Master of Science Mathematics

(M.Sc. Mathematics)

MMT-204

PARTIAL DIFFERENTIAL EQUATIONS

Semester-II

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COURSE TITLE	**:**	**PARTIAL DIFFERENTIAL EQUATIONS**
COURSE CODE	**:**	**MMT-204**
COURSE CREDIT	**:**	**4**

COURSE OBJECTIVES

While studying the **PARTIAL DIFFERENTIAL EQUATIONS**, the Learner shall be able to:

CO 1: Develop an understanding of formation of partial differential equations.

CO 2: Discuss the method of separation of variables to solving partial differential equations.

CO 3: Describe about to find the elementary solutions of Laplace equation.

CO 4: Represent the motion of the string is governed by one-dimensional wave equation.

CO 5: Solve the diffusion equation by using Integral transform technique.

COURSE LEARNING OUTCOMES

After completion of the **PARTIAL DIFFERENTIAL EQUATIONS**, the Learner will be able to:

CLO 1: Apply and analyse to describe real world system using Partial Differential Equations

CLO 2: Master the basic ideas and ability to solve the physical problems.

CLO 3: Analyze the theory of Green's function for Laplace equation.

CLO 4: Obtain the general solution for wave equation.

CLO 5: Obtain the basic knowledge of diffusion equation and find the solution of diffusion equation in cylindrical coordinates and spherical coordinates.

BLOCKI: PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Partial Differential Equations – Origins of First Order Differential Equations – Cauchy's Problem for first order equations – Linear Equations of the first order – Nonlinear partial differential equations of the first order – Cauchy's method of characteristics – Compatible system of First order Equations – Solutions satisfying Given Condition, Jacobi's method

BLOCK II: PARTIAL DIFFERENTIAL EQUATIONS OF THE 2nd ORDER

The Origin of Second Order Equations – Linear partial Differential Equations with constant coefficients –

Equations with variable coefficients – Separation of variables – The method of Integral Transforms – Non – linear equations of the second order.

BLOCK III: LAPLACE'S EQUATION

Elementary solutions of Laplace equation – Families of Equipotential Surfaces – Boundary value problems – Separation of variables – Surface Boundary Value Problems – Separation of Variables – Problems with Axial Symmetry – The Theory of Green's Function for Laplace Equation.

BLOCK IV: THE WAVE EQUATION

The Occurrence of the wave equation in Physics – Elementary Solutions of the One – dimensional Wave equations – Vibrating membrane, Application of the calculus of variations – Three dimensional problem – General solutions of the Wave equation.

BLOCK V: THE DIFFUSION EQUATION

Elementary Solutions of the Diffusion Equation – Separation of variables – The use of Integral Transforms – The use of Green's functions

REFERENCE BOOKS :

1. Ian Sneddon – Elements of Partial Differential Equations – McGrawHill International Book Company, New Delhi, 1983
2. M.D. Raisinghania Advanced Differential Equations S. Chand and Company Ltd., New Delhi, 2001
3. K. Sankara Rao, Introduction to Partial Differential Equations, Second edition – Prentice – Hall of India, New Delhi 2006
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Contents

BLOCK-I

UNIT 1

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER-I

Structure

Overview

In this unit, we will illustrate the basic concepts of differential equations and Origins of Partial differential equations

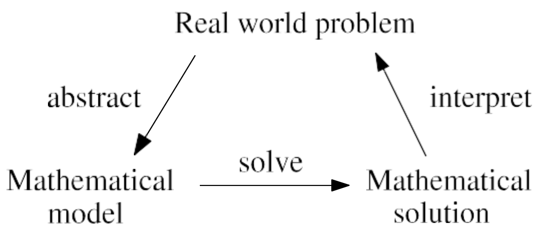
Objectives

After successful completion of this lesson, students will be able to

- understand the basic concepts of Partial Differential Equations.
- form PDE by eliminating arbitrary constants.
- form of PDE by eliminating arbitrary functions.
- understand the concept of Cauchy's Problem for first-order equations.
- to solve linear equations of first order.

1.1 Introduction

In this section, we present the basic concepts of differential equations. To start with, we recall the quote given by V.I. Arnold, "Differential equations form the basis for the scientific view of the world". Next, we discuss the basic cycle of real world problem,



Naturally, all the phenomena can be governed by differential equations. So, we begin with the definition and classification of differential equations.

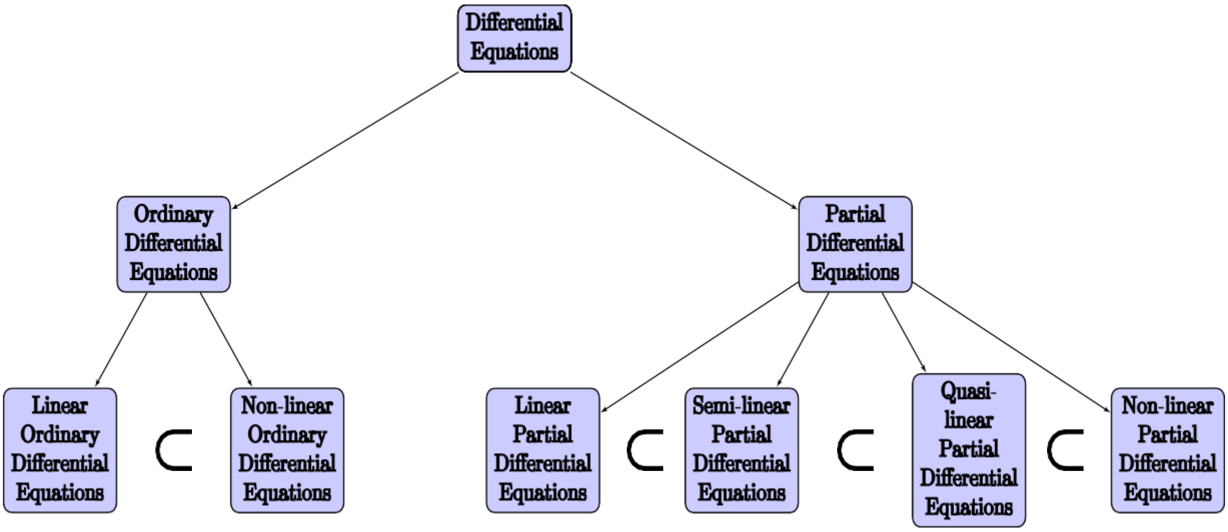
Differential Equation

An equation involves unknown function and its derivatives (differential coefficients).

or

An equation involves independent variables, dependent variables and derivatives of dependent variables with respect to independent variables.

Classification of Differential Equations



Ordinary Differential Equations

An ordinary differential equations is a differential equation in which a single independent variable enters either explicily or implicitly.

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

which is a general second order equation.

Linear Ordinary Differential Equations

The degree of the dependent variable and its derivatives is one.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$$

Nonlinear Ordinary Differential Equations

The degree of the dependent variable and its derivatives is more than one.

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0.$$

Next section we discuss the partial differential equations and its classifications.

1.2 Partial Differential Equations

Many physical phenomena arise in nature can be governed by differential equations, especially, the problems in science and engineering are expressed by means of partial differential equations. Partial differential equations arise in geometry and physics when the number of independent variables in the problem under discussion is two or more. When such is the case, any dependent variable is likely to be a function of more than one variable, so that it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several variables. For instance, in the study of thermal effects in a solid body the temperature θ may vary from point to point in the solid as well as from time to time, and, as a consequence, the derivatives

$$\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial t},$$

will, in general, be nonzero.

Partial Differential Equations (PDEs)

A partial differential equation is a differential equation in which more than one independent variables.

Order of PDE

The order of a partial differential equation is the highest partial derivative in the equation.

Degree of PDE

The degree of a partial differential equation is the highest power of the highest partial derivative in the equation.

Example

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$$

or

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0$$

which is a general second order equation.

Furthermore in any particular problem it may happen that higher derivatives of the types

$$\frac{\partial^2 \theta}{\partial x^2}, \frac{\partial^2 \theta}{\partial x \partial t}, \frac{\partial^3 \theta}{\partial x^2 \partial t}, \text{ etc.}$$

may be of physical significance.

Linear PDE

A PDE which is linear in the unknown function and all its derivatives with coefficients depending on the independent variables alone is called a Linear PDE.

Examples:

- $P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$ (First Order)
- $A(x, y)\frac{\partial z^2}{\partial x^2} + B(x, y)\frac{\partial z^2}{\partial x \partial y} + C(x, y)\frac{\partial z^2}{\partial y^2} + D(x, y)\frac{\partial z}{\partial x} + E(x, y)\frac{\partial z}{\partial y} + F(x, y)z + H(x, y) = 0$ (Second order)
- $\sum A_i(x, y) z_{x_1 x_2 \cdots x_m} + \sum B_i(x, y) z_{x_1 x_2 \cdots x_{m-1}} + \cdots + F(x, y)z + H(x, y) = 0$ (m^{th} order) where $x_i = x$ or y and all coefficients $A_i, B_i, \ldots, F, H$ are functions of independent variables x and y alone.

Semi-linear PDE

In a PDE, the coefficients of derivatives of order m are functions of the independent variables alone is called a Semi-linear PDE.

Examples:

- $P(x, y)p + Q(x, y)q = R(x, y, z)$ (First Order)
- $A(x,y)\dfrac{\partial z^2}{\partial x^2} + B(x,y)\dfrac{\partial z^2}{\partial x \partial y} + C(x,y)\dfrac{\partial z^2}{\partial y^2} + F\left(x,y,z,\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}\right) = 0$ (Second order)
- $\sum A_i(x, y) z_{x_1 x_2 \cdots x_m} + F\left(x, y, z, z_{x_i}, z_{x_i x_j}, \ldots, z_{x_1 x_2 \cdots x_{m-1}}\right) = 0$ (m^{th} order).

Quasi-linear PDE

A PDE of order m is called Quasi-linear if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function or order strictly less than m.

Examples:

- $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$ (First Order)
- $A(x,y,z)\dfrac{\partial z^2}{\partial x^2} + B(x,y,z)\dfrac{\partial z^2}{\partial x \partial y} + C(x,y,z)\dfrac{\partial z^2}{\partial y^2} + F\left(x,y,z,\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}\right) = 0$ (Second order)
- $\sum A_i(x, y, z) z_{x_1 x_2 \cdots x_m} + F\left(x, y, z, z_{x_i}, z_{x_i x_j}, \ldots, z_{x_1 x_2 \cdots x_{m-1}}\right) = 0$ (m^{th} order).

Nonlinear PDE

A PDE is called Nonlinear if it does not comes under the above three types, namely, linear, semi-linear and quasi-linear.

Examples:

- $F(x, y, z, p, q) = 0$ (First Order)
- $F\left(x,y,z,\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y},\dfrac{\partial z^2}{\partial x^2},\dfrac{\partial z^2}{\partial x \partial y},\dfrac{\partial z^2}{\partial y^2}\right) = 0$ (Second order)
- $F\left(x, y, z, z_{x_i}, z_{x_i x_j}, \ldots, z_{x_1 x_2 \cdots x_{m-1}}, z_{x_1 x_2 \cdots x_m}\right) = 0$ (m^{th} order).

In the main we shall suppose that there are two independent variables x and y and that the dependent variable is denoted by z. If we write

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \tag{1}$$

then the first order partial differential equation can be written in the symbolic form

$$f(x, y, z, p, q) = 0. \tag{2}$$

1.3 Origins of First-order Partial Differential Equations

In this section, we discuss the formation of partial differential equations. Mainly, there are two methods to form a partial differential equations

(i) Eliminating arbitrary constants,

(ii) Eliminating arbitrary functions.

1.3.1 Formation of PDE by eliminating arbitrary constants

Let

$$F(x, y, z, a, b) = 0 \tag{1}$$

where a and b denote arbitrary constants. If we differentiate this equation with respect to x and y, we obtain the relation

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \qquad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0. \tag{2}$$

The set of equations (1) and (2) constitute three equations involving two arbitrary constants a and b, and, in the general case, it will be possible to eliminate a and b from these equations to obtain a relation of the kind

$$f(x, y, z, p, q) = 0 \tag{3}$$

showing that the system of surfaces (1) gives rise to a partial differential equation (3) of the first order.

Problem 1.3.1. Find the PDE of the family of spheres whose centres lie on the $z-$ axis and radius $a.$

Solution. Let $x^2 + y^2 + (z - c)^2 = a^2$ (1)

be the family of spheres whose centres lie on the $z-$ axis and radius $a.$

Differentiating equation (1) partially with respect to x and $y,$ we get

$$2x + 2(z - c)\frac{\partial z}{\partial x} = 0 \Rightarrow x + p(z - c) = 0 \Rightarrow z - c = -\frac{x}{p} \tag{2}$$

$$2y + 2(z - c)\frac{\partial z}{\partial y} = 0 \Rightarrow y + q(z - c) = 0 \Rightarrow z - c = -\frac{y}{q}. \tag{3}$$

From (2) and (3), we have

$$-\frac{x}{p} = -\frac{y}{q}$$

$$\boxed{yp - xq = 0}$$

which is the required PDE. ■

Problem 1.3.2. Find the PDE of the family of right circular cones whose axes coincide with the line $Oz.$

Solution. Let $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ (1)

be the family of right circular cones whose axes coincide with the line $Oz.$

Differentiating equation (1) partially with respect to x and $y,$ we get

$$2x = 2(z - c)\frac{\partial z}{\partial x}\tan^2 \alpha \Rightarrow x = (z - c)p\tan^2 \alpha \Rightarrow \frac{x}{p} = (z - c)\tan^2 \alpha \tag{2}$$

$$2y = 2(z-c)\frac{\partial z}{\partial y}\tan^2\alpha \Rightarrow y = (z-c)q\tan^2\alpha \Rightarrow \frac{y}{q} = (z-c)\tan^2\alpha. \tag{3}$$

From (2) and (3), we have

$$\frac{x}{p} = \frac{y}{q}$$

$$\boxed{yp - xq = 0}$$

which is the required PDE. ■

Problem 1.3.3. Eliminate the constants a and b form the equation $z = (x+a)(y+b)$.

Solution. Given $z = (x+a)(y+b)$. (1)

Differentiating (1) partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = (y+b) \Rightarrow (y+b) = p \tag{2}$$

$$q = \frac{\partial z}{\partial y} = (x+a) \Rightarrow (x+a) = q. \tag{3}$$

Using (2) and (3) in (1), we get

$$z = qp$$

$$\boxed{pq = z}$$

which is the required PDE. ■

Problem 1.3.4. Find the PDE of the family of spheres of unit radius whose centres lie on the

$xy-$ plane.

Solution. Let $(x-a)^2 + (y-b)^2 + z^2 = 1$ (1)

be the family of spheres of unit radius whose centres lie on the $xy-$ plane.

Differentiating equation (1) partially with respect to x and y, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) = -zp \tag{2}$$

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow (y-b) = -zq \tag{3}$$

Using (2) and (3) in (1), we have

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$\boxed{z^2(p^2+q^2+1) = 1}$$

which is the required PDE. ■

1.3.2 Formation of PDE by eliminating arbitrary functions

In this subsection, we explain the formulation of PDE by eliminating the arbitrary functions.

Problem 1.3.5. Eliminate the arbitrary function f from the equation $z = f(x^2 + y^2)$.

Solution. Given $z = f(x^2+y^2)$, (1)

where the function f is arbitrary. Now if we write $x^2+y^2 = u$ and differentiate equation (1) with respect to x and y, respectively, we obtain the relations

$$p = \frac{\partial z}{\partial x} = 2x f'(u), \tag{2}$$

$$q = \frac{\partial z}{\partial y} = 2y f'(u), \tag{3}$$

where $f'(u) = \frac{df}{du}$ and by eliminating the arbitrary function $f(u)$,

$$\frac{(2)}{(3)} \Rightarrow \qquad \frac{p}{q} = \frac{2x f'(u)}{2y f'(u)}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$\boxed{py = qx}$$

which is the required PDE. ■

Problem 1.3.6. Form the PDE by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$.

Solution. Given $z = f\left(\frac{xy}{z}\right)$. (1)

Differentiating equation (1) partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) y \left[\frac{z \cdot 1 - x \cdot \frac{\partial z}{\partial x}}{z^2}\right] = f'\left(\frac{xy}{z}\right) y \frac{z - px}{z^2} \tag{2}$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) x \left[\frac{z \cdot 1 - y \cdot \frac{\partial z}{\partial y}}{z^2}\right] = f'\left(\frac{xy}{z}\right) x \frac{z - qy}{z^2} \tag{3}$$

$$\frac{z^2}{z-qy}$$

(2)

(3) $\Rightarrow$

$$\frac{p}{q} = \frac{f'\left(\frac{xy}{z}\right) y \frac{z-px}{z^2}}{f'\left(\frac{xy}{z}\right) x \frac{z-qy}{z^2}} = \frac{y}{x}\,\frac{z-px}{z^2}$$

$$px(z - qy) = qy(z - px)$$

$$pxz - pqxy = qyz - pqxy$$

$$pxz = qyz$$

$$\boxed{px = qy}$$

which is the required PDE. ■

1.3.3 Formation of Partial Differential Equations by elimination of arbitrary function F from $F(u, v) = 0$, where u and v are functions of x, y and z

Let

$$F(u, v) = 0, \tag{1}$$

where u and v are known functions of x, y and z and F is an arbitrary function of u and v.

To form a differential equation by eliminating the arbitrary function F, we differentiate equation (1) partially with respect to x and y, we obtain the equations

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \tag{2}$$

and

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0. \tag{3}$$

If we now eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ for these equations (2) and (3), we obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q \end{vmatrix} = 0$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) - \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) = 0$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}q+\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}p+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}pq-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}q-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}p-\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}pq=0$$

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right)q=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

which gives

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)}.$$

This is a linear PDE of the type

$$Pp+Qq=R, \tag{4}$$

where

$$P=\frac{\partial(u,v)}{\partial(y,z)},\quad Q=\frac{\partial(u,v)}{\partial(z,x)},\quad R=\frac{\partial(u,v)}{\partial(x,y)}.$$

Equation (4) is called Lagrange's PDE of first order.

Problem 1.3.7. Eliminate the arbitrary function f from the equation $f(x^2+y^2+z^2,\ z^2-2xy)=0.$

Solution. The given relation is of the form

$$F(u,v)=0,$$

where $u=x^2+y^2+z^2$, $v=z^2-2xy$.

Hence, the required PDE is of the form

$$Pp+Qq=R, \qquad \text{(Lagrange equation)}$$

where

$$P=\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\\ \cdots & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2y & -2x\\ \cdots & \cdots\end{vmatrix}=4yz+4xz=4z(x+y)$$

∂z ·

$2z \quad 2z$

$\partial(y, z) \quad \cdot \frac{\partial u}{\partial z}$

$$Q = \frac{\partial(u, v)}{\partial(z, x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2z \\ 2x & -2y \end{vmatrix} = -4yz - 4xz = -4z(x+y)$$

and

$$R = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & -2x \end{vmatrix} = -4x^2 + 4y^2 = -4(x^2 - y^2).$$

Therefore

$$4z(x+y)p - 4z(x+y)q = -4(x^2 - y^2)$$

$$\cancel{4}z\cancel{(x+y)}p - \cancel{4}z\cancel{(x+y)}q = -\cancel{4}\cancel{(x+y)}(x-y)$$

$$\Rightarrow \qquad zp - zq = y - x$$

which is the required PDE. ■

Check Your Progress

1. Eliminate the constants a and b from the following equations:

   (a) $z = (x+a)(y+b)$

   (b) $2z = (ax+y)^2 + b$

   (c) $ax^2 + by^2 + z^2 = 1.$

2. Eliminate the arbitrary function f from the equations:

   (a) $z = xyf(x^2 + y^2)$

   (b) $z = x + y + f(xy)$

(c) $z = f\,\dfrac{xy}{z}$

(d) $z = f(x - y)$

(e) $f(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0.$

1.4 Cauchy's Problem for First-order Equations

Cauchy's Problem.

