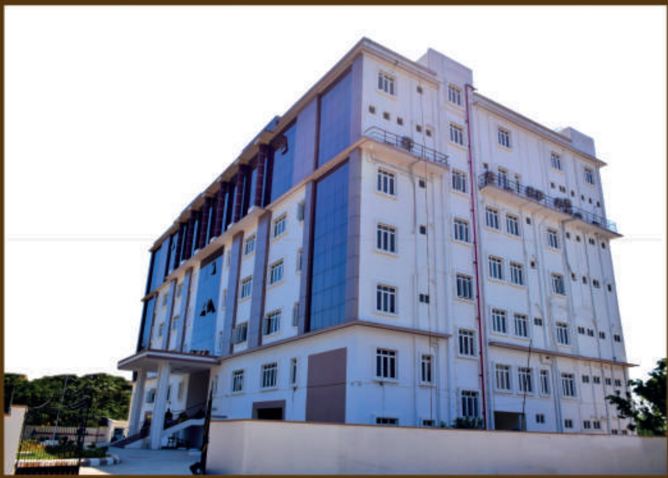
## About School of Sciences



School of Sciences, established in 2004, has been offering B.Sc. and M.Sc. programmes in Mathematics since 2005 and B.Sc., Mathematics with Computer Application since 2007. B.Sc. Programmes in Physics, Chemistry, Botany, and Zoology were introduced in 2017, and M.Sc. Programmes in Physics, Chemistry, Botany, and Zoology were launched in 2018. As per the Academic Restructure -2020, the Department of Geography and Department of Apparel and Fashion Design were merged with the School and these Departments are also offering B.Sc. and M.Sc. Programmes.

Currently, School of Sciences consists of following seven Departments, viz.

1. Mathematics
2. Physics
3. Chemistry
4. Botany
5. Zoology
6. Geography, and
7. Apparel and Fashion Design

All the above said Departments are offering the various academic Programmes from Certificate to Research Degree level (M.Phil. & Ph.D.) in their concerned disciplines.

The main objective of the School is to excite the brains and hearts of rural students through constant inquiry and active participation in Science stream. The School of study has blazed a trail of information transmission, generation, and graduating over 25000 Science students across the Nation. It has built a niche for itself in the core areas of teaching, research, consultation, administration, and community services over the last two decades.

The Department of Mathematics offers the following Programmes in the School

- B.Sc. Mathematics (Semester)
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