If

(a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are functions which, together with their first derivatives, are continuous in the interval M defined by $\mu_1 < \mu < \mu_2$;

(b) And if $F(x, y, z, p, q)$ is a continuous function of x, y, z, p and q in a certain region U of the $xyzpq$ space, then it is required to establish the existence of a function $\varphi(x, y)$ with the following properties:

(1) $\varphi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

(2) For all values of x and y lying in R, the point $\{x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)\}$ lies in U and

$$F[x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)] = 0.$$

(3) For all μ belonging to the interval M, the point $\{x_0(\mu), y_0(\mu)\}$ belongs to the region R, and

$$\varphi\{x_0(\mu), y_0(\mu)\} = z_0.$$

Geometrically, there exists a surface $z = \varphi(x, y)$ which passes through the curve Γ whose parametric equations are

$$x = x_0(\mu),\ y = y_0(\mu),\ z = z_0(\mu) \tag{1}$$

and at every point of which the directionl $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q) = 0. \tag{2}$$

The above theorem is only one form of the Cauchy problem.

To prove the existence of a solution of equation (2) passing through a curve with equations (1) it is necessary to make some further assumptions about the form of the function F and the nature of the curve Γ.

Theorem 1.4.1. *If $g(y)$ and all its derivatives are continuous for $|y - y_0| < \delta$, if x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$, and if $f(x, y, z, q)$ and all its partial derivatives are continuous in a region S defined by*

$$|x - x_0| < \delta, \ |y - y_0| < \delta, \ |q - q_0| < \delta,$$

then there exists a unique function $\varphi(x, y)$ such that:

(a) *$\varphi(xy)$ and all its partial derivatives are continuous in a region R defined by $|x - x_0| < \delta_1$, $|y - y_0| < \delta_2$;*

(b) *For all (x, y) in R, $z = \varphi(x, y)$ is a solution of the equation*

$$\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right).$$

(c) *For all values of y in the interval $|y - y_0| < \delta_1$, $\varphi(x_0, y) = g(y)$.*

1.5 Linear Equations of the First Order

Consider the partial differential equations of the form

$$Pp + Qq = R, \tag{1}$$

where P, Q and R are given functions of x, y and z (which do not involve p or q), p denotes $\dfrac{\partial z}{\partial x}$, q denotes $\dfrac{\partial z}{\partial y}$. This equation is known as *Lagrange's equation*.

Theorem 1.5.1. *The general solution of the linear partial differential equation*

$$Pp + Qq = R \tag{1}$$

is

$$F(u, v) = 0 \tag{2}$$

where F *is an arbitrary function and* $u(x, y, z) = c_1$ *and* $v(x, y, z) = c_2$ *form a solution of the equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

Proof. To prove this theorem in two stages:

(a) We shall show that all integral surfaces of the equation (1) are generated by the integral curves of the equations (3);

(b) and then we shall prove that all surfaces generated by integral curves of the equations (3) are integral surfaces of the equation (1).

Equation (2) consists of a set of two independent ordinary differential equations, that is, a two parameter family of curves in space, one such set can be written as

$$\frac{dy}{dx} = \frac{Q(x, y, z)}{P(x, y, z)} \tag{3}$$

which is referred to as "characteristic curve".

We know that the total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \tag{4}$$

The matrix form of the equations (1) and (4) can be written as

$$\begin{bmatrix} P & Q \\ dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} R \\ dz \end{bmatrix}. \tag{5}$$

Both the equations must hold on the integral surface. For the existence of finite solutions of equation (5), we have

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = \begin{vmatrix} P & R \\ dx & dz \end{vmatrix} = \begin{vmatrix} R & Q \\ dz & dy \end{vmatrix} = 0$$

on expanding the determinants

$$Pdy - Qdx = 0 \qquad Pdz - Rdx = 0 \qquad Rdy - Qdz = 0$$

$$Pdy = Qdx \qquad Pdz = Rdx \qquad Rdy = Qdz$$

$$\frac{dy}{Q} = \frac{dx}{P} \qquad \frac{dz}{R} = \frac{dx}{P} \qquad \frac{dy}{Q} = \frac{dz}{R}.$$

Combining all the above, we get

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \tag{6}$$

which are called auxiliary equations for a given PDE.

Next, we have to show that any surface generated by the integral curves of equation (6) has an equation of the form $F(u, v) = 0$.

Let

$$u(x, y, z) = C_1 \quad \text{and} \quad v(x, y, z) = C_2 \tag{7}$$

be two independent integrals of the ordinary differential equations (6). If equations (7) satisfy equation (6), then we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

and

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0.$$

Solving these equations, we find

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dx}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

which can be rewritten as

$$\frac{dx}{\frac{\partial(u, v)}{\partial(y, z)}} = \frac{dy}{\frac{\partial(u, v)}{\partial(z, x)}} = \frac{dz}{\frac{\partial(u, v)}{\partial(x, y)}}. \tag{8}$$

The relation $F(u, v) = 0$, where F is an arbitrary function, leads to the partial differntial equation

$$p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}. \tag{9}$$

The equation (8) can be written as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

The solution of these equations are known to be $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$. Hence $F(u, v) = 0$ is the required solution. ■

Next we generalize the Lagrange's equation to n independent variables is obviously the equation

$$X_1 p_1 + X_2 p_2 + \cdots + X_n p_n = Y, \tag{4}$$

where $X_1, X_2, \ldots, X_n$ and Y are functions of n independent variables $x_1, x_2, \ldots, x_n$ and a dependent variable f; p_i denotes $\frac{\partial f}{\partial x_i}$ $(i = 1, 2, \ldots, n)$.

We now state the theorem for obtaining the general solution of generalized Lagrange's equation.

Theorem 1.5.2. *If* $u_i(x_1, x_2, \ldots, x_n, z) = c_i$ $(i = 1, 2, \ldots, n)$ *are independent solutions of the equations*

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R},$$

then the relation $\Phi(u_1, u_2, \ldots, u_n) = 0$, *in which the function* Φ *is arbitrary, is a general solution of the linear partial differential equation*

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R.$$

Proof. If the solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R} \tag{5}$$

are

$$u_i(x_1, x_2, \ldots, x_n, z) = c_i \qquad i = 1, 2, \ldots, n, \tag{6}$$

then the n equations

$$\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} dx_j + \frac{\partial u_i}{\partial z} dz = 0 \qquad i = 1, 2, \ldots, n \tag{7}$$

must be compatible with the equations (5). In other words, we must have

$$\sum_{j=1}^{n} P_j \frac{\partial u_i}{\partial x_j} + R\frac{\partial u_i}{\partial z} = 0. \tag{8}$$

Solving the set of n equations (8) for P_i, we find that

$$\frac{P_i}{\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,\ldots,x_{i-1},z,x_{i+1},\ldots,x_n)}} = \frac{R}{\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(u_1,u_2,\ldots,u_n)}} \qquad i = 1, 2, \ldots, n, \tag{9}$$

where $\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(u_1, u_2, \ldots, u_n)}$ denotes the Jacobian

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Consider the relation

$$\Phi(u_1, u_2, \ldots, u_n) = 0. \tag{10}$$

Differentiating it with respect to x_i, we obtain the equation

$$\sum_{j=1} \frac{\partial \Phi}{\partial u_j}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial z}\frac{\partial z}{\partial x_i}\right) = 0$$

and there are n such equations, one for each value of i. Eliminating the n quantities $\frac{\partial \Phi}{\partial u_1}, \ldots, \frac{\partial \Phi}{\partial u_n}$ from these equations, we obtain the relation

$$\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)} + \sum_{j=1} \frac{\partial z}{\partial x_j}\frac{\partial(u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n)}{\partial(x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n)} = 0. \tag{11}$$

Substituting from equations (9) into the equation (11), we see that the function z defined by the relation (10) is a solution of the equation

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R. \tag{12}$$

This completes the proof. ■

Methods for Solving Lagrange's Auxiliary Equation

In this section, we explain how to solve the linear PDE $Pp + Qq = R$ using the auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

There are two methods to solve the above Lagrange's auxiliary equation.

- Method of grouping
- Method of multipliers

1.5.1 Method of grouping

If it is possible to take two fractions $\frac{dx}{P} = \frac{dz}{R}$ from which y can be cancelled or is absent, leaving equations in x and z only. If so integrate it by giving $u(x, z) = C_1$.

Similarly take another two fractions say $\frac{dx}{P} = \frac{dy}{Q}$, which may give $v(x, y) = C_2$. Therefore, the solution of (1) is

$$F(u, v) = 0.$$

1.5.2 Method of multipliers

Choose any three multipliers l, m, n which may be functions of x, y and z

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} = k(say)$$

such that the expression $lP + mQ + nR = 0$.

Since

$$ldx + mdy + ndz = k(lP + mQ + nR).$$

we obtain

$$ldx + mdy + ndz = 0.$$

On integration, we get $u(x, y, z) = C_1$.

Similarly, we can choose three multipliers l', m', n', we get $v(x, y, z) = C_2$. Therefore, the solution of (1) is

$$F(u, v) = 0.$$

Example 1.5.1. Find the general solution of the differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z.$$

Solution. Given $x^2p + y^2q = (x+y)z.$ (1)

Comparing (1) with $Pp + Qq = R$, we have $P = x^2, Q = y^2, R = (x+y)z.$

The integral surfaces of this equation are generated by the integral curves of the equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}. \tag{2}$$

The first equation of this set has obviously the integral

$$x^{-1} - y^{-1} = c_1 \tag{3}$$

and it follows immediately from the equations that

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

which has the integral

$$\frac{x-y}{z} = c_2. \tag{4}$$

Combining the solutions (3) and (4), we see that the integral curves of the equations (1) are given by equation (4) and the equation

$$\frac{xy}{z} = c_3 \tag{5}$$

and that the curves given by these equations generate the surface

$$F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0,$$

where the function F is arbitrary. ■

Note. The above surface can be expressed as

$$z = xyf\left(\frac{x-y}{z}\right)$$

or

$$z = xyg\left(\frac{x-y}{xy}\right),$$

where f and g are arbitrary functions.

Problem 1.5.1. Find the general integral of the linear partial differential equations $y^2p - xyq = x(z-2y)$.

Solution. Given $y^2p - xyq = x(z-2y)$. (1)

Comparing (1) with $Pp + Qq = R$, we have $P = y^2, Q = -xy, R = x(z-2y)$.

The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}. \tag{2}$$

The first two members of equation (2) give us

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad xdx = -ydy,$$

which on integration gives

$$\frac{x^2}{2} = -\frac{y^2}{2} + C \quad \text{or} \quad x^2 + y^2 = C_1. \tag{3}$$

The last two members of equation (2) give

$$\frac{dy}{-y} = \frac{dz}{z - 2y} \quad \text{or} \quad zdy - 2ydy = -ydz.$$

we have

$$2ydy = ydz + zdy$$

which on integration yields

$$y^2 = yz + C_2 \quad \text{or} \quad y^2 - yz = C_2. \tag{4}$$

Hence, the curves given by equations (3) and (4) generate the required integral surface as

$$F(x^2 + y^2, y^2 - yz) = 0.$$ ■

Problem 1.5.2. Find the general integral of the linear partial differential equations $(y + zx)p - (x + yz)q = x^2 - y^2$.

Solution. Given $(y + zx)p - (x + yz)q = x^2 - y^2.$ (1)

Comparing (1) with $Pp + Qq = R$, we have $P = y + zx$, $Q = -(x + yz)$, $R = x^2 - y^2$.

The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}. \tag{2}$$

To get the first integral curve, let us consider the first combination as

$$\frac{xdx+ydy}{xy+zx^2-xy-y^2z} = \frac{dz}{x^2-y^2}$$

or

$$\frac{xdx+ydy}{z(x^2-y^2)} = \frac{dz}{x^2-y^2}.$$

That is,

$$xdx+ydy = zdz.$$

On integration, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2} = C \quad \text{or} \quad x^2+y^2-z^2 = C_1. \tag{3}$$

To get the second integral curve, let us consider the combination such as

$$\frac{ydx+xdy}{y^2+xyz-x^2-xyz} = \frac{dz}{x^2-y^2}$$

or

$$ydx+xdy+dz = 0$$

which on integration results in

$$xy+z = C_2. \tag{4}$$

Thus, the curves given by equations (3) and (4) generate the required integral surface as

$$F(x^2 + y^2 - z^2, xy + z) = 0. \qquad \blacksquare$$

Example 1.5.2. If u is a function of x, y and z which satisfies the partial differential equation

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$$

show that u contains x, y and z only in combinations $x+y+z$ and $x^2+y^2+z^2$.

Solution. Given

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0 \tag{1}$$

The integral surfaces of this equation are generated by the integral curves of the equations

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0} \tag{2}$$

and they are equivalent to the three relations

$$du = 0$$

$$dx + dy + dz = 0$$

$$xdx + ydy + zdz = 0$$

On integration, we obtain the integrals

$$u = c_1, \;\; x + y + z = c_2, \;\; x^2 + y^2 + z^2 = c_3.$$

Hence the general solution is of the form

$$u = f(x+y+z,\ x^2+y^2+z^2). \qquad \blacksquare$$

Check Your Progress

Find the general integrals of the linear partial differential equations:

1. $z(xp - yq) = y^2 - x^2$
2. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$
3. $px(x+y) = qy(x+y) - (x-y)(2x+2y+z)$
4. $y^2 p - xyq = x(z - 2y)$
5. $(y+zx)p - (x+yz)q = x^2 - y^2$
6. $x(x^2+3y^2)p - y(3x^2+y^2)q = 2z(y^2 - x^2)$

Let us Sum up:

In this unit, the students acquired knowledge to

- classify the differential equations.
- find the order and degree of the PDE's.
- formation of PDE's by eliminating arbitrary functions/constants.
- solve the linear differential equations.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-I

UNIT 2

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER-II

Structure

Objective

Overview

Let us Sum up:

In this unit, the students acquired knowledge to

- classify the differential equations.
- find the order and degree of the PDE's.

- formation of PDE's by eliminating arbitrary functions/constants.
- solve the linear differential equations.

2.1 Nonlinear Partial Differential Equations of the First Order

In this section, we discuss the problem of finding the solutions of the partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

in which the function F is not necessarily linear in p and q.

The solution of the partial differential equation (1) has two-parameter family of integral curves

$$f(x, y, z, a, b) = 0 \tag{2}$$

Any envelope of the system (2) touches at each of its points a member of the system.

We now classify the integrals (solutions) of a partial differential equation (1):

(a) Two-parameter systems of surfaces

$$f(x, y, z, a, b) = 0.$$

Such an integral is called a *complete integral*.

(b) If we take any one-parameter subsystem

$$f\{x, y, z, a, \varphi(a)\} = 0$$

of the system (2), and form its envelope, we obtain a solution of equation (1). When the function $\varphi(a)$ which defines this subsystem is arbitrary, the solution obtained is called the *general integral* of (1) corresponding to the complete integral (2). When a definite function $\varphi(a)$ is used, we obtain a particular case of the general integral.

(c) If the envelope of the two-parameter system (2) exists, it is also a solution of the equation (1); it is called the *singular integral* of the equation.

Illustration of the above three kinds of solution:

In problem 1.3.4 of section 1, the partial differential equation

$$z^2(1 + p^2 + q^2) = 1. \tag{3}$$

obtained from the two parameter family of surface

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \tag{4}$$

where a and b are arbitrary parameters. Since it contains two arbitrary constants, the solution (4) is thus a *complete integral* of the equation (3).

Substitute $b = a$ in equation (4), we obtain the one-parameter system

$$(x - a)^2 + (y - a)^2 + z^2 = 1$$

whose envelope is obtained by eliminating a between this equation and

$$x + y - 2a = 0$$

so that it has equation

$$(x - y)^2 + 2z^2 = 2. \tag{5}$$

Differentiating both sides of this equation with respect to x and y, respectively, we obtain the relations

$$2zp = y - x, \quad 2zq = x - y$$

from which it follows immediately that (5) is an integral surface of the equation (3). It is a solution of type (b); i.e., it is a *general integral* of the equation (3).

The envelope of the two-parameter system (3) is obtained by eliminating a and b from equation (4) and the two equations

$$x - a = 0 \qquad y - b = 0$$

i.e., the envelope consists of the pair of planes $z = \pm 1$. It is readily verified that these planes are integral surfaces of the equation (3); since they are of type (c) they constitute the *singular integral* of the equation.

CHECK YOUR PROGRESS

1. Verify that $z = ax + by + a + b - ab$ is a complete integral of the partial differential equation

$$zpx + qy + p + q - pq,$$

where a and b are arbitrary constants. Show that the envelope of all planes corresponding to complete integrals provides a singular solution of the differential equation, and determine a general solution by finding the envelope of those planes that pass through the origin.

2. Verify that the equations

(a) $z = \sqrt{2x + a} + \sqrt{2y + b}$

(b) $z^2 + \mu = 2(1 + \lambda^{-1})(x + \lambda y)$

are both complete integrals of the partial differential equation

$$z = \frac{1}{p} + \frac{1}{q}.$$

Show, further, that the complete integral (b) is the envelope of the one-parameter subsystem obtained by taking

$$b = -\frac{a}{\lambda} - \frac{\mu}{1 + \lambda}$$

in the solution (a).

2.2 Cauchy's Method of Characteristics

In this section, we explain the methods of solving the nonlinear partial differential equation using characteristics strip, due to Cauchy.

$$F\left(x,\ y,\ z,\ \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y}\right) = 0. \tag{1}$$

Theorem 2.2.1. *A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.*

Proof. The plane passing through the point $P(x_0, y_0, z_0)$ with its normal parallel to the direction n defined by the direction ratios $(p_0, q_0, -1)$ is uniquely specified by the set of numbers $D(x_0, y_0, z_0, p_0, q_0)$. Conversely any such set of five real numbers defines a plane in three-dimensional space. For this reason a set of five numbers $D(x, y, z, p, q)$ is called a *planeelement* of the space. In particular a plane element $(x_0, y_0, z_0, p_0, q_0)$ whose components satisfy an equation

$$F(x, y, z, p, q) = 0 \tag{2}$$

is called an integral element of the equation (2) at the point (x_0, y_0, z_0). It is theoretically possible to solve an equation of the type (2) to obtain an expression

$$q = G(x, y, z, p) \tag{3}$$

from which to calculate q when x, y, z and p are known. Keeping x_0, y_0 and z_0 fixed and varying p, we obtain a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p)\}$, which depend on the single parameter p. As p varies, we obtain a set of plane elements all of which pass through the point P and which therefore envelop a cone with vertex P; the cone so generated is called the elementary cone of equation (2) at the point P (cf. Fig. 16).

Consider now a surface S whose equation is

$$z = g(x, y). \tag{4}$$

If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$, $g_y(x, y)$ are continuous in a certain region R of the xy plane, then the tangent plane at each point of S determines a plane element of the type

$$\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\} \tag{5}$$

which we shall call the tangent element of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$. ■

A curve C with parametric equations

$$x = x(t),\ y = y(t),\ z = z(t) \tag{6}$$

lies on the surface (4) if

$$z(t) \equiv g\{x(t), y(t)\}$$

for all values of t in the appropriate interval I. If P_0 is a point on this curve determined by the parameters t_0, then the direction ratios of the tangent line P_0P_1 (cf. Fig. 17) are $\{x'(t_0), y'(t_0), z'(t_0)\}$, where $x'(t_0)$ denotes the value of $\frac{dx}{dt}$ when $t = t_0$, etc. This direction will be perpendicular to the direction $(p_0, q_0, -1)$ if

$$z'(t_0) = p_0 x_0'(t_0) + q_0 y_0'(t_0).$$

For this reason we say that any set

$$\{x(t), y(t), z(t), p(t), q(t)\} \tag{7}$$

of five real functions satisfying the condition

$$z'(t) = p(t)x'(t) + q(t)y'(t) \tag{8}$$

defines a strip at the point (x, y, z) of the curve C. If such a strip is also an integral element of equation (2), we say that it is an *integral strip* of equation (2); i.e., the set of functions (7) is an integral strip of equation (2) provided they satisfy condition (8) and the further condition

$$F\{x(t), y(t), z(t), p(t), q(t)\} \equiv 0 \tag{9}$$

for all t in I.

If at each point the curve (6) touches a generator of the elementary cone, we say that the corresponding strip is a characteristic strip. We shall now derive the equations determining a characteristic strip. The point $(x + dx, y + dy, z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = pdx + qdy, \tag{10}$$

where p, q satisfy the relation (2). Differentiating (10) with respect to p, we obtain

$$0 = dx + \frac{dq}{dp}\, dy, \tag{11}$$

where, from (2),

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0. \tag{12}$$

Solving the equations (10), (11) and (12) for the ratios of dy, dz to dx, we obtain

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} \tag{13}$$

so that along a characteristic strip $x'(t), y'(t), z'(t)$ must be proportional to $F_p, F_q, pF_p + qF_q$, respectively. If we choose the parameter t in such a way that

$$x'(t) = F_p, \qquad y'(t) = F_q, \tag{14}$$

then

$$z'(t) = pF_p + qF_q. \tag{15}$$

Along a characteristic strip p is a function of t so that

$$\begin{aligned} p'(t) &= \frac{\partial p}{\partial x} x'(t) + \frac{\partial p}{\partial y} y'(t) \\ &= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q} \\ &= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q} \end{aligned}$$

since $\frac{\partial p}{\partial y} \equiv \frac{\partial p}{\partial x}$. Differentiating equation (2) with respect to x, we find that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

so that on a characteristic strip

$$p'(t) = -(F_x + pF_z) \tag{16}$$

and it can be shown similarly that

$$q'(t) = -(Fy + qF_z). \tag{17}$$

Collecting equations (14) to (17) together, we see that we have the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t) = F_p,\ \ y'(t) = F_q,\ \ z'(t) = pF_p + qF_q,\ \ p'(t) = -F_x - pF_z,\ \ q'(t) = -F_y - qF_z. \quad (18)$$

These equations are known as the *characteristic equations* of the differential equation (2). The characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_0 of t.

Theorem 2.2.2. *Along every characteristic strip of the equation* $F(x, y, z, p, q) = 0$ *the function* $F(x, y, z, p, q)$ *is a constant.*

Proof. Along a characteristic strip (18), we have

$$\begin{aligned}
&\frac{d}{dt}F\{x(t), y(t), z(t), p(t), q(t)\}\\
&= F_x x' + F_y y' + F_z z' + F_p p' + F_q q'\\
&= F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z)\\
&= 0
\end{aligned}$$

so that $F(x, y, z, p, q) = k$, a constant along the strip. ■

Solution of Partial Differential Equation

The partial differential equation (1) which passes through a curve F whose parameteric equations are

$$x = \theta(v),\ \ y = \varphi(v),\ \ z = \chi(v), \quad (19)$$

then in the solution

$$x = x(p_0, q_0, x_0, y_0, z_0, t_0, t), \text{ etc.} \quad (20)$$

of the characteristic equations (18), taking

$$x_0 = \theta(v),\ y_0 = \varphi(v),\ z_0 = \chi(v)$$

as the initial values of x, y, z. The corresponding initial values of p_0, q_0 are determined by the relations

$$\chi'(v) = p_0\theta'(v) + q_0\varphi'(v)$$
$$F\{\theta(v), \varphi(v), \chi(v), p_0, q_0\} = 0.$$

Substituting these values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 in equation (20), we obtain x, y, z can be expressed in terms of the two parameters t, v, to give

$$x = X_1(v, t),\ y = Y_1(v, t),\ z = Z_1(v, t) \tag{21}$$

Eliminating v, t from these three equations yields

$$\psi(x, y, z) = 0$$

which is the equation of the integral surface of equation (1) through the curve Γ.

Problem 2.2.1. Find the solution of the equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$$

which passes through the x-axis.

Solution. The initial values are (for $x-$ axis)

$$x_0 = v,\ y_0 = 0,\ z_0 = 0 \text{ with } t_0 = 0 \tag{1}$$

then the solution in the parametric form is

$$x = x(v,t),\ \ y = y(v,t),\ \ z = z(v,t).$$

Then, the differential equation becomes

$$F = z_0 - \frac{1}{2}(p_0^2 + q_0^2) + (p_0 - v)(q_0 - y_0) = -\frac{1}{2}(p_0^2 + q_0^2) + (p_0 - x_0)q_0 = -\frac{1}{2}(p_0 - q_0)^2 - vq_0 = 0$$

and the strip condition

$$\frac{dz}{dv} = p\frac{dx}{dv} + q\frac{dy}{dv}$$

or

$$\frac{dz_0}{dv} = p_0\frac{dx_0}{dv} + q_0\frac{dy_0}{dv}$$

gives

$$0 = p_0 \cdot 1 + q_0 \cdot 0 \quad \text{or} \quad p_0 = 0. \qquad \text{(by(1))}$$

Using (1) in (2), we obtain

$$p_0 = 0,\ \ q_0 = s \qquad \text{(unique initial strip)}.$$

The characteristic equations for this partial differential equation are

$$\frac{dx}{dt} = p + q - y$$
$$\frac{dy}{dt} = p + q - x$$

$$\frac{dz}{dt} = p(p + q - y) + q(p + q - x)$$

$$\frac{dp}{dt} = p + q - y$$

$$\frac{dq}{dt} = p + q - x$$

from above equations

$$\frac{dx}{dt} = \frac{dp}{dt} \qquad\qquad \frac{dy}{dt} = \frac{dq}{dt}$$

$$x = p + c_1 \qquad\qquad y = q + c_2$$

Using the initial conditions

$$x_0 = v,\ y_0 = 0,\ z_0 = 0,\ p_0 = 0,\ q_0 = 2v$$

then the constants of integration c_1 and c_2 are

$$c_1 = v \text{ and } c_2 = -2v$$

yields

$$x = v + p,\ y = q - 2v.$$

Also we have

$$\frac{d}{dt}(p + q - x) = p + q - x,\ \frac{d}{dt}(p + q - y) = p + q - y$$

gives

$$p + q - x = ve^t, \quad p + q - y = 2ve^t.$$

Hence

$$x = v(2e^t - 1),\ y = v(e^t - 1),\ p = 2v(e^t - 1),\ q = v(e^t + 1) \tag{22}$$

Substituting in the third of the characteristic equations, we have

$$\frac{dz}{dt} = 5v^2e^{2t} - 3v^2e^t$$

with solution

$$z = \frac{5}{2}v^2(e^{2t} - 1) - 3v^2(e^t - 1) \tag{23}$$

Now, we obtain the expressions for t and v

$$e^t = \frac{y - x}{2y - x}, \quad v = x - 2y$$

so that substituting in (23),

$$z = \frac{1}{2}y(4x - 3y)$$

which is the required integral surface. ■

Problem 2.2.2. Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the parabola $x = 0, y^2 = z$.

Solution. The initial data curve (Parabola $x = 0, y^2 = z$) is

$$x_0 = 0,\ y_0 = v,\ z_0 = v^2.$$

then the parametric form of solution is

$$x = x(v, t),\ y = y(v, t),\ z = z(v, t)$$

From the differential equation,

$$F = p_0 q_0 - z_0 = p_0 q_0 - v^2 = 0.$$

and the strip condition

$$\frac{dz}{dv} = p\frac{dx}{dv} + q\frac{dy}{dv}$$

or

$$\frac{dz_0}{dv} = p_0\frac{dx_0}{dv} + q_0\frac{dy_0}{dv}$$

gives

$$2s = p_0 \cdot 0 + q_0 \cdot 1 \quad \text{or} \quad q_0 - 2s = 0$$

Therefore,

$$q_0 = 2s \quad \text{and} \quad p_0 = \frac{z_0}{q_0} = \frac{s^2}{2s} = \frac{s}{2} \qquad \text{(unique initial strip)}.$$

Now, the characteristic equations of the given PDE are given by

$$\frac{dx}{dt} = q, \ \frac{dy}{dt} = p, \ \frac{dz}{dt} = 2pq, \ \frac{dp}{dt} = p, \ \frac{dq}{dt} = q.$$

From the characteristics equations

$$\frac{dp}{dt} = p \qquad \frac{dq}{dt} = q \qquad \frac{dx}{dt} = q \qquad \frac{dy}{dt} = p \qquad \frac{dz}{dt} = 2pq$$

$$\int \frac{dp}{p} = dt \qquad \int \frac{dq}{q} = dt \qquad \frac{dx}{dt} = c_2e^t \qquad \frac{dy}{dt} = c_1e^t \qquad \frac{dz}{dt} = 2c_1c_2e^t$$

$$\log p = t + a \qquad \log q = t + b \qquad \int dx = \int c_2e^tdt \qquad \int dy = \int c_1e^tdt \qquad \int dz = \int 2c_1c_2e^tdt$$

$$p = c_1e^t \qquad q = c_2e^t \qquad x = c_2e^t + c_3 \qquad y = c_1e^t + c_4 \qquad z = 2c_1c_2e^t + c_5$$

Using the initial conditions

$$x_0 = 0, \ y_0 = v, \ z_0 = v^2, \ p_0 = \frac{v}{2}, \ q_0 = 2v$$

we obtain

$$c_1 = \frac{v}{2}, c_2 = 2v, c_3 = -2v, c_4 = \frac{v}{2}, c_5 = 0$$

Therefore,

$$p = \frac{v}{2}e^t, \ q = 2ve^t, \ x = 2v(e^t - 1), \ y = \frac{v}{2}(e^t + 1) \ z = v^2e^{2t}.$$

Eliminating v and t from x, y and z in the above equation

$$16z = (4y + x)^2$$

which is the required integral surface. ■

Problem 2.2.3. Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0$, $y = 0$.

Solution. The initial data (Parabola $4z + x^2 = 0$, $y = 0$.) is

$$x_0 = v,\ y_0 = 0,\ z_0 = -\frac{v^2}{4}$$

then the parametric form of the solution is

$$x = x(v, t),\ y = y(v, t),\ z = z(v, t).$$

From the differential equation,

$$F = p_0^2 - q_0^2 - z_0 = p_0^2 - q_0^2 + \frac{v^2}{4} = 0$$

and the strip condition

$$\frac{dz}{dv} = p\frac{dx}{dv} + q\frac{dy}{dv}.$$

or

$$\frac{dz_0}{dv} = p_0\frac{dx_0}{dv} + q_0\frac{dy_0}{dv}.$$

gives

$$-\frac{v}{2} = p_0 \cdot 1 + q_0 \cdot 0 \quad \text{or} \quad p_0 = -\frac{v}{2}.$$

Therefore,

$$q_0 = \pm\frac{v}{\sqrt{2}},\ p_0 = -\frac{v}{2} \qquad \text{(unique initial strip).}$$

The characteristic equations for this partial differential equations are

$$\frac{dx}{dt} = 2p,\ \frac{dy}{dt} = 2q,\ \frac{dz}{dt} = -2p^2 + 2q^2,\ \frac{dp}{dt} = -p,\ \frac{dq}{dt} = -q.$$

From the characteristics equations

$$\frac{dp}{dt} = -p \qquad \int \frac{dp}{p} = -\int dt \qquad \log p = -t + a \qquad p = c_1 e^{-t}$$

$$\frac{dq}{dt} = -q \qquad \int \frac{dq}{q} = -\int dt \qquad \log q = -t + b \qquad q = c_2 e^{-t}$$

$$\frac{dx}{dt} = -2p \qquad \frac{dx}{dt} = -2c_1 e^{-t} \qquad \int dx = -2\int c_1 e^{-t} dt \qquad x = 2c_1 e^{-t} + c_3$$

$$\frac{dy}{dt} = -2q \qquad \frac{dy}{dt} = 2c_2 e^{-t} \qquad \int dy = 2\int c_2 e^{-t} dt \qquad y = -2c_2 e^{-t} + c_4$$

$$\frac{dz}{dt} = -2(p^2 - q^2) \qquad \frac{dz}{dt} = -2z \qquad \int \frac{dz}{z} = -2\int dt \qquad z = c_5 e^{-2t}$$

Using the initial conditions

$$x_0 = v,\ y_0 = 0,\ z_0 = -\frac{v^2}{4},\ p_0 = -\frac{v}{2},\ q_0 = \pm\frac{v}{\sqrt{2}},$$

we obtain

$$c_1 = -\frac{v}{2},\ c_2 = \frac{v}{\sqrt{2}},\ c_3 = 2s,\ c_4 = \sqrt{2}v,\ c_5 = -\frac{v^2}{4}$$

Therefore,

$$p = -\frac{v}{2}\exp(-t),\ q = \pm\frac{v}{\sqrt{2}}\exp(-t),\ x = v(2 - \exp(-t)), y = \sqrt{2}v(1 - \exp(-t)),\ z = -\frac{v^2}{4}\exp(-2t).$$

Eliminating the parameters v and t from x, y and z in the above equation

$$4z + (x - \sqrt{2}y)^2 = 0$$

which is the required integral surface. ■

Check Your Progress

1. Write down, and integrate completely, the equations for the characteristics of

$$(1+q^2)z = px$$

expressing x, y, z and p in terms of φ, where $q = \tan\varphi$, and determine the integral surface which passes through the parabola $x^2 = 2z$, $y = 0$.

2. Integrate the equations for the characteristics of the equation

$$p^2 + q^2 = 4z$$

expressing x, y, z and p in terms of q, and then find the solutions of this equation which reduce to $z = x^2 + 1$ when $y = 0$.

2.3 Compatible Systems of First-order Equations

Definition 2.3.1. Let $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ be the two first order partial differential equations. We say that the two partial differential equations are compatible if every solution of the first equation is also a solution of the second equation.

Theorem 2.3.1. *Let $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ be the two first order partial differential equations. Then the necessary and sufficient conditions for the two partial differential equations are compatible is* $\dfrac{\partial(f,g)}{\partial(x,p)} + p\dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\dfrac{\partial(f,g)}{\partial(z,q)} = 0.$

Proof. Given

$$f(x, y, z, p, q) = 0 \tag{1}$$

and

$$g(x, y, z, p, q) = 0 \tag{2}$$

are two first order partial differential equations.

We have to find the conditions for compatible of (1) and (2).

If

$$J = \frac{\partial(f, g)}{\partial(p, q)} \quad 0. \tag{3}$$

Solving the equations (1) and (2) to obtain the explicit expressions

$$p = \varphi(x, y, z), \;\; q = \psi(x, y, z) \tag{4}$$

for p and q. The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations should be completely integrable, i.e., that the differential relation

$$pdx + qdy = dz$$

or

$$\varphi(x, y, z)dx + \psi(x, y, z)dy = dz \tag{1}$$

should be integrable, for which the necessary condition is

$$\tilde{X} \cdot \text{curl}\tilde{X} = 0$$

where $X = \{\varphi, \psi, -1\}$. That is,

$$(\varphi\tilde{i} + \psi\tilde{j} - \tilde{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi & \psi & -1 \end{vmatrix} = 0$$

or

$$\varphi(-\psi_z) + \psi(\varphi_z) = \psi_x - \varphi_y$$

which can be rewritten as

$$\psi_x + \varphi\psi_z = \varphi_y + \psi\varphi_z. \tag{2}$$

Now, differentiating equation (1) with respet to x and z, we get

$$f_x + f_p\frac{\partial p}{\partial x} + f_q\frac{\partial q}{\partial x} = 0$$

and

$$f_z + f_p\frac{\partial p}{\partial z} + f_q\frac{\partial q}{\partial z} = 0.$$

But, from equation (1), we have

$$\frac{\partial p}{\partial x} = \frac{\partial \varphi}{\partial x}, \quad \frac{\partial q}{\partial x} = \frac{\partial \psi}{\partial x}$$

and so on.

Using these results, the above equations can be recast into

$$f_x + f_p\varphi_x + f_q\psi_x = 0$$

and

$$f_z + f_p\varphi_z + f_q\psi_z = 0.$$

Multiplying the second one of the above pair by φ and adding to the first one, we readily obtain

$$(f_x + \varphi f_z) + f_p(\varphi_x + \varphi\varphi_z) + f_q(\psi_x + \varphi\psi_z) = 0.$$

Similarly, from equation (1), we can deduce that

$$(g_x + \varphi g_z) + g_p(\varphi_x + \varphi\varphi_z) + g_q(\psi_x + \varphi\psi_z) = 0.$$

Solving the above pair of equations for $(\psi_x + \varphi\psi_z)$, we have

$$\frac{(\psi_x + \varphi\psi_z)}{f_p(g_x + \varphi g_z) - g_p(f_x + \varphi f_z)} = \frac{1}{f_q g_p - g_q f_p} = \frac{1}{J}$$

or

$$\begin{aligned} \psi_x + \varphi\psi_z &= \frac{1}{J}[(f_p g_x - g_p f_x) + \varphi(f_p g_z - g_p f_z)] \\ &= \frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)} + \varphi\frac{\partial(f,g)}{\partial(z,p)}\right] \end{aligned} \qquad (3)$$

where J is defined in equation (3). Similarly, differentiating equation (**??**) with respect to y and z and using equation (**??**), we can show that

$$\varphi_x + \psi\varphi_z = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right]. \qquad (4)$$

Finally, substituting the values of $\psi_x + \varphi\psi_z$ and $\varphi_x + \psi\varphi_z$ from equations (3) and (4) into equation (2), we obtain

$$\frac{\partial(f,g)}{\partial(x,p)} + \varphi\frac{\partial(f,g)}{\partial(z,p)} = -\left(\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right).$$

In view of equations (4), we can replace φ and ψ by p and q, respectively to get

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0.$$

This is the desired compatibility condition.

∎

Problem 2.3.1. Show that the equations

$$xp = yq, \;\; z(xp + yq) = 2xy$$

are compatible and solve them.

Solution. Let

$$f = xp - yq = 0 \tag{1}$$

and

$$g = z(xp + yq) - 2xy = 0. \tag{2}$$

Then,

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} p & x \\ pz-2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -px^2 - qxy,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} -q & -y \\ qz-2x & yz \end{vmatrix} = -2xy,$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ xp+yq & yz \end{vmatrix} = pxy + qy^2$$

and we find

$$\begin{aligned}
\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} &= 2xy + p(-px^2 - qxy) - 2xy + q(pxy + qy^2) \\
&= 2\cancel{xy} - p^2x^2 - \cancel{pqxy} - 2\cancel{xy} + \cancel{pqxy} + q^2y^2 \\
&= -p^2x^2 + p^2x^2 \\
&= 0.
\end{aligned}$$

Hence the given PDEs are compatible.

Solving equations (1) and (2) for p and q, we obtain

$$\frac{p}{2xy^2} = \frac{q}{2x^2y} = \frac{1}{2xyz}$$

from which we get

$$p = \frac{2xy^2}{2xyz} = \frac{y}{z}$$

and

$$q = \frac{2x^2y}{2xyz} = \frac{x}{z}.$$

In order to get the solution of the given system, we have to integrate

$$dz = \frac{y}{z}\,dx + \frac{x}{z}\,dy$$

or

$$zdz = ydx + xdy.$$

On integration, we get

$$\frac{z^2}{2} = xy + c_1$$

The solution of the given system is

$$z^2 = 2xy + c$$

which is one parameter family. ■

Problem 2.3.2. Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find the solution.

Solution. Let

$$f = xp - yq - x = 0 \tag{1}$$

and

$$g = x^2 p + q - xz = 0. \tag{2}$$

Then,

$$\frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} (p-1) & x \\ (2xp - z) & x^2 \end{vmatrix} = px^2 - x^2 - 2x^2 p + xz = xz - x^2 p - x^2,$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = x^2,$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q,$$

$$\frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy$$

and we find

$$\begin{aligned} \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} &= xz - x^2 p - x^2 + px^2 - q - qxy \\ &= xz - q - qxy - x^2 \end{aligned}$$

$$= \quad xz - q - x^2 p$$

$$= \ 0.$$

Hence the given PDEs are compatible.

Next we have to find the solution.

Solving equations (1) and (2) for p and q, we obtain

$$\frac{p}{xyz + x} = \frac{q}{-x^3 + x^2 z} = \frac{1}{x + x^2 y}$$

from which we get

$$p = \frac{x(1 + yz)}{x(1 + xy)} = \frac{1 + yz}{1 + xy}$$

and

$$q = \frac{x^2(z - x)}{x(1 + xy)} = \frac{x(z - x)}{1 + xy}.$$

In order to get the solution of the given system, we have to integrate

$$dz = \frac{1 + yz}{1 + xy}\, dx + \frac{x(z - x)}{1 + xy}\, dy$$

or

$$dz - dx = \frac{y(z - x)}{1 + xy}\, dx + \frac{x(z - x)}{1 + xy}\, dy$$

or

$$\frac{dz - dx}{z - x} = \frac{xdx + ydy}{1 + xy}.$$

On integration, we get

$$\ln(z - x) = \ln(1 + xy) + \ln c.$$

That is,

$$z - x = c(1 + xy).$$

The solution of the given system is

$$z = x + c(1 + xy)$$

which is one parameter family. ■

Problem 2.3.3. Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ that is homogeneous in x, y and z.

Solve completely the simultaneous equations

$$z = px + qy, \quad 2xy(p^2 + q^2) = z(yp + xq).$$

Solution. (i) Given that differential equation

$$f(x, y, z, p, q) = 0 \tag{1}$$

is homogeneous in x, y, z. If $f(x, y, z, p, q)$ is a homogeneous function in x, y, z of degree n, then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n\, f. \tag{2}$$

Here

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0.$$

Taking $g(x, y, z, p, q) = px + qy - z = 0$. Then

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p}\\ p & x\end{vmatrix}=x\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial p}\\ -1 & x\end{vmatrix}=x\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\frac{\partial f}{\partial y} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix}\frac{\partial f}{\partial y} & \frac{\partial f}{\partial q}\\ q & y\end{vmatrix}=y\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial q},$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial q}\\ -1 & y\end{vmatrix}=y\frac{\partial f}{\partial z}+\frac{\partial f}{\partial q}$$

and we find

$$\begin{aligned}\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} &= x\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial p}+p\left(x\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\right)+y\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial q}\\ &\quad +q\left(y\frac{\partial f}{\partial z}+\frac{\partial f}{\partial q}\right)\\ &= x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+(px+qy)\frac{\partial f}{\partial y}\\ &= x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial y}\end{aligned}$$

$$= 0.$$

Hence, the differential equation $z = px + qy$ is compatible with any differential equation $f(x, y, z, p, q)$ that is homogeneous in $x, y, z.$

(ii) $f(x, y, z, p, q) = 2xy(p^2 + q^2) - z(yp + xq) = 0$ (3)

$g(x, y, z, p, q) = px + qy - z = 0$ (4)

From (4),

$$q = \frac{z - px}{y} \tag{5}$$

Using (5) in (4), we get

$$2x(x^2 + y^2)p^2 - z(3x^2 + y^2)p + xz^2 = 0$$

$$(2xz - p)[(x^2 + y^2)p - xz] = 0$$

so that

$$p = \frac{z}{2x}, q = \frac{z}{2y} \tag{6}$$

and

$$p = \frac{xz}{(x^2 + y^2)}, q = \frac{yz}{(x^2 + y^2)}. \tag{7}$$

For $p = \frac{z}{2x}, q = \frac{z}{2y}$, then

$$dz = pdx + qdy$$

$$dz = \frac{z}{2x}dx + \frac{z}{2y}dy$$

$$2\frac{dz}{z} = \frac{dx}{x} + \frac{dy}{y}$$

$$2\log z = \log x + \log y + \log c_1$$

$$\boxed{z^2 = c_1 xy.} \tag{8}$$

For $p = \frac{xz}{(x^2+y^2)}$, $q = \frac{yz}{(x^2+y^2)}$, then

$$dz = pdx + qdy$$

$$dz = \frac{xz}{(x^2+y^2)}dx + \frac{yz}{(x^2+y^2)}dy$$

$$2\frac{dz}{z} = \frac{2xdx + 2ydy}{(x^2+y^2)}$$

$$2\log z = \log(x^2+y^2) + \log c_2$$

$$\boxed{z^2 = c_1(x^2+y^2).} \tag{9}$$

Equations (8) and (9) are two common solutions of (3) and (4). ■

Check Your Progress

1. Show that the equations $f(x, y, p, q) = 0$, $g(x, y, p, q) = 0$ are compatible if

$$\frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} = 0.$$

Verify that the equations $p = P(x, y)$, $q = Q(x, y)$ are compatible if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

2. If $u_1 = \dfrac{\partial u}{\partial x}$, $u_2 = \dfrac{\partial u}{\partial y}$, $u_3 = \dfrac{\partial u}{\partial z}$, show that the equations

$$f(x, y, z, u_1, u_2, u_3) = 0, \qquad g(x, y, z, u_1, u_2, u_3) = 0$$

are compatible if

$$\frac{\partial(f, g)}{\partial(x, u_1)} + \frac{\partial(f, g)}{\partial(y, u_2)} + \frac{\partial(f, g)}{\partial(z, u_3)} = 0.$$

2.4 Solutions Satisfying Given Conditions

In this section, we explain the outline of three important concepts.

(i) Determination of surfaces satisfies the partial differential equation and passing through a given surface.

(ii) Determination of surfaces satisfies the partial differential equation and circumscribing a given surface.

(iii) Derivation of one complete integral from the other.

Definition 2.4.1. A curve which touches each member of a given family of curves is called **envelope** of that family.

Note 2.4.1. (i) Envelope of one parameter family of curves can be obtained by differentiating the equation with respect to the parameter and eliminating the parameter from the given equation and equation obtained by differentiation gives the envelope of the given one parameter family.

(ii) If the given equation of curve is quadratic in terms of parameter, i.e. $A\alpha^2 + B\alpha + c = 0$, then envelope is given by discriminant is equal to zero ($B^2 - 4AC = 0$).

(iii) Envelope of two parameter family of curves and a relation connecting the two parameters obtained by differentiating the given equation and relation with respect to one parameter and eliminating the parameter from the given equation and equation obtained by differentiation gives the envelope of the given two parameter family.

2.4.1 Solution passing through a given Surface

Consider a first order partial differential equation

$$F(x, y, z, p, q) = 0. \tag{1}$$

Now, we determine the solution of (1) which passes through a given curve C which has parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{2}$$

t being a parameter.

If the integral surface of the equation (1) through the curve C exists, then it may be one of the three possible cases

(a) A particular case of the complete integral

$$f(x, y, z, a, b) = 0 \tag{3}$$

obtained by giving a or b particular values; or

(b) A particular case of the general integral corresponding to (3), i.e., the envelope of a one-parameter subsystem of (3); or

(c) The envelope of the two-parameter system (3).

To determine the solution (surface E) of (1) which passes through a given curve C, that is, E is the envelope of a one-parameter subsystem of (3) each of whose members touches the curve C, provided that such a subsystem exists.

We must find E such that the subsystem made up of those members of the family (3) which touch the curve C, that is, the points of intersection of the surface (3) and the curve C are determined in terms of the parameter t by the equation

$$f\{x(t), y(t), z(t), a, b\} = 0 \tag{4}$$

and the condition that the curve C should touch the surface (3) is that the equation (4) must have two equal roots and the equation

$$\frac{\partial}{\partial t} f\{x(t), y(t), z(t), a, b\} = 0 \tag{5}$$

should have a common root. The condition for this to be so is the eliminant of t from (4) and (5),

$$\psi(a, b) = 0 \tag{6}$$

which is a relation between a and b alone. The factors of (6) leads to

$$b = \varphi_1(a), \;\; b = \varphi_2(a), \ldots \tag{7}$$

each of which defines a subsystem of one parameter. The envelope of each of these one-parameter subsystems is a solution of the problem.

Problem 2.4.1. Find a complete integral of the partial differential equation

$$(p^2 + q^2)x = pz$$

and deduce the solution which passes through the curve $x = 0,\; z^2 = 4y$.

Solution. Given partial differential equation

$$F(x, y, z, p, q) = (p^2 + q^2)x - pz = 0 \tag{1}$$

By Charpit's method, the auxillary equations of the given PDE are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_z)} = \frac{dq}{-(F_y + qF_z)}. \tag{2}$$

From (1), we have

$$F_x = p^2 + q^2, F_y = 0, F_z = -p, F_p = 2px - z, F_q = 2qx,$$

then (2) becomes

$$\frac{dx}{2px - z} = \frac{dy}{2qx} = \frac{dz}{2(p^2 + q^2) - pz} = \frac{dp}{-q^2} = \frac{dq}{pq}. \tag{3}$$

From (3),

$$\frac{dp}{-q^2} = \frac{dq}{pq}$$

$$-pdp = qdq$$

On integration

$$p^2 + q^2 = a^2 \tag{4}$$

Using (4) in (1), we obtain

$$p = \frac{a^2 x}{z} \text{ and } q = \frac{a}{z}\sqrt{z^2 - a^2x^2}$$

and

$$dz = pdx + qdy$$

Substituting p and q in the above equation

$$dz = \frac{a^2 x}{z}dx + \frac{a}{z}\sqrt{z^2 - a^2x^2}\,dy$$

$$zdz - a^2xdx = a\sqrt{z^2 - a^2x^2}\,dy$$

$$\frac{zdz - a^2xdx}{\sqrt{z^2 - a^2x^2}} = ady$$

$$\frac{d(z^2 - a^2x^2)}{2\sqrt{z^2 - a^2x^2}} = ady$$

$$\sqrt{z^2 - a^2x^2} = ay + b$$

$$\Rightarrow \quad z^2 = a^2x^2 + (ay + b)^2 \tag{5}$$

which is the required complete integral of (1).

The parametric equations of the given curve ($x = 0,\ z^2 = 4y$) are

$$x = 0, \quad y = t^2, \quad z = 2t. \tag{6}$$

The intersections condition of (5) and (6) gives

$$4t^2 = (at^2 + b)^2$$

$$a^2t^4 + (2ab - 4)t^2 + b^2 = 0$$

and this equation has equal roots if

$$(ab - 2)^2 = a^2b^2ab = 1,$$

then the one-parameter subsystem is

$$z^2 = a^2x^2 + \left(ay + \frac{1}{a}\right)^2 \quad a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0$$

then the envelope of the surface becomes

$$(2y - z^2)^2 = 4(x^2 + y^2).$$

which is the required surface passing through the given surface. ■

2.4.2 Derivation of one complete integral from the other

Let

$$f(x, y, z, a, b) = 0 \tag{8}$$

and

$$g(x, y, z, h, k) = 0 \tag{9}$$

be the two complete integral.

Steps

(i) Choosing a curve Γ on the surface (9) in whose equations the constants h, k appear as independent parameters.

(ii) The intersection of (8) and curve Γ with the equal roots conditions gives the one parameter family of subsystem.

(iii) Find the envelope of the one-parameter subsystem.

This envelope contains two arbitrary constants h and k, it gives (9), it is a complete integral.

Problem 2.4.2. Show that the equation

$$xpq + yq^2 = 1$$

has complete integrals

(a) $(z + b)^2 = 4(ax + y)$

(b) $kx(z + h) = k^2y + x^2$

and deduce (b) from (a).

Solution. The two complete integrals may be derived from the characteristic equations.

Consider the curve

$$y = 0, \quad x = k(z + h) \tag{1}$$

on the surface (b).

The intersections of (a) and (1), we have

$$(z + b)^2 - 4ak(z + b) + 4ak(b - h) = 0$$

and this has equal roots if

$$a^2k^2 = ak(b - h)$$

this implies $ak = 0$ or $b = h + ak.$

If we consider $a = 0,$ then the envelope of the subsystem formed does not depend on h and $k.$

So, we consider the second subsystem formed by substituting $b = h + ak$ in (a), we obtain

$$(z + h + ak)^2 = 4(ax + y)$$

$$k^2a^2 + 2a\{k(z + h) - 2x\} + (z + h)^2 - 4y = 0$$

and this has envelope

$$\{k(z + h) - 2x\}^2 = \{(z + h)^2 - 4y\}k^2.$$

On simplification, we get the complete integral (b), i.e.,

$$kx(z + h) = k^2y + x^2. \qquad \blacksquare$$

2.4.3 Solution circumscribes a given Surface

In this subsection, we explain the determination of an integral surface which circumscribes a given surface.

Definition 2.4.2. Two surfaces are said to **circumscribe** each other if they touch along a curve.

Example: A conicoid and its enveloping cylinder.

Consider a first order partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

whose complete integral is given by

$$f(x, y, z, a, b) = 0. \tag{2}$$

To determine a surface E, satisfies the partial differential equation (1), that is, complete integral

(2),

$$u(x, y, z) = 0 \tag{3}$$

which circumscribes

$$\psi(x, y, z) = 0. \tag{4}$$

Hence the surface E is the envelope of the one parameter subsystem S of the two-parameter system (2), it is touched at each of its points, and, in particular, at each point P of Γ, is a curve where the surface E touches the given surface Σ.

Steps:

(i) To find the complete integral $f(x, y, z, a, b) = 0$ of given PDE $F(x, y, z, p, q) = 0$.

(ii) Given circumscribing $\psi(x, y, z) = 0$.

(iii) To determine the equation

$$\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z}.$$

(iv) Eliminating x, y and z from complete integral $f(x, y, z, a, b) = 0$, circumscribing equation $\psi(x, y, z) = 0$ and the above equation, we obtain a relation $\chi(a, b) = 0$.

(v) Factorizing $\chi(a, b) = 0$, we have $b = \varphi_1(a)$, $b = \varphi_2(a)$, $\ldots$.

(vi) Using each factors, we find a one parameter subsystem of complete integral.

(vii) To find the envelope of the above one parameter subsystem, which is the required integral surface E circumscribing the surface Σ along a curve Γ.

Problem 2.4.3. Show that the only integral surace of the equation

$$2q(z - px - qy) = 1 + q^2$$

which is circumscribed about the paraboloid $2x = y^2 + z^2$ is the enveloping cylinder which touches it along its section by the plane $y + 1 = 0$.

Solution. Given PDE

$$2q(z - px - qy) = 1 + q^2$$

can be written as

$$z = px + qy + \frac{q^2 + 1}{2q} \tag{1}$$

which is in the form of Clairaut type

$$z = px + qy + f(p, q).$$

Then the complete integral is

$$z = ax + by + \frac{b^2 + 1}{2b}. \tag{2}$$

Given circumscribing equation

$$2x = y^2 + z^2 \tag{3}$$

The consistent condition is

$$\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z} \Rightarrow \frac{a}{2} = \frac{b}{-2y} = \frac{-1}{-2z}$$

which give the relations

$$y = -\frac{b}{a}, \qquad z = \frac{1}{a}. \tag{4}$$

Eliminating x between equations (2), (3) and (4), we have

$$aby^2 + 2b^2y + abz^2 - 2bz + b^2 + 1 = 0.$$

Eliminating y and z from this equation and the equations (4), we obtain

$$(b-a)(b^2+1)=0$$

If we take $b^2=-1$, then we does not obtain the one parameter family of subsystem.

Therefore, we consider the relation $b=a$, a one parameter family of subsystem

$$\{2(x+y)+1\}a^2-2az+1=0.$$

The envelope of the above subsystem is

$$z^2=2(x+y)+1 \tag{5}$$

which is the enveloping cylinder touches the surface (2),

$$(y+1)^2=0 \;\Rightarrow\; y+1=0$$

is the plane section. ■

CHECK YOUR PROGRESS

1. Find a complete integral of the equation $p^2x+qy=z$, and hence derive the equation of an integral surface of which the line $y=1,\ x+z=0$ is a generator.
2. Show that the integral surface of the equation

$$z(1-q^2)=2(px+qy)$$

which passes through the line $x = 1,\ \ y = hz + k$ has equation

$$(y - kx)^2 = z^2\{(1 + h^2)x - 1\}.$$

3. Show that the differential equation

$$2xz + q^2 = x(xp + yq)$$

has a complete integral

$$z + a^2 x = axy + bx^2$$

and deduce that

$$x(y + hx)^2 = 4(z - kx^2)$$

is also a complete integral.

4. Find the complete integral of the differential equation

$$xp(1 + q) = (y + z)q$$

corresponding to that integral of Charpit's equations which involves only q and $x,$ and deduce that

$$(z + hx + k)^2 = 4hx(k - y)$$

is also a complete integral.

5. Find the integral surface of the differential equation

$$(y + zq)^2 = z^2(1 + p^2 + q^2)$$

circumscribed about the surface $x^2 - z^2 = 2y.$

6. Show that the integral surface of the equation $2y(1 + p^2) = pq$ which is circumscribed about the cone $x^2 + z^2 = y^2$ has equation

$$z^2 = y^2(4y^2 + 4x + 1).$$

2.5 Jacobi's Method

Consider a first order partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

and let us assume the solution of (1) with the following relation

$$u(x, y, z) = 0, \tag{2}$$

then

$$p = -\frac{u_1}{u_3}, \qquad q = -\frac{u_2}{u_3}, \tag{3}$$

where u_i denotes $\dfrac{\partial u}{\partial x_i}$ $(i = 1, 2, 3)$ and $x_1 = x$, $x_2 = y$ and $x_3 = z$. Substitute from equations (3) into equation (1), we obtain

$$f(x, y, z, u_1, u_2, u_3) = 0. \tag{4}$$

The main idea of Jacobi's method is the introduction of two further partial differential equations of the first order

$$g(x, y, z, u_1, u_2, u_3, a) = 0, \quad h(x, y, z, u_1, u_2, u_3, b) = 0 \tag{5}$$

involving two arbitrary constants a and b and such that:

(a) Equations (4) and (5) can be solved for u_1, u_2, u_3 ;

(b) The equation

$$du = u_1 dx + u_2 dy + u_3 dz \tag{6}$$

obtained from these values of u_1, u_2, u_3 is integrable, for which the conditions are

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial y}, \quad \frac{\partial u_3}{\partial y} = \frac{\partial u_2}{\partial z}, \quad \frac{\partial u_1}{\partial z} = \frac{\partial u_3}{\partial x}.$$

We have to find two equations which are compatible with (4), it is clear that g and h have to be solutions of the linear partial differential equation

$$f_{u_1}\frac{\partial g}{\partial x} + f_{u_2}\frac{\partial g}{\partial y} + f_{u_3}\frac{\partial g}{\partial z} - f_x\frac{\partial g}{\partial u_1} - f_y\frac{\partial g}{\partial u_2} - f_z\frac{\partial g}{\partial u_3} = 0 \qquad (7)$$

which has subsidiary equations

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}. \qquad (8)$$

Steps

1. Given equation $F(x, y, z, p, q) = 0$.
2. Find $f(x, y, z, u_1, u_2, u_3) = 0$ by substituting $p = -\frac{u_1}{u_3}$ and $q = -\frac{u_2}{u_3}$ in $F(x, y, z, p, q) = 0$.
3. Form an auxiliary equation
$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}.$$
4. Find two solutions $g(x, y, z, u_1, u_2, u_3, a) = 0$ and $h(x, y, z, u_1, u_2, u_3, b) = 0$ of the auxiliary equations.
5. Verify the condition $(g, h) = \sum_{r=1} \left(\frac{\partial g}{\partial x_r}\frac{\partial h}{\partial u_r} - \frac{\partial g}{\partial u_r}\frac{\partial h}{\partial x_r}\right) = 0.$
6. Find the values of u_1, u_2 and u_3
7. Find the solution from $du = u_1dx+u_2dy+u_3dz.$ On integration, we get the required solution.

Problem 2.5.1. Solve the equation $p^2x + q^2y = z.$

Solution. Given $p^2x + q^2y = z.$ (1)

Let $p = -\frac{u_1}{u_3}, \quad q = -\frac{u_2}{u_3}.$ (2)

Substitute the equation (2) in equation (1), we get

$$f(x, y, z, u_1, u_2, u_3) = xu_1^2 + yu_2^2 - zu_3^2 = 0 \tag{3}$$

then

$$f_x = u_1^2,\ f_y = u_2^2,\ f_z = -u_3^2,\ f_{u_1} = 2u_1x,\ f_{u_2} = 2u_2y,\ f_{u_3} = -2u_3z.$$

Auxiliary equations are

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}$$

$$\frac{dx}{2u_1x} = \frac{dy}{2u_2y} = \frac{dz}{-2u_3z} = \frac{du_1}{-u_1^2} = \frac{du_2}{-u_2^2} = \frac{du_3}{u_3^2}$$

with solutions

$$xu_1^2 = a, \qquad yu_2^2 = b$$

whence

$$u_1 = \left(\frac{a}{x}\right)^{\frac{1}{2}}, \quad u_2 = \left(\frac{b}{y}\right)^{\frac{1}{2}}$$

From (3), we have

$$u_3 = \left(\frac{a+b}{z}\right)^{\frac{1}{2}}$$

Substitute the values of u_1, u_2 and u_3 in

$$du = u_1dx + u_2dy + u_3dz$$

$$= \left(\frac{a}{x}\right)^{\frac{1}{2}} dx + \left(\frac{b}{y}\right)^{\frac{1}{2}} dy + \left(\frac{}{z}\right)^{\frac{1}{2}} dz.$$

On integration

$$u = 2(ax)^{\frac{1}{2}} + 2(by)^{\frac{1}{2}} + 2\{(a+b)z\}^{\frac{1}{2}} + c.$$

Taking $b = 1$, $c = b$, we get the complete integral

$$\{(1+a)z\}^{\frac{1}{2}} = (ax)^{\frac{1}{2}} + y^{\frac{1}{2}} + b.$$

■

Generalization of Jacobi's method

Solve an equation of the type

$$f_1(x_1, x_2, \ldots, x_n, u_1, \ldots, u_n) = 0,$$

where u_i denotes $\dfrac{\partial u}{\partial x_i}$ $(i = 1, 2, \ldots, n)$, then we find $n-1$ auxiliary functions $f_2, f_3, \ldots, f_n$ from the subsidiary equations

$$\frac{dx_1}{f_{u_1}} = \frac{dx_2}{f_{u_2}} = \cdots = \frac{dx_n}{f_{u_n}} = \frac{du_1}{-f_{x_1}} = \frac{du_2}{-f_{x_2}} = \cdots = \frac{du_n}{-f_{x_n}}$$

involving $n-1$ arbitrary constants. Solving these for $u_1, u_2, \ldots, u_n$, we determine u by integrating the Pfaffian equation

$$du = \sum_{i=1}^{n} u_i dx_i$$

the solution so obtained containing n arbitrary constants.

Check Your Progress

1. Show that a complete integral of the equation

$$f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$$

is

$$u = ax + by + \theta(a, b)z + c,$$

where a, b and c are arbitrary constants and $f(a, b, \theta) = 0$.

Find a complete integral of the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}.$$

2. Show how to solve, by Jacobi's method, a partial differential equation of the type

$$f\left(x, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial z}\right) = g\left(y, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

and illustrate the method by finding a complete integral of the equation

$$2x^2y\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial z} = x^2\frac{\partial u}{\partial y} + 2y\left(\frac{\partial u}{\partial x}\right)^2.$$

3. Prove that an equation of the "Clairaut" form

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

is always soluble by Jacobi's method.

Hence solve the equation

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 1.$$

Let us Sum up:

In this unit, the students acquired knowledge to

- solve nonlinear PDE's of the first order.
- derive of one complete integral from the other.
- Jacobi's method.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-II

UNIT 3

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER-I

Overview

In this unit, we discuss the partial differential equations of the second order with constant and variable coefficients. We also explain method of solving partial differential equations using the separation of variables. Finally, we discuss the method of integral transforms.

Notations:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

3.1 Origin of Second-order Equations

In this section, we discuss the formation of second order partial differential equations by eliminating arbitrary functions.

Problem 3.1.1. Form a second order PDE by eliminating arbitrary functions from $z = f(u) + g(v) + w$, where f and g are arbitrary functions of u and v, respectively, and u, v, and w are prescribed functions of x and y.

Solution. Given

$$z = f(u) + g(v) + w \tag{1}$$

Differentiating (1) partially with respect to x and y, we get

$$p = f'(u)u_x + g'(v)v_x + w_x \tag{2}$$

$$q = f'(u)u_y + g'(v)v_y + w_y \tag{3}$$

Again differentiating, we have

$$r = f''(u)u_x^2 + g''(v)v_x^2 + f'(u)u_{xx} + g'(v)v_{xx} + w_{xx} \tag{4}$$

$$s = f''(u)u_xu_y + g''(v)v_xu_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy} \tag{5}$$

$$t = f''(u)u_y^2 + g''(v)v_y^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}. \tag{6}$$

Eliminate the four quantities f', f'', g' and g'' from the equations (2) to (6), we obtain

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \tag{7}$$

which involves only the derivatives p, q, r, s, t and known functions of x and y, therefore a partial differential equation of the second order.

Furthermore if we expand the determinant on the left-hand side of equation (7) in terms of the elements of the first column, we get

$$Rr + Ss + Tt + Pp + Qq = W, \tag{8}$$

where R, S, T, P, Q, W are known functions of x and y. Therefore the relation (1) is a solution of the second-order linear partial differential equation (8). ■

Problem 3.1.2. Form a second order PDE by eliminating arbitrary functions from $z = f(x+ay) + g(x - ay)$, where f and g are arbitrary functions.

Solution. Given

$$z = f(x + ay) + g(x - ay). \tag{1}$$

Differentiating (1) twice partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = f'(x + ay) + g'(x - ay)$$
$$r = \frac{\partial^2 z}{\partial x^2} = f''(x + ay) + g''(x - ay). \tag{2}$$

Differentiating (1) twice partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = af'(x + ay) - ag'(x - ay)$$
$$t = \frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 g''(x - ay)$$
$$t = a^2(f''(x + ay) + g''(x - ay)). \tag{3}$$

From (2) and (3), we obtain

$$t = a^2 r$$

which is the required second-order linear partial differential equation. ■

Problem 3.1.3. Prove that if f and g are arbitrary functions of a single variable, then $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

provided that $\alpha^2 = 1 - v^2/c^2$.

Solution. Given

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y). \tag{1}$$

Differentiating (1) twice partially with respect to x, we get

$$\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y). \tag{2}$$

Similarly differentiating (1) twice partially with respect to y and t respectively, we get

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right] \tag{3}$$

and

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]. \tag{4}$$

Adding (2) and (3) and using (4), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (1 - \alpha^2) \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]$$

$$= \frac{1 - \alpha^2}{v^2} \frac{\partial^2 u}{\partial t^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}} \tag{5}$$

where $\alpha^2 = 1 - v^2/c^2$.

Thus given u is a solution of the partial differential equation (5). ■

Problem 3.1.4. If $z = f\left(x^2 - y\right) + g\left(x^2 + y\right)$, where the functions f and g are arbitrary, prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2 \frac{\partial^2 z}{\partial y^2}.$$

Solution. Given

$$z = f\left(x^2 - y\right) + g\left(x^2 + y\right). \tag{1}$$

Differentiating (1) twice partially with respect to x, we get

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2xf'\left(x^2 - y\right) + 2xg'\left(x^2 + y\right)\\ &= 2x\left[f'\left(x^2 - y\right) + g'\left(x^2 + y\right)\right] \qquad (2)\\ \frac{\partial^2 z}{\partial x^2} &= 2\left[f'\left(x^2 - y\right) + g'\left(x^2 + y\right)\right] + 4x^2\left[f''\left(x^2 - y\right) + g''\left(x^2 + y\right)\right]. \qquad (3)\end{aligned}$$

Differentiating (1) twice partially with respect to y, we get

$$\begin{aligned}\frac{\partial z}{\partial y} &= f'\left(x^2 - y\right) - g'\left(x^2 + y\right) \qquad (4)\\ \frac{\partial^2 z}{\partial y^2} &= f''\left(x^2 - y\right) + g''\left(x^2 + y\right). \qquad (5)\end{aligned}$$

From (2), we have

$$f'\left(x^2 - y\right) + g'\left(x^2 + y\right) = \frac{1}{2x}\frac{\partial z}{\partial x}. \tag{6}$$

From (3), (5) and (6), we obtain

$$\boxed{\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2\frac{\partial^2 z}{\partial y^2}}$$

which is the required partial differential equation. ■

Note 3.1.1. If $z = \sum_{r=1}^{n} f_r(v_r)$, where the functions f_r are arbitrary and the functions v_r are known, then it leads to a linear partial differential equation of the nth order.

Problems

1. Verify that the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x}$$

   is satisfied by

$$z = \frac{1}{x}\varphi(y-x) + \varphi'(y-x)$$

   where φ is an arbitrary function.

2. If $u = f(x+iy) + g(x-iy)$, where the functions f and g are arbitrary, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

3. A variable z is defined in terms of variables x, y as the result of eliminating t from the equations

$$z = tx + yf(t) + g(t)$$
$$0 = x + yf'(t) + g'(t)$$

   Prove that, whatever the functions f and g may be, the equation

$$rt - s^2 = 0$$

   is satisfied.

3.2 Linear Partial Differential Equations with Constant Co-efficients

Consider a linear partial differential equation with constant coefficients of the form

$$F(D, D')\, z = f(x, y), \tag{1}$$

where $F(D, D')$ denotes a differential operator of the type

$$F(D, D') = \sum_r \sum_s c_{rs} D^r D'^s \tag{2}$$

and the quantities c_{rs} are constants; and $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$.

Solution:

The most general solution of the corresponding homogeneous linear partial differential equation

$$F(D, D')\, z = 0 \tag{3}$$

is called the complementary function of the equation (1), which contains the correct number of arbitrary elements (functions), as in the case of ordinary differential equations.

Similarly, any particular solution of the equation (1) which contains no arbitrary constants or functions is called a particular integral of (1).

Thus, the general solution of (1) is the sum of complementary function (C.F) and the particular integral (P.I) of (1), i.e.,

$$z = C.F + P.I.$$

Theorem 3.2.1. *If u is the complementary function and z_1 a particular integral of a linear partial differential equation, then $u + z_1$ is a general solution of the equation.*

Proof. Consider a linear PDE

$$F(D, D')z = f(x, y), \tag{1}$$

Given u is the complementary function of (1), i.e.,

$$F(D, D')u = 0. \tag{2}$$

Also given z_1 a particular integral of a linear PDE (1), i.e.,

$$F(D, D')z_1 = f(x, y), \tag{3}$$

To prove: $u + z_1$ is the general solution of (1).

$$\begin{aligned} F(D, D')(u + z_1) &= F(D, D')u + F(D, D')z_1 \\ &= 0 + f(x, y) \end{aligned}$$

$$F(D, D')(u + z_1) = f(x, y).$$

$\Rightarrow$ $u + z_1$ satisfies the equation (1). Therefore, $u + z_1$ is the general solution of (1). ■

Theorem 3.2.2. *If $u_1, u_2, \ldots, u_n$, are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then*

$$\sum_{r=1}^{n} c_r u_r$$

where the c_r 's are arbitrary constants, is also a solution.

Proof. The given homogeneous linear partial differential equation is

$$F(D, D')z = 0 \tag{1}$$

Given that $u_1, u_2, \ldots, u_n$ are the solution of (1).

$$F(D, D')u_1 = 0$$
$$F(D, D')u_2 = 0$$
$$\vdots$$
$$F(D, D')u_n = 0.$$

Also,

$$F(D, D')c_r u_r = c_r F(D, D')u_r.$$

Now,

$$\begin{aligned} F(D, D')\left(\sum_{r=1}^{n} c_r u_r\right) &= \sum_{r=1}^{n} F(D, D')c_r u_r \\ &= \sum_{r=1}^{n} c_r F(D, D')u_r \\ &= c_1 F(D, D')u_1 + c_2 F(D, D')u_2 + \cdots + c_n F(D, D')u_n \\ &= 0. \end{aligned}$$

Therefore, $\sum_{r=1}^{n} c_r u_r$ is the solution of (1). ■

Classification of linear differential operators

Reducible The operator $F(D, D')$ is said to reducible if it can be factorized into the linear factor of the type $D + aD' + b$, where a and b are constants.

Example: $D^2 - D'^2 = (D + D')(D - D')$.

Irreducible The operator $F(D, D')$ is said to irreducible if it is not reducible.

Example: $D^2 - D'$.

Theorem 3.2.3. *If the operator $F(D, D')$ is reducible, the order in which the linear factors occur is unimportant.*

Proof. For proving this theorem, we have to prove : $(\alpha_r D + \beta_r D' + \gamma_r)(\alpha_s D + \beta_s D' + \gamma_s)$ $= (\alpha_s D + \beta_s D' + \gamma_s)(\alpha_r D + \beta_r D' + \gamma_r)$. Now

$$\begin{aligned}(\alpha_r D + \beta_r D' + \gamma_r)(\alpha_s D + \beta_s D' + \gamma_s) &= \alpha_r\alpha_s D^2 + \alpha_s\beta_r DD' + \alpha_r\beta_s DD' + \beta_r\beta_s D'^2 + \gamma_s\alpha_r D \\ &\quad + \gamma_r\alpha_s D + \gamma_s\beta_r D' + \gamma_r\beta_s D' + \gamma_r\gamma_s \\ &= \alpha_r\alpha_s D^2 + (\alpha_s\beta_r + \alpha_r\beta_s) DD' + \beta_r\beta_s D'^2 + (\gamma_s\alpha_r + \gamma_r\alpha_s) D \\ &\quad + (\gamma_s\beta_r + \gamma_r\beta_s) D' + \gamma_r\gamma_s. \qquad (1)\end{aligned}$$

Also,

$$\begin{aligned}(\alpha_s D + \beta_s D' + \gamma_s)(\alpha_r D + \beta_r D' + \gamma_r) &= \alpha_r\alpha_s D^2 + (\alpha_s\beta_r + \alpha_r\beta_s) DD' + \beta_r\beta_s D'^2 + (\gamma_s\alpha_r + \gamma_r\alpha_s) D \\ &\quad + (\gamma_s\beta_r + \gamma_r\beta_s) D' + \gamma_r\gamma_s. \qquad (2)\end{aligned}$$

From (1) and (2), we get

$$\left(\alpha_r D + \beta_r D' + \gamma_r\right)\left(\alpha_s D + \beta_s D' + \gamma_s\right) = \left(\alpha_s D + \beta_s D' + \gamma_s\right)\left(\alpha_r D + \beta_r D' + \gamma_r\right).$$

$\therefore$ For any reducible operator can be written in the form

$$F\left(D, D'\right) = \prod_{r=1}^{n}\left(\alpha_r D + \beta_r D' + \gamma_r\right). \qquad \blacksquare$$

Theorem 3.2.4. *If $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D, D')$ and $\varphi_r(\xi)$ is an arbitrary function of the single variable ξ, then if $\alpha_r \neq 0$,*

$$u_r = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\varphi_r\left(\beta_r x - \alpha_r y\right)$$

is a solution of the equation $F(D, D')\, z = 0$.

Proof. Let

$$F\left(D, D'\right) z = 0 \tag{1}$$

be a partial differential equation. Since (1) is reducible

$$F\left(D, D'\right) z = \prod_{r=1}^{n}\left(\alpha_r D + \beta_r D' + \gamma_r\right) z. \tag{2}$$

If z satisfies $(\alpha_r D + \beta_r D' + \gamma_r)\, z = 0, r = 0, 1, 2, \ldots n$, then it gives us complementary function.

Now

$$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$$

is a linear first order partial differential equation and the auxiliary equation

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}. \tag{3}$$

Consider the first and second term, we get

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$$

$$\Rightarrow \qquad c_{r1} = \beta_r x - \alpha_r y,$$

c_{r1} being a constant. Also

$$\frac{1}{z}\frac{dz}{dx} = \frac{-\gamma_r}{\alpha_r} \qquad \Rightarrow \qquad z = c_{r2} \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

where c_{r2} is a constant. Therefore the solution of (3) is

$$c_{r2} = \varphi_r(c_{r1})$$

implies

$$z = \varphi_r\,(c_{r1}) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) = \varphi_r\,(\beta_r x - \alpha_r y) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right).$$

If $\alpha_r \neq 0$, therefore

$$\text{C.F.} = \varphi_r\,(\beta_r x - \alpha_r y) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

φ_r is an arbitrary function and hence it is a solution of $(\alpha_r D + \beta_r D' + \gamma_r)\, z = 0$. Now

$$F(D, D')\, u_r = \prod_{s=1}^{n} (\alpha_s D + \beta_s D' + \gamma_s)\, (\alpha_r D + \beta_r D' + \gamma_r)\, u_r. \tag{6}$$

Combining equations (5) and (6), we get

$$F(D, D')\, u_r = 0.$$

Thus $u_r = \exp\left(-\dfrac{\gamma_r x}{\alpha_r}\right) \varphi_r\,(\beta_r x - \alpha_r y)$ is a solution of (1). Ths completes the proof. ■

Theorem 3.2.5. *If $\beta_r D' + \gamma_r$ is a factor of $F\,(D, D')$ and $\varphi_r(\xi)$ is an arbitrary function of the single variable ξ, then if $\beta_r \neq 0$,*

$$u_r = \exp\left(-\frac{\gamma_r y}{\beta_r}\right) \varphi_r\,(\beta_r x)$$

is a solution of the equation $F\,(D, D') = 0$.

Proof. Let

$$F(D, D')\, z = 0 \tag{1}$$

be a partial differential equation. Since (1) is reducible

$$F(D, D')\, z = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)\, z. \tag{2}$$

If z satisfies $(\beta_r D' + \gamma_r)\, z = 0,\ r = 0, 1, 2, \ldots\ldots n$, then it gives us complementary function. Now

$$0\frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$$

is a linear first order partial differential equation and the auxiliary equation

$$\frac{dx}{0} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}. \tag{3}$$

Consider the first and second term, we get

$$\frac{dx}{0} = \frac{dy}{\beta_r}$$

$$\Rightarrow \qquad c_{r1} = \beta_r x,$$

c_{r1} being a constant. Also, consider the second and third of (3), we get

$$\frac{dz}{-\gamma_r z} = \frac{dy}{\beta_r} \qquad \Rightarrow \qquad z = c_{r2} \exp\left(\frac{-\gamma_r x}{\beta_r}\right)$$

where c_{r2} is a constant. Therefore the solution of (3) is

$$c_{r2} = \varphi_r(c_{r1})$$

implies

$$z = \varphi_r\,(c_{r1}) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) = \varphi_r\,(\beta_r x) \exp\left(\frac{-\gamma_r x}{\beta_r}\right).$$

Therefore

$$\text{C.F.} = \varphi_r(\beta_r x)\exp\left(\frac{-\gamma_r x}{\beta_r}\right)$$

φ_r is an arbitrary function and hence it is a solution of $(\beta_r D' + \gamma_r)\,z = 0.$ Now

$$F(D, D')\,u_r = \prod_{s=1}^{n}(\alpha_s D + \beta_s D' + \gamma_s)(\beta_r D' + \gamma_r)\,u_r. \tag{6}$$

Combining equations (5) and (6), we get

$$F(D, D')\,u_r = 0.$$

Thus $u_r = \exp\left(-\frac{\gamma_r x}{\beta_r}\right)\varphi_r(\beta_r x)$ is a solution of (1). This completes the proof. ■

Theorem 3.2.6. *If* $(\alpha_r D + \beta_r D' + \gamma_r)^n$ $(\alpha_r \neq 0)$ *is a factor of* $F(D, D')$ *and if the functions* $\varphi_{r1}, \ldots, \varphi_{rn}$ *are arbitrary, then*

$$\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{n} x^{s-1}\varphi_{rs}(\beta_r x - \alpha_r y)$$

is a solution of $F(D, D') = 0$.

Proof. In the decomposition of $F(D, D')$ into linear factors, we may get multiple factors of the type $(\alpha_r D + \beta_r D' + \gamma_r)^n$.

For $n = 1$, the solution corresponding to a factor of this type can be obtained from Theorems 2.2.4 and 2.2.5.

For $n = 2$, we have to find the solutions of the equation

$$\left(\alpha_r D + \beta_r D' + \gamma_r\right)^2 z = 0 \tag{1}$$

Let

$$Z = \left(\alpha_r D + \beta_r D' + \gamma_r\right)^2 z \tag{2}$$

then

$$\left(\alpha_r D + \beta_r D' + \gamma_r\right) Z = 0. \tag{3}$$

By theorem 2.2.4, (3) has the solution of the form

$$Z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \varphi_r\left(\beta_r x - \alpha_r y\right),$$

if $\alpha_r \neq 0$.

To find the corresponding function z we have to solve the first-order linear partial differential equation

$$\alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \varphi_r\left(\beta_r x - \alpha_r y\right).$$

which is a Lagrange's linear equation of the form $Pp + Qq = R$, then the auxiliary equations are

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z + e^{-\gamma' x/\alpha_r}\varphi_r\left(\beta_r x - \alpha_r y\right)} \tag{4}$$

Consider the first and second term, on integration, we get

$$\beta_r x - \alpha_r y = c_1$$

Next we consider the first and last term and substituting $c_1 = \beta_r x - \alpha_r y$ in the last term, we get the

$$\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z + e^{-\gamma_r x/\alpha_r}\varphi_r(c_1)}$$

$$\frac{dz}{dx} + \frac{\gamma_r}{\alpha_r} z = \frac{1}{\alpha_r} e^{-\frac{\gamma_r}{\alpha_r}x}\varphi_r(c_1). \qquad (5)$$

which is a first-order linear equation of the form

$$\frac{dy}{dx} + Py = Q$$

and whose solution of the form

$$y e^{\int Pdx} = \int Q e^{\int Pdx} + c$$

Therefore the solution of (5) is

$$z = \frac{1}{\alpha_r} \cdot \varphi_r(c_1)\, x + c_2 \cdot e^{-\frac{\gamma_r}{\alpha_r}x}.$$

Therefore the solution of (1) is

$$z = \{x\varphi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_r x - \alpha_r y)\}\, e^{-\frac{\gamma_r x}{\alpha_r}}$$

where the functions φ_r, ψ_r are arbitrary.

By induction, the result holds. This completes the proof. ■

Theorem 3.2.7. *If* $(\beta_r D' + \gamma_r)^m$ *is a factor of* $F(D, D')$ *and if the functions* $\varphi_{r1}, \ldots \varphi_{rm}$ *are arbitrary, then*

$$\exp\left(-\frac{\gamma_r y}{\beta_r}\right) \sum_{s=1}^{m} x^{s-1} \varphi_{rs}(\beta_r x)$$

is a solution of $F(D, D')z = 0$.

Proof. The proof is similar to proof of Theorem 2.2.6. ■

Note 3.2.1.

- By the above theorems, any reducible operator $F(D, D')$ is of the form

$$F(D, D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)^{m_r}$$

and if none of the α_r 's is zero, then the corresponding complementary function is

$$u = \sum_{r=1}^{n} \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \sum_{s=1}^{m_r} x^{s-1} \varphi_{rs}(\beta_r x - \alpha_r y)$$

where the functions φ_{rs} $(s = 1, \ldots, n_r; r = 1, \ldots, n)$ are arbitrary.

- If some of the α_r 's are zero, the necessary modifications to the above expression can be made by means of Theorems 2.2.5 and 2.2.7.

Problem 3.2.1. Solve the equation

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2}.$$

Solution. Given

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2} \Rightarrow \frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} - 2\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0 \Rightarrow \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)^2 = 0$$

can be written as

$$(D^2 - D'^2)^2 = 0$$

$$\left[(D + D')(D - D')\right]^2 z = 0$$

$$(D + D')^2 (D - D')^2 z = 0.$$

Therefore, the solution is

$$z = x\varphi_1(x - y) + \varphi_2(x - y) + x\psi_1(x + y) + \psi_2(x + y)$$

where the functions $\varphi_1, \varphi_2, \psi_1, \psi_2$ are arbitrary. ■

Particular Integral

Consider a non-homogeneous linear partial differential equation with constant coefficients of the form

$$F(D, D')z = f(x, y), \tag{1}$$

where $F(D, D')$ denotes a reducible linear differential operator of the form

$$F(D, D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r).$$

We discussed the complementary function of equation (1). Now we need to find a particular integral to complete the solution.

If

$$z_1 = \prod_{r=2}^{n} \left(\alpha_r D + \beta_r D' + \gamma_r\right) z \tag{2}$$

then equation (1) is

$$\left(\alpha_1 D + \beta_1 D' + \gamma_1\right) z_1 = f(x, y)$$

$$\Rightarrow \quad \alpha_1 \frac{\partial z_1}{\partial x} + \beta_1 \frac{\partial z_1}{\partial y} + \gamma_1 z_1 = f(x, y)$$

which is a Lagrange's equation, substituting the value of z_1 in (2) and repeating the process, until the last first-order equation for z.

Problem 3.2.2. Find the solution of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$.

Solution. Given $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$ (1)

can be written as

$$\left(D^2 - D'^2\right) z = x - y$$

$$\left(D - D'\right)\left(D + D'\right) z = x - y.$$

The complementary function is

$$C.F = \varphi_1(x + y) + \varphi_2(x - y)$$

where φ_1 and φ_2 are arbitrary.

To find the particular integral

$$\left(D - D'\right)\left(D + D'\right) z = x - y. \tag{2}$$

Let

$$z_1 = (D + D')z \tag{3}$$

then the equation (2) becomes

$$(D - D')z_1 = x - y$$

$$\frac{\partial z_1}{\partial x} - \frac{\partial z_1}{\partial y} = x - y$$

which is a first-order linear equation of the form $Pp + Qq = R$ with the auxillary equation

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_1}{x - y}$$

Solving the above equation, we obtain

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\int dx = -\int dy$$

$$x = -y + c_1$$

$$x + y = c_1$$

$$\frac{dx - dy}{1 - (-1)} = \frac{dz_1}{x - y}$$

$$\frac{1}{2}(x - y)(dx - dy) = dz_1$$

$$\frac{1}{2}\frac{(x - y)^2}{2} = z_1 + c_2$$

$$\frac{(x - y)^2}{4} = z_1 + c_2$$

Then the solution is

$$c_2 = f(c_1) \quad \text{where } f \text{ is an arbitrary function}$$

$$z_1 - \frac{(x - y)^2}{4} = f(x + y)$$

$$z_1 = \frac{(x - y)^2}{4} + f(x + y)$$

Take $f = 0$

$$z_1 = \frac{(x - y)^2}{4}$$

Substituting this value of z_1 into (3), we have

$$(D + D')z = \frac{(x - y)^2}{4}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{(x - y)^2}{4}$$

which is a first-order linear equation of the form $Pp + Qq = R$ with the auxillary equation

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{(x-y)^2}{4}}$$

On solving, we get Solving the above equation, we obtain

$$z = \frac{1}{4}x(x - y)^2 + f(x - y)$$

in which f is arbitrary. Taking $f \equiv 0$, we obtain the particular integral

$$z = \frac{1}{4}x(x - y)^2$$

Hence the general solution of (1) is

$$z = C.F + P.I$$

$$z = \varphi_1(x+y) + \varphi_2(x-y) + \frac{1}{4}x(x-y)^2.$$

where the functions φ_1 and φ_2 are arbitrary. ■

Theorem 3.2.8. $F(D, D')\, e^{ax+by} = F(a, b)e^{ax+by}$.

Proof. Let $F(D, D') = c_{rs}D^rD'^s$. Then

$$D^r e^{ax+by} = a^r e^{ax+by}$$

$$D'^s e^{ax+by} = b^s e^{ax+by}.$$

Now,

$$\begin{aligned} F(D, D')\, e^{ax+by} &= c_{rs}D^r D'^s e^{ax+by} \\ &= c_{rs}a^r b'^s e^{ax+by} \\ &= F(a, b)e^{ax+by}. \end{aligned}$$

This completes the theorem. ■

Theorem 3.2.9. $F(D, D')\left\{e^{ax+by}\varphi(x, y)\right\} = e^{ax+by}F(D+a, D'+b)\,\varphi(x, y)$.

Proof. Let $F(D, D') = c_{rs}D^rD'^s$. By Leibnitz's theorem for the r^{th} derivative of a product, we have

$$
\begin{aligned}
D^r\left(e^{ax}\varphi\right) &= \sum_{\rho=0}^{r} {}^rC_\rho \left(D^\rho e^{ax}\right) D^{r-\rho}\varphi \\
&= e^{ax} \sum_{\rho=0}^{r} {}^rC_\rho a^\rho D^{r-\rho} \varphi \\
&= e^{ax}(D+a)^r \varphi \\
D'^s\left(e^{by}\varphi\right) &= \sum_{\rho=0}^{s} {}^sC_\rho \left(D'^\rho e^{by}\right) D'^{r-\rho}\varphi \\
&= e^{by} \sum_{\rho=0}^{s} {}^sC_\rho b^\rho D'^{r-\rho} \varphi \\
&= e^{by}(D'+b)^s \varphi.
\end{aligned}
$$

Now,

$$
\begin{aligned}
F\left(D, D'\right) &= c_{rs} D^r D'^s e^{ax+by} \varphi(x, y) \\
&= c_{rs} e^{ax} e^{by} (D+a)^r (D'+b)^s \varphi(x, y) \\
&= e^{ax+by} c_{rs} (D+a)^r (D'+b)^s \varphi(x, y) \\
&= e^{ax+by} c_{rs} F\left(D+a, D'+b\right) \varphi(x, y).
\end{aligned}
$$

This completes the proof. ■

Problem 3.2.3. Show that the equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial z}{\partial t}$ possesses solutions of the form $\sum_{n=0}^{\infty} c_n \cos\left(nx + \varepsilon_n\right) e^{-kn^2 t}$.

Solution. Given

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}. \tag{1}$$

Let us assume the solution of the form

$$z = e^{ax+bt} \tag{2}$$

Substitute (2) in (1), we get

$$a^2 = \frac{b}{k}$$

and this relation is satisfied if we take $a = \pm in$, $b = -kn^2$. Then the general solution is

$$z = \sum_{n=0} c_n \cos(nx + \varepsilon_n)\, e^{-kn^2 t}. \qquad \blacksquare$$

Particular integral for irreducible differential operator

Consider a non-homogeneous linear partial differential equation with constant coefficients of the form

$$F(D, D')z = f(x, y), \tag{1}$$

where $F(D, D')$ denotes a reducible / irreducible linear differential operator.

To find the particular integral of the equation (1), we write

$$z = \frac{1}{F(D, D')} f(x, y)$$

Expand the operator F^{-1} by the binomial theorem and integrations with respect to x and y are made for the respective operators D^{-1} and D'^{-1}.

Problem 3.2.4. Find a particular integral of the equation $(D^2 - D')z = 2y - x^2$.

Solution. Given $(D^2 - D')z = 2y - x^2$. (1)

Then the particular integral is

$$P.I = \frac{1}{D^2 - D'}(2y - x^2)$$

$$= -\left(1 - \frac{D^2}{D'}\right)^{-1} \frac{1}{D'}\left(2y - x^2\right)$$

$$= -\left[\frac{1}{D'} - \frac{D^2}{D'^2} - \cdots \frac{D^4}{D'^3} - \cdots\right]\left(2y - x^2\right)$$

$$z = -\frac{1}{D'}\left(2y - x^2\right) - \frac{1}{D^2} D^2\left(2y - x^2\right)$$

$$= -y^2 + x^2 y + \frac{1}{D^2}(2)$$

$$\boxed{P.I = x^2 y.}$$

■

Particular integral for $f(x, y) = e^{ax+by}$

Consider a non-homogeneous linear partial differential equation with constant coefficients of the form

$$F\left(D, D'\right) z = f(x, y), \tag{1}$$

where $F(D, D')$ denotes a reducible / irreducible linear differential operator.

To find the particular integral of the equation (1), we write

$$z = \frac{1}{F(D, D')} e^{ax+by}$$

except if it happens that $F(a, b) \equiv 0$.

Problem 3.2.5. Find a particular integral of the equation $\left(D^2 - D'\right) z = e^{2x+y}$.

Solution. Given $\left(D^2 - D'\right) z = e^{2x+y}$. (1)

Here $F(D, D') = D^2 - D'$, $a = 2$, and $b = 1$, so that $F(a, b) = 3$, and the particular integral is

$$P.I = \frac{1}{3} e^{2x+y}.$$

■

Problem 3.2.6. Find a particular integral of the equation $\left(D^2 - D'\right) z = e^{x+y}$.

Solution. Given $(D^2 - D')z = e^{x+y}$. (1)

Here $F(D, D') = D^2 - D'$, $a = 1, b = 1$, and $F(a, b) = 0$. However,

$$F(D + a, D' + b) = (D + 1)^2 - (D' + 1) = D^2 + 2D - D'$$

Then the particular integrals are

$$\frac{1}{2} x e^{x+y} \text{ and } -y e^{x+y}.$$ ■

Particular integral for $f(x, y) = \sin(ax + by)$ **or** $\cos(ax + by)$

When the function $f(x, y)$ is of the form of a trigonometric function, it is possible to make use of the last two methods by expressing it as a combination of exponential functions with imaginary exponents, but it is often simpler to use the method of undetermined coefficients.

Problem 3.2.7. Find a particular integral of the equation $(D^2 - D')z = A\cos(lx + my)$, where A, l, m are constants.

Solution. Given $(D^2 - D')z = A\cos(lx + my)$. (1)

Let us assume the particular integral of the form

$$z = c_1 \cos(lx + my) + c_2 \sin(lx + my). \quad (2)$$

Substitute (2) in (1) and equating the coefficient of the sine to zero and that of the cosine to A, we obtain the equations

$$mc_1 - l^2c_2 = 0$$

$$-l^2c_1 + mc_2 = A$$

for the determination of c_1 and c_2. Solving these equations for c_1 and c_2, we obtain the particular integral

$$z = \frac{A}{m^2 - l^4}\left\{m \sin(lx + my) + l^2 \cos(lx + my)\right\} \blacksquare$$

Working Procedure for Complementary Function

Consider a non-homogeneous linear partial differential equation with constant coefficients of the form

$$F\left(D, D'\right) z = f(x, y), \tag{1}$$

where $F(D, D')$ denotes a reducible / irreducible linear differential operator.

By putting $m = \frac{D}{D'}$ then the auxiliary equation can be written as

$$F(m, 1) = 0$$

(i) If the roots of auxiliary equation (A.E.) are $m_1, m_2, m_3, \ldots$ (all distinct), then

$$\text{C.F.} = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \ldots$$

where $f_1, f_2, f_3, \ldots$ are all arbitrary functions.

(ii) If two roots of A.E. are equal i.e. $m_2 = m_1$, then

$$\text{C.F.} = f_1(y + m_1x) + x f_2(y + m_1x) + f_3(y + m_3x) + \ldots,$$

where $f_1, f_2, f_3, \ldots$ are all arbitrary functions.

(iii) If three roots of A.E. are equal i.e. $m_3 = m_2 = m_1$, then

$$\text{C.F} = f_1(y + m_1x) + x f_2(y + m_1x) + x^2 f_3(y + m_1x) + f_4(y + m_4x) + \ldots, \text{ C.F.}$$
$$= f_1(y + m_1x) + x f_2(y + m_1x) + x^2 f_3(y + m_1x) + \cdots,$$

where $f_1, f_2, f_3, \ldots$ are all arbitrary functions.

Working Procedure for Particular Integral (P.I.)

The Particular Integral (P.I.) of the equation

$$F(D, D')\, z = F(x, y)$$

where

$$F(D, D') = D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots a_n D^n$$

is given by P.I. $= \dfrac{1}{F(D, D')} F(x, y)$.

Method I (Particular Cases for $f(x, y)$)

(i) When $f(x, y) = e^{ax+by}$, then P.I. $= \dfrac{1}{F(D, D')} e^{ax+by}$, $F(a, b) \neq 0$.
If $F(a, b) = 0$, then it is called a case of failure.

(ii) When $f(x, y) = \sin(ax + by)$

$$\text{P.I.} = \frac{1}{F(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by)$$

provided $F(-a^2, -ab, -b^2) \neq 0$ otherwise it is called a case of failure. A similar rule holds for $F(x, y) = \cos(ax + by)$.

(iii) When $f(x, y) = x^m y^n$, m, n are positive integers, then

$$\text{P.I.} = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$$

If $m < n$, we expand binomially $[F(D, D')]^{-1}$ in powers of D/D' and for $m > n$. If $m < n$, we expand binomially $[\varphi(D, D')]^{-1}$ in powers of D'/D. Also we have

$$\frac{1}{D} f(x, y) = \int_{y\text{-constant}} f(x, y)\,dx \quad \text{and} \quad \frac{1}{D'} f(x, y) = \int_{x\text{-constant}} f(x, y)\,dy.$$

(iv) When $f(x, y) = e^{ax+by}\varphi(x, y)$, then

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D, D')} e^{ax+by}\varphi(x, y) \\ &= e^{ax+by}\frac{1}{F(D + a, D' + b)}\varphi(x, y). \end{aligned}$$

for $\dfrac{1}{F(D + a, D' + b)}\varphi(x, y)$ can be evaluate using any one of the above steps (i), (ii) and (iii).

Method II. (General Method)

This method is applicable to all cases where $f(x, y)$ is not of the form.

Now $F(D, D')$ can be factorized, in general, into n-linear factors, therefore

$$\begin{aligned} \text{P.I.} &= \frac{1}{F(D, D')} f(x, y) \\ &= \frac{1}{(D - m_1 D')(D - m_2 D')\ldots(D - m_n D')} f(x, y) \\ &= \frac{1}{D - m_1 D'} \cdot \frac{1}{D - m_2 D'} \cdots \frac{1}{D - m_n D'} f(x, y). \end{aligned}$$

We find that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)dx,$$

where c is replaced by $y+mx$ after integration. Thus P.I. can be evaluated by repeated application of the above rule.

Problem 3.2.8. Solve the equation $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} - \dfrac{\partial^3 z}{\partial x \partial y^2} + 2\dfrac{\partial^3 z}{\partial y^3} = e^{x+y}$.

Solution. Given

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$$

can be written as

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y} \tag{1}$$

To find complementary function:

Auxiliary equation is

$$m^3 - 2m^2 - m + 2 = 0$$

$$(m+1)(m-1)(m-2) = 0$$

$$m = -1, m = 1, m = 2$$

$$\boxed{\text{C.F} = \varphi_1(y - x) + \varphi_2(y + x) + \varphi_3(y + 2x)}$$

To find Particular Integral:

$$\begin{aligned}
\text{P.I} &= \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} e^{x+y} \\
&= \frac{1}{(D - 2D')(D + D')(D - D')} e^{x+y} \qquad (D = 1, D' = 1) \\
&= \frac{1}{(-1)(2)(D - D')} e^{x+y} \\
&= -\frac{1}{2}\frac{1}{(D - D')} e^{x+y} \\
&= -\frac{1}{2}\int e^{x+c-x} \qquad \text{where } y = c - x \\
&= -\frac{1}{2}\int e^c dx \\
&= -\frac{1}{2} e^c \int dx \\
&= -\frac{1}{2} e^c x
\end{aligned}$$

$$\boxed{P.I = -\frac{1}{2} x e^{x+y}.}$$

The general solution is

$$z = C.F + P.I$$

$$\boxed{z = \varphi_1(y - x) + \varphi_2(y + x) + \varphi_3(y + 2x) \cdot \frac{1}{2} x e^{x+y}.}$$ ∎

Problems for Practice

1. Show that the equation $\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}$ possesses solutions of the form $\sum_{r=0}^{\infty} c_r e^{-kt} \cos(x_r x + \varepsilon_r)\cos(\omega_r t + \delta_r)$, where $c_r, \alpha_r, \varepsilon_r, \delta_r$ are constants and $\omega_r^2 = \alpha_r^2 c^2 - k^2$.
2. Solve the equations
   (a) $r + s - 2t = -e^{x+y}$.
   (b) $r - s + 2q - z = x^2 y^2$.
   (c) $r + s - 2t - p - 2q = 0$.
3. Find the solution of the equation $\nabla_1^2 z = e^{-x}\cos y$ which tends to zero as $x \to \infty$ and has the value $\cos y$ when $x = 0$.
4. Show that a linear partial differential equation of the type $\sum_{r,s} c_{rs} x^r y^s \frac{\partial^{r+s} z}{\partial x^r \partial y^s} = f(x, y)$ may be reduced to one with constant coefficients by the substitutions $\xi = \log x$, $\eta = \log y$. Hence solve the equation $x^2 r - y^2 t + xp - yq = \log x$.

3.3 Equations with Variable Coefficients

Consider a second order partial differential equation with variable coefficients of the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \tag{1}$$

which may be written in the form

$$L(z) + f(x, y, z, p, q) = 0 \tag{2}$$

where L is the differential operator defined by the equation

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} \tag{3}$$

and R, S, T are continuous functions of x and y possessing continuous partial derivatives.

The equation (1) is said to be

(i) Elliptic if $S^2 - 4RT < 0$

(ii) Parabolic if $S^2 - 4RT = 0$, and

(iii) Hyperbolic if $S^2 - 4RT > 0$

at a point (x_0, y_0)

If this is true at all points in a domain Ω, then (1) is said to be elliptic, parabolic or hyperbolic in that domain.

Canonical Forms

Consider the transformation of the independent variables x and y of equation (1) to new variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{4}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \quad 0 \tag{5}$$

in the domain Ω where equation (1) holds.

By chain rule of partial differential equation, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \xi_x z_\xi + \eta_x z_\eta$$

$$q = \frac{\partial z}{\partial y} = \xi_y z_\xi + \eta_y z_\eta$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\xi_x z_\xi + \eta_x z_\eta\right) = \xi_x^2 z_{\xi\xi} + z_\xi \xi_{xx} + \eta_x^2 z_{\eta\eta} + z_\eta \eta_{xx} + 2\xi_x \eta_x z_{\xi\eta}$$

$$t = \frac{\partial^2 z}{\partial y^2} = z_u \xi_y^2 + 2\xi_y \eta_y z_{\xi\eta} + z_{\eta\eta}\eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = z_{\xi_y}\xi_x\xi_y + z_{\eta\eta}\eta_x\eta_y + z_{\xi\eta}\xi_x\eta_y + z_{\xi\eta}\eta_x\xi_y + z_\eta\eta_x\eta_y + z_\xi\xi_x\xi_y$$

Substituting the values of p, q, r, s and t in (1), we get

$$A\left(\xi_x, \xi_y\right) z_y + 2B\left(\xi_x, \xi_y, \eta_x, \eta_y\right) z_{\xi_\eta} + A\left(\eta_x, \eta_y\right) z_{\eta\eta} = F\left(\xi_1, \eta, z_z, z_z, z_\eta\right) \tag{6}$$

where

$$A(u, v) = Ru^2 + S\,uv + Tv^2$$

$$2B\,(u_1, v_1, u_2, v_2) = 2Ru_1u_2 + S\,(u_1v_2 + u_2v_1) + 2Tv_1v_2$$

Then

$$2B^2\left(\xi_k, \xi_w\eta_k\eta_v\right) - A\left(\xi_e, \xi_y\right) A\left(\eta_k\eta_y\right) = (S^a - 4RT)\,J. \tag{7}$$

Case I: $S^2 - 4RT > 0$.

Under the condition $S^2 - 4RT > 0$, the equation

$$R\lambda^2 + S\lambda + T = 0$$

has real and distinct roots and the roots λ_1 and λ_2 are given by

$$\lambda_1, \lambda_2 = \frac{-S \pm \sqrt{S^2 - 4RT}}{2R}.$$

Choose ξ and η such that

$$\xi_x = \lambda_1\xi_y, \qquad \eta_x = \lambda_2\eta_y. \tag{8}$$

Now $\xi_x = \lambda_1 \xi_y \Rightarrow \xi_x - \lambda_1 \xi_y = 0$, we have

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

$\therefore \quad d\xi = 0 \Rightarrow \xi =$ constant and

$$\frac{dy}{-\lambda_1} = \frac{dx}{1} \Rightarrow \frac{dy}{dx} + \lambda_1(x, y) = 0. \tag{9}$$

Similarly

$$\frac{dy}{dx} + \lambda_2(x, y) = 0 \tag{10}$$

which is known as characteristic equations. Thus, $f_1(x, y) =$ constant and $f_2(x, y) =$ constant implies

$$\xi = f_1(x, y) \qquad \text{and} \qquad \eta = f_2(x, y). \tag{11}$$

Now

$$\begin{aligned} A\ \xi_x, \xi_y &= R\xi_x^2 + S\,\xi_x\xi_y + T\xi_y^2 \\ &= \xi_y^2\ R\lambda_1^2 + S\lambda_1 + T \\ &= \xi_y^2 0 \\ &= 0 \end{aligned}$$

since λ_1 is a root of $R\lambda^2 + S\lambda + T = 0.$ (12)

Similarly $A\ \eta_x, \eta_y = 0$, as λ_2 is also a root of equation (12).

$$B^2 = S^2 - 4RT \quad J \neq 0$$

Equation (6) reduces to

$$z_{\xi\eta} = g\ \ \xi, \eta, z, z_\xi, z_\eta$$

which is a required canonical form for the hyperbolic PDE.

Case II: $S^2 - 4RT = 0.$

Under the condition $S^2 - 4RT = 0$, the equation

$$R\lambda^2 + S\lambda + T = 0$$

has equal roots $\lambda_1 = \lambda_2 = \lambda$ (say). Choose $\xi = f_1(x, y)$, $f_1(x, y) =$ constant is a solution of

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

Since $A\ \xi_x, \xi_y\ = 0, S^2 - 4RT = 0$, therefore from (7), we have $B = 0$.

However, $A\ \eta_x, \eta_y\quad 0$, otherwise η will depend upon ξ.

Substituting $A = B = 0$ in equation (6) reduces to

$$z_{\eta\eta} = g\ \xi, \eta, z, z_\xi, z_\eta$$

which is the required canonical form for the parabolic partial differential equation.

Case III: $S^2 - 4RT < 0.$

Under the condition $S^2 - 4RT < 0$, the equation

$$R\lambda^2 + S\lambda + T = 0$$

has imaginary roots and therefore ξ and η will be complex.

Let $\xi = \alpha + i\beta, \eta = \alpha - i\beta; \alpha, \beta$ are equal.

$$\alpha = \frac{1}{2}(\xi + \eta), \beta = \frac{i}{2}(\eta - \xi)$$

With this transformation, we have

$$z_{\xi\eta} = \frac{1}{4}\ z_{\alpha\alpha} + z_{\beta\beta}$$

and proceeding on the similar lines as in Case I, we get

$$z_{\alpha\alpha} + z_{\beta\beta} = \varphi\ \alpha, \beta, z, z_\alpha, z_\beta$$

which is the required canonical form for the elliptic partial differential equation.

Problem 3.3.1. Reduce the equation $\dfrac{\partial^2 z}{\partial x^2} = x^2\dfrac{\partial^2 z}{\partial y^2}$ to canonical form.

Solution. The given equation is

$$\frac{\partial^2 z}{\partial x^2} = x^2\frac{\partial^2 z}{\partial y^2} \quad\Rightarrow\quad \frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = 0. \tag{1}$$

Comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

we have $R = 1$, $S = 0$, $T = -x^2$, then

$$S^2 - 4RT = 0 - 4(1)(-x^2) = 4x^2 > 0$$

for all x and y. Hence the given equation is hyperbolic everywhere.

Then the characteristic equations to find ξ and η are

$$\frac{dy}{dx} + \lambda_1 = 0 \;\Rightarrow\; \frac{dy}{dx} = -\left(\frac{-S + \sqrt{S^2 - 4RT}}{2R}\right) = -x$$

$$\frac{dy}{dx} + \lambda_2 = 0 \;\Rightarrow\; \frac{dy}{dx} = -\left(\frac{-S - \sqrt{S^2 - 4RT}}{2R}\right) = x.$$

On integration

$$y = -\frac{x^2}{2} + c_1, \qquad y = +\frac{x^2}{2} + c_2$$

$$c_1 = y + \frac{x^2}{2} \qquad c_2 = y - \frac{x^2}{2}$$

$$\therefore\quad \xi = y + \frac{x^2}{2} \qquad \eta = y - \frac{x^2}{2}.$$

Then $\xi_x = x$, $\xi_y = 1$, $\xi_{xx} = 1$, $\xi_{xy} = 0$, $\xi_{yy} = 0$ and $\eta_x = -x$, $\eta_y = 1$, $\eta_{xx} = -1$, $\eta_{xy} = 0$, $\eta_{yy} = 0$. Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x = x(z_\xi - z_\eta)$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y = z_\xi + z_\eta$$

$$z_{xx} = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi \xi_{xx} + z_\eta \eta_{xx} = x^2(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}) + (z_\xi - z_\eta)$$

$$z_{xy} = z_{\xi\xi}\xi_x\xi_y + z_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + z_{\eta\eta}\eta_x\eta_y + z_\xi \xi_{xy} + z_\eta \eta_{xy} = x(z_{\xi\xi} - z_{\eta\eta})$$

$$z_{yy} = z_{\xi\xi}\xi_y^2 + 2z_{\xi\eta}\xi_y\eta_y + z_{\eta\eta}\eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy} = z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}.$$

Substituting these values in (1), we get

$$\boxed{\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}\left(\frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}\right)}$$

which is the required canonical form of the given equation. ■

Problem 3.3.2. Reduce the partial differential equation $y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$ to canonical form and hence solve it.

Solution. The given equation is

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} - \frac{y^2}{x}\frac{\partial z}{\partial x} - \frac{x^2}{y}\frac{\partial z}{\partial y} = 0. \qquad (1)$$

Comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

we have $R = y^2, S = -2xy, T = x^2$, then

$$S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$$

for all x and y. Hence the given equation is parabolic everywhere.

Then the characteristic equation to find ξ is

$$\frac{dy}{dx} + \lambda = 0 \Rightarrow \frac{dy}{dx} = \left(\frac{-S + \sqrt{S^2 - 4RT}}{2R}\right) = \frac{S}{2R} = -\frac{x}{y}.$$

On integration

$$y^2 = -x^2 + c_1$$

$$c_1 = x^2 + y^2$$

$$\therefore \quad \xi = x^2 + y^2.$$

Choose η independent of ξ, we take

$$\xi == x^2 - y^2.$$

Then $\xi_x = 2x, \xi_y = 2y, \xi_{xx} = 2, \xi_{xy} = 0, \xi_{yy} = 2y$ and $\eta_x = 2x, \eta_y = -2y, \eta_{xx} = 2, \eta_{xy} = 0, \eta_{yy} = -2.$

Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x = 2x(z_\xi + z_\eta)$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y = 2y(z_\xi - z_\eta)$$

$$z_{xx} = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi\xi_{xx} + z_\eta\eta_{xx} = 4x^2(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}) + 2(z_\xi + z_\eta)$$

$$z_{xy} = z_{\xi\xi}\xi_x\xi_y + z_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + z_{\eta\eta}\eta_x\eta_y + z_\xi\xi_{xy} + z_\eta\eta_{xy} = 4xy(z_{\xi\xi} - z_{\eta\eta})$$

$$z_{yy} = z_{\xi\xi}\xi_y^2 + 2z_{\xi\eta}\xi_y\eta_y + z_{\eta\eta}\eta_y^2 + z_\xi\xi_{yy} + z_\eta\eta_{yy} = 4y^2(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}) + 2(z_\xi - z_\eta).$$

Substituting these values in (1), we get

$$\boxed{z_{\eta\eta} = 0}$$

which is the required canonical form of the given equation.

To solve the canonical form

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

$$\frac{\partial z}{\partial \eta} = A$$

$$z = A\eta + B,$$

where A and B are arbitrary functions of ξ. Therefore

$$z = \eta A(\xi) + B(\xi)$$

$$\boxed{z = (x^2 - y^2)\, A(x^2 + y^2) + B(x^2 + y^2)}$$

which is the required solution of the given equation. ■

Problem 3.3.3. Reduce the equation $(n-1)^2\dfrac{\partial^2 z}{\partial x^2} - y^{2n}\dfrac{\partial^2 z}{\partial y^2} = ny^{2n-1}\dfrac{\partial z}{\partial y}$ to canonical form and find its general solution.

Solution. The given equation is

$$(n-1)^2\frac{\partial^2 z}{\partial x^2} - y^{2n}\frac{\partial^2 z}{\partial y^2} - ny^{2n-1}\frac{\partial z}{\partial y} = 0. \tag{1}$$

Comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

we have $R = (n-1)^2, S = 0, T = -y^{2n}$, then

$$S^2 - 4RT = 4(n-1)^2y^{2n} = (2(n-1)y^n)^2 > 0$$

for all x and y. Hence the given equation is hyperbolic everywhere.

Then the characteristic equations to find ξ and η are

$$\frac{dy}{dx} + \lambda_1 = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{-S + \sqrt{S^2 - 4RT}}{2R}\right) = -\frac{y^n}{(n-1)}$$

$$\frac{dy}{dx} + \lambda_2 = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{-S - \sqrt{S^2 - 4RT}}{2R}\right) = \frac{y^n}{(n-1)}.$$

On integration

$$(n-1)y^{-n}dy = -dx \qquad\qquad (n-1)y^{-n}dy = dx$$

$$y^{1-n} = -x + c_1, \qquad\qquad y^{1-n} = x + c_2$$

$$c_1 = x + y^{1-n} \qquad\qquad c_2 = x - y^{1-n}$$

$$\therefore \quad \xi = x + y^{1-n} \qquad\qquad \eta = x - y^{1-n}.$$

Then $\xi_x = 1, \xi_y = (1-n)y^{-n}, \xi_{xx} = 0, \xi_{xy} = 0, \xi_{yy} = -n(1-n)y^{-n-1}$ and $\eta_x = 1$, $\eta_y =$

$-(1-n)y^{-n}, \eta_{xx} = 0, \eta_{xy} = 0, \eta_{yy} = n(1-n)y^{-n-1}$. Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x = z_\xi + z_\eta$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y = (1-n)y^{-n}(z_\xi - z_\eta)$$

$$z_{xx} = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi \xi_{xx} + z_\eta \eta_{xx} = z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}$$

$$z_{xy} = z_{\xi\xi}\xi_x\xi_y + z_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + z_{\eta\eta}\eta_x\eta_y + z_\xi \xi_{xy} + z_\eta \eta_{xy}$$

$$z_{yy} = z_{\xi\xi}\xi_y^2 + 2z_{\xi\eta}\xi_y\eta_y + z_{\eta\eta}\eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy} = (1-n)^2 y^{-2n}(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}) - n(1-n)y^{-1-n}(z_\xi - z_\eta)$$

Substituting these values in (1), we get

$$4(n-1)^2 z_{\xi\eta} = 0$$

$$\boxed{z_{\xi\eta} = 0}$$

which is the required canonical form of the given equation.

To find the solution of the given equation, we can solve the canonical form,

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

$$\frac{\partial z}{\partial \xi} = A(\xi)$$

$$z = A(\xi)\xi + B(\eta)$$

$$z = f_1(\xi) + f_2(\eta)$$

$$\boxed{z = f_1(x + y^{1-n}) + f_2(x + y^{1-n})} \qquad (f_1 \text{ and } f_2 \text{ are arbitrary functions})$$

which is the required solution of the given equation ■

Problem 3.3.4. Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

Solution. The given equation is

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0. \tag{1}$$

Comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

we have $R = 1, S = 2, T = 1$, then

$$S^2 - 4RT = 4 - 4 = 0$$

for all x and y. Hence the given equation is parabolic everywhere.

Then the characteristic equation to find ξ is

$$\frac{dy}{dx} + \lambda = 0 \Rightarrow \frac{dy}{dx} = -\frac{-S + \sqrt{S^2 - 4RT}}{2R} = \frac{S}{2R} = 1.$$

On integration

$$y = x + c_1$$

$$c_1 = x - y$$

$$\therefore \quad \xi = x - y.$$

Choose η independent of ξ, we take

$$\eta = x + y.$$

Then $\xi_x = 1, \xi_y = -1, \xi_{xx} = 0, \xi_{xy} = 0, \xi_{yy} = 0$ and $\eta_x = 1, \eta_y = 1, \eta_{xx} = 0, \eta_{xy} = 0, \eta_{yy} = 0.$ Now

$$z_x = z_\xi \xi_x + z_\eta \eta_x = z_\xi + z_\eta$$

$$z_y = z_\xi \xi_y + z_\eta \eta_y = -z_\xi + z_\eta$$

$$z_{xx} = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi\xi_{xx} + z_\eta\eta_{xx} = z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}$$

$$z_{xy} = z_{\xi\xi}\xi_x\xi_y + z_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + z_{\eta\eta}\eta_x\eta_y + z_\xi\xi_{xy} + z_\eta\eta_{xy} = z_{\eta\eta} - z_{\xi\xi}$$

$$z_{yy} = z_{\xi\xi}\xi_y^2 + 2z_{\xi\eta}\xi_y\eta_y + z_{\eta\eta}\eta_y^2 + z_\xi\xi_{yy} + z_\eta\eta_{yy} = z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}.$$

Substituting these values in (1), we get

$$\boxed{z_{\eta\eta} = 0}$$

which is the required canonical form of the given equation.

To solve the canonical form

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

$$\frac{\partial z}{\partial \eta} = A$$

$$z = A\eta + B,$$

where A and B are arbitrary functions of ξ. Therefore

$$z = \eta A(\xi) + B(\xi)$$

$$\boxed{z = (x + y)\ A(x - y) + B(x - y)}$$

which is the required solution of the given equation. ■

Problem 3.3.5. Reduce the equation $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form.

Solution. The given equation is

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0. \qquad (1)$$

Comparing (1) with

$$Rr + S s + Tt + f(x, y, z, p, q) = 0$$

we have $R = 1, S = 0, T = x^2$, then

$$S^2 - 4RT = 0 - 4(1)(x^2) = -4x^2 < 0$$

for all x and y. Hence the given equation is elliptic everywhere.

Then the characteristic equations to find ξ and η are

$$\frac{dy}{dx}+\lambda_1=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{-S+\sqrt{S^2-4RT}}{2R}\right)=ix$$

$$\frac{dy}{dx}+\lambda_2=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{-S-\sqrt{S^2-4RT}}{2R}\right)=-ix.$$

On integration

$$iy=-\frac{x^2}{2}+c_1, \qquad -iy=-\frac{x^2}{2}+c_2$$

$$c_1=\frac{x^2}{2}+iy \qquad c_2=\frac{x^2}{2}-iy$$

$$\therefore \quad \xi=\frac{x^2}{2}+iy \qquad \eta=\frac{x^2}{2}-iy.$$

Now, we introduce the second transformation

$$\alpha=\frac{\xi+\eta}{2} \quad \text{and} \quad \beta=\frac{\xi-\eta}{2},$$

we obtain

$$\alpha=\frac{1}{2}x^2, \qquad \beta=y.$$

Then $\alpha_x=x, \alpha_y=0, \alpha_{xx}=1, \alpha_{xy}=0, \alpha_{yy}=0$ and $\beta_x=0, \beta_y=1, \beta_{xx}=0, \beta_{xy}=0, \beta_{yy}=0$. Now

$$z_x=z_\alpha\alpha_x+z_\beta\beta_x=xz_\alpha$$

$$z_y=z_\alpha\alpha_y+z_\beta\beta_y=z_\beta$$

$$z_{xx}=z_{\alpha\alpha}\alpha_x^2+2z_{\alpha\beta}\alpha_x\beta_x+z_{\beta\beta}\beta_x^2+z_\alpha\alpha_{xx}+z_\beta\beta_{xx}=x^2z_{\alpha\alpha}$$

$$z_{xy} = z_{\alpha\alpha}\alpha_x\alpha_y + z_{\alpha\beta}(\alpha_x\beta_y + \alpha_y\beta_x) + z_{\beta\beta}\beta_x\beta_y + z_\alpha\alpha_{xy} + z_\beta\beta_{xy} = xz_{\alpha\beta}$$

$$z_{yy} = z_{\alpha\alpha}\alpha_y^2 + 2z_{\alpha\beta}\alpha_y\beta_y + z_{\beta\beta}\beta_y^2 + z_\alpha\alpha_{yy} + z_\beta\beta_{yy} = z_{\beta\beta}.$$

Substituting these values in (1), we get

$$\boxed{\frac{\partial^2\zeta}{\partial\alpha^2} + \frac{\partial^2\zeta}{\partial\beta^2} = -\frac{1}{2\alpha}\frac{\partial\zeta}{\partial\alpha}}$$

which is the required canonical form of the given equation. ■

Examples

- The one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$$

  is hyperbolic with canonical form

$$\frac{\partial^2\zeta}{\partial\xi\partial\eta} = 0.$$

- The one-dimensional diffusion equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$

  is parabolic with canonical form.

- The two-dimensional harmonic equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

  is elliptic and in canonical form.

Check Your Progress

1. Show how to find a solution containing two arbitrary functions of the equation $s = f(x, y)$. Hence solve the equation $s = 4xy + 1$.

2. Show that, by a simple substitution, the equation $Rr + Pp = W$ can be reduced to a linear partial differential equation of the first order, and outline a procedure for determining the solution of the original equation.

   Illustrate the method by finding the solutions of the equations:

   (a) $xr + 2p = -2y$

   (b) $s - q = e^{x+y}$.

3. If the functions R, P, Z contain y but not x, show that the solution of the equation $Rr + Pp + Zz = W$ can be obtained from that of a certain second-order ordinary differential equation with constant coefficients. Hence solve the equation $yr + (y^2 + 1)p + yz = e^x$.

3.4 Separation of Variables

Consider a second-order linear partial differential equation

$$Rr + Ss + Tt + Pp + Qq + Zz = F. \tag{1}$$

Let us assume a solution of the form

$$z = X(x)Y(y). \tag{2}$$

Substituting (2) in (1) it is possible to write the equation (1) in the form

$$\frac{1}{X} f(D)X = \frac{1}{Y} g(D')Y, \tag{3}$$

where $f(D)$, $g(D')$ are quadratic functions of $D = \partial/\partial x$ and $D' = \partial/\partial y$, respectively, we say that the equation (2) is separable in the variables x, y. In equation 3,

$$\frac{1}{X} f(D)X = \frac{1}{Y} g(D') Y = \lambda$$

$$f(D)X = \lambda X, \quad g(D)Y = \lambda Y \tag{4}$$

Problem 3.4.1. Solve the one-dimensional diffusion equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}$$

using the method of separable of variables.

Solution. Given

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t} \tag{1}$$

Let us assume a solution of the form

$$z(x, t) = X(x)T(t) \tag{2}$$

Substituting equation (2) in equation (1), we get

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = \lambda, \quad \text{(a separation constant)}$$

Then we have

$$\frac{d^2X}{dx^2} - \lambda X = 0$$

$$\frac{dT}{dt} - \lambda kT = 0.$$

The following three cases arises:

Case I Let $\lambda > 0$, then $\lambda = n^2$, we get

$$\frac{d^2X}{dx^2} - n^2X = 0 \quad \text{and} \quad \frac{dT}{dt} - kn^2T = 0.$$

which gives

$$X = c_1e^{nx} + c_2e^{-nx}, \qquad Y = c_3e^{kn^2t}.$$

Case II Let $\lambda < 0$, then $\lambda = -n^2$, we get

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + kn^2 T = 0.$$

which gives

$$X = c_1 \cos nx + c_2 \sin nx, \qquad Y = c_3e^{-kn^2t}.$$

Case III Let $\lambda = 0$. Then

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0.$$

which gives

$$X = c_1x + c_2, \qquad Y = c_3.$$

Thus, various possible solutions of the heat conduction equation (1) are

$$
\begin{aligned}
z(x, t) &= (c'_1 e^{nx} + c'_2 e^{-nx})e^{kn^2 t} \\
z(x, t) &= (c'_1 \cos nx + c'_2 \sin nx)e^{-kn^2 t} \\
z(x, t) &= c'_1 x + c'_2
\end{aligned}
$$

where

$$c'_1 = c_1 c_3, \qquad c'_2 = c_2 c_3.$$

If a solution tends to zero as $t \to \infty$, then it is possible to take the second solution on simplification,

$$z(x, t) = c_n \cos (nx + \varepsilon_n)\, e^{-n^2 kt},$$

where c_n is a constant, is a solution of the partial differential (1) for all values of n. Hence expressions formed by summing over all values of n,

$$z(x, t) = \sum_{n=0}^{\infty} c_n \cos (nx + \varepsilon_n)\, e^{-n^2 kt}.$$

As $z \to 0$ as $t \to \infty$, we get

$$z(x, 0) = \sum_{n=0}^{\infty} c_n \cos (nx + \varepsilon_n). \qquad \blacksquare$$

Problem 3.4.2. Solve the two-dimensional diffusion equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{k}\dfrac{\partial z}{\partial t}$.

Solution. Given

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{k}\frac{\partial z}{\partial t}. \tag{1}$$

Let us assume the solution of the form

$$z = X(x)Y(y)T(t). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{k}\frac{T'}{T} = -n^2.$$

Then $T' + kn^2 T = 0$ whose solution is

$$T = e^{-kn^2 t}$$

and

$$\frac{X''}{X} = -\left(n^2 + \frac{Y''}{Y}\right) = -l^2 \text{ (say)}.$$

Hence,

$$\begin{aligned} X'' + l^2 X &= 0 \\ \frac{Y''}{Y} = -n^2 + l^2 &= -m^2(\text{say}) \Rightarrow Y'' + m^2 Y = 0. \end{aligned}$$

which gives

$$X = A\cos lx + B\sin lx = c_l\cos(lx + \varepsilon_1)$$

and

$$Y = C\cos my + D\sin my = c_m\cos(my + \varepsilon_2)$$

Thus, the general solution of the given PDE is

$$z(x, y, t) = c_l \cos(lx + \varepsilon_1) c_m \cos(my + \varepsilon_2) e^{-k(l^2+m^2)t}$$

$$= c_{lm} \cos(lx + \varepsilon_1) \cos(my + \varepsilon_2) e^{-k(l^2+m^2)t}$$

where

$$n^2 = l^2 + m^2 \text{ and } c_{lm} = c_l c_m.$$

By the principle of superposition, the most general solution is

$$z(x, y, t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} c_{lm} \cos(lx + \varepsilon_l) \cos(my + \varepsilon_m) e^{-k(l^2+m^2)t}. \quad \blacksquare$$

Check Your Progress

1. By separating the variables, show that the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2}$$

   has solutions of the form $A \exp(\pm inx \mp inct)$, where A and n are constants. Hence show that functions of the form

$$z(x, t) = \sum_r \left\{ A_r \cos \frac{r\pi ct}{a} + B_r \sin \frac{r\pi ct}{a} \right\} \sin \frac{r\pi x}{a}$$

   where the A_r 's and B_r 's are constants, satisfy the wave equation and the boundary conditions $z(0, t) = 0, z(a, t) = 0$ for all t.

2. By separating the variables, show that the equation $\nabla_1^2 V = 0$ has solutions of the form

$A\exp(\pm nx \pm iny)$; where A and n are constants. Deduce that functions of the form

$$V(x,y) = \sum_r A_r e^{-r\pi x/a} \sin\frac{r\pi y}{a} \qquad x \geq 0,\ 0 < y \leq a$$

where the A_r 's are constants, are plane harmonic functions satisfying the conditions $V(x,0) = 0$, $V(x,a) = 0$, $V(x,y) \to 0$ as $x \to \infty$.

3. Show that if the two-dimensional harmonic equation $\nabla_1^2 V = 0$ is transformed to plane polar coordinates r and θ, defined by $x = r\cos\theta, y = r\sin\theta$ it takes the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

and deduce that it has solutions of the form $(Ar^n + Br^{-n})\, e^{\pm in\theta}$, where A, B, and n are constants.

Determine V if it satisfies $\nabla_1^2 V = 0$ in the region $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$ and satisfies the conditions:

(i) V remains finite as $r \to 0$;

(ii) $V = \sum_r c_n \cos(n\theta)$ on $r = a$.

4. Show that in cylindrical coordinates ρ, z, φ Laplace's equation has solutions of the form $R(\rho)e^{\pm mz \pm in\varphi}$, where $R(\rho)$ is a solution of Bessel's equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(m^2 - \frac{n^2}{\rho^2}\right) R = 0$$

If $R \to 0$ as $z \to \infty$ and is finite when $\rho = 0$, show that, in the usual notation for Bessel functions 1 the appropriate solutions are made up of terms of the form $J_n(m\rho)e^{-mz \pm in\varphi}$.

5. Show that in spherical polar coordinates r, θ, φ Laplace's equation possesses solutions of the form

$$\left\{Ar^n + \frac{B}{r^{n+1}}\right\} \Theta(\cos\theta)e^{\pm im\varphi}$$

where A, B, m, and n are constants and $\Theta(\mu)$ satisfies the ordinary differential equation

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0.$$

Let us Sum up:

In this unit, the students acquired knowledge to

- solve linear PDE's with constant coefficients.
- solve linear PDE's with variable coefficients.
- solve PDE's by method of separation of variable techniques.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-II

UNIT 4

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER-II

Overview

In this unit, we will illustrate the method of integral transforms and Laplace transform techniques.

4.1 The Method of Integral Transforms

In this section, we explain the method of integral transforms to find the solution of partial differential equations.

To determine a function u which depends on the independent variables $x_1, x_2, \dots, x_n$ and governed by the linear partial differential equation

$$a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u + Lu = f(x_1, x_2, \dots, x_n), \tag{1}$$

where L is a linear differential operator in the variables $x_2, \dots, x_n$ and $\alpha \le x_1 \le \beta$. Let

$$\bar{u}(\xi, x_2, \dots, x_n) = \int_\alpha^\beta u(x_1, x_2, \dots, x_n)\, K(\xi, x_1)\, dx_1 \tag{2}$$

Multiply $a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u$ by $K(\xi, x_1)$ and integrating with respect to x_1 from α to β, we have

$$\int_\alpha^\beta \left(a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u \right) K(\xi, x_1)\, dx_1 \tag{3}$$

Then, the integration by parts gives

$$\begin{aligned}
\int_\alpha^\beta (aK)\frac{d^2u}{dx^2}\,dx &= \int_\alpha^\beta (aK)\frac{d}{dx}(u')\,dx \\
&= [u'Ka]_\alpha^\beta - \int_\alpha^\beta (aK)'u'\,dx \\
&= [u'Ka]_\alpha^\beta - [u(aK)']_\alpha^\beta + \int_\alpha^\beta u(aK)''\,dx \\
\int_\alpha^\beta (bK)\frac{du}{dx}\,dx &= [u(bK)]_\alpha^\beta - \int_\alpha^\beta u(bK)'\,dx \\
\int_\alpha^\beta (cK)u\,dx &= \int_\alpha^\beta u(cK)\,dx.
\end{aligned}$$

Using the above equations, we obtain

$$\int_\alpha^\beta \left(a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u \right) K(\xi, x_1)\, dx_1$$
$$= g(\xi, x_2, \ldots, x_n) + \int_\alpha^\beta u \left(\frac{\partial^2}{\partial x_1^2}(aK) - \frac{\partial}{\partial x_1}(bK) + cK \right) dx_1 \quad (4)$$

where
$$g(\xi, x_2, \ldots, x_n) = \left[a\frac{\partial u}{\partial x_1} K(\xi, x_1) + u\left(bK - \frac{\partial}{\partial x_1}(aK) \right) \right]_\alpha^\beta. \quad (5)$$

From equation (4), we can get the idea of choosing the function $K(\xi, x_1)$
$$\frac{\partial^2}{\partial x_1^2}(aK) - \frac{\partial}{\partial x_1}(bK) + cK = \lambda K, \quad (6)$$

where λ is a constant.

Multiplying equation (1) by $K(\xi, x_1)$ and integrating with respect to x_1 from α to β, we find that the function $\bar{u}(\xi, x_2, \ldots, x_n)$, defined by equation (2), satisfies the equation

$$(L + \lambda)\bar{u}(\xi, x_2, \ldots, x_n) = F(\xi, x_2, \ldots, x_n), \quad (7)$$

where

$$F(\xi, x_2, \ldots, x_n) = \bar{f}(\xi, x_2, \ldots, x_n) - g(\xi, x_2, \ldots, x_n),$$
$$\bar{f}(\xi, x_2, \ldots, x_n) = \int_\alpha^\beta f(x_1, x_2, \ldots, x_n) K(\xi, x_1)\, dx_1.$$

Definition 4.1.1. The integral transform of u is defined as

$$\bar{u}(\xi, x_2, \ldots, x_n) = \int_\alpha^\beta u(x_1, x_2, \ldots, x_n) K(\xi, x_1)\, dx_1,$$

where $K(\xi, x_1)$ is the kernel of the transform.

Note 4.1.1.

- The effect of employing the integral transform defined by the equations (2) and (6) is to reduce the partial differential equation (1) in n independent variables $x_1, x_2, \ldots, x_n$

to one in $n-1$ independent variables $x_2, \ldots, x_n$ and a parameter ξ.

- The successive use of integral transforms of this type the given partial differential equation may eventually be reduced to an ordinary differential equation.

Definition 4.1.2. Inverse Integral Transform The inverse integral transform of $\bar{u}(\xi, x_2, \ldots, x_n)$ is given by

$$u(x_1, x_2, \ldots, x_n) = \int_{\gamma}^{\delta} \tilde{u}(\xi, x_2, \ldots, x_n)\, H(\xi, x_1)\, d\xi,$$

where $\bar{u}(\xi, x_2, \ldots, x_n)$ is defined by (2).

Table 4.1: Inversion Theorems for Integral Transforms

Transform	(α, β)	$K(\xi, x)$	(γ, δ)	$H(\xi, x)$
Fourier	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}} e^{i\xi x}$	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}} e^{-i\xi x}$
Fourier cosine	$(0, \infty)$	$\sqrt{\frac{2}{\pi}} \cos(\xi x)$	$(0, \infty)$	$\sqrt{\frac{2}{\pi}} \cos(\xi x)$
Fourier sine	$(0, \infty)$	$\sqrt{\frac{2}{\pi}} \sin(\xi x)$	$(0, \infty)$	$\sqrt{\frac{2}{\pi}} \sin(\xi x)$
Laplace	$(0, \infty)$	$e^{-\xi x}, R(\xi) > c$	$(\gamma - i\infty, \gamma + i\infty)$	$\frac{1}{2\pi i} e^{\xi x}, \gamma > c$
Mellin	$(0, \infty)$	$x^{\xi - 1}$	$(\gamma - i\infty, \gamma + i\infty)$	$\frac{1}{2\pi i} x^{-\xi}$
Hankel	$(0, \infty)$	$xJ_{\nu}(\xi x), \nu \geq -\frac{1}{2}$	$(0, \infty)$	$\xi J_{\nu}(\xi x)$

Solution of Partial Differential Equations by using Integral Transform Technique

Steps

(i) The calculation of the function $\bar{f}(\xi, x_2, \ldots, x_n)$ by simple integration;

(ii) The construction of the equation (4) for the transform $\bar{u}$;

(iii) The solution of this equation;

(iv) The calculation of u from the expression for $\tilde{u}$ by means of the appropriate inversion theorem.

Problem 4.1.1. Derive the solution of the equation:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

for the region $r \geq 0, z \geq 0$, satisfying the conditions:

(i) $V \to 0$ as $z \to \infty$ and as $r \to \infty$

(ii) $V = f(r)$ on $z = 0, r \geq 0.$

Solution. Given

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{1}$$

By the definition of Hankel transform, we have

$$\bar{V} = \int_0^\infty rV(r, z)J_0(\xi r)dr.$$

then, integration by parts and using the condition (i), we get

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) rJ_0(\xi r)dr = -\xi^2 \bar{V}$$

where $J_0(\xi r)$ is a solution of Bessel's differential equation

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \xi^2 f = 0.$$

Then the Hankel transform of (1) is

$$\frac{d^2\bar{V}}{dz^2} - \xi^2\bar{V} = 0$$

where, as a result of the boundary conditions, we know that $\bar{V} \to 0$ as $z \to \infty$ and that $\bar{V} = \bar{f}(\xi)$ on $z = 0$, $\bar{f}(\xi)$ denoting the Hankel transform (of zero order) of (r). The appropriate solution of the equation for $\bar{V}$ is therefore

$$\bar{V} = \bar{f}(\xi)e^{-\xi z}.$$

From the inversion theorem for the Hankel transform, we know that

$$V(r, z) = \int_0^\infty \xi\bar{V}(\xi, z)J_0(\xi r)d\xi$$

so that the required solution is

$$V(r, z) = \int_0^\infty \xi f(\xi)e^{-\xi z}J_0(\xi r)d\xi.$$

If the form of $f(r)$ is given explicitly, $f(\xi)$ can be calculated so that $V(r, z)$ can be obtained as the result of a single integration. ■

Problem 4.1.2. Determine the solution of the equation

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0.$$

($-\infty$ « $x < \infty, y$ ∴ 0) satisfying the conditions:

(i) z and its partial derivatives tend to zero as $x \cdot > 1, \infty$;

(ii) $z \cdots f(x)$, $\partial z/\partial y : -0$ on $y = 0$.

Solution. Given

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0.$$

By the definition of Fourier transform,

$$Z(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-}^{\infty} z(x, y) e^{i\xi x} dx$$

for which, as a result of an integration by parts taking account of (i), we have

$$\frac{1}{\sqrt{2\pi}} \int_{-}^{\infty} \frac{\partial^1 z}{\partial x^4} e^{i\xi x} dx : -\xi^4 \bar{z}$$

so that the equation determining the Fourier transform $\bar{z}$ is

$$\frac{d^2 Z}{dy^2} + \xi^4 Z = 0$$

with $Z = F(\xi)$, $dZ/dy = 0$ when $y = 0$. Therefore

$$Z = F(\xi) \cos \xi^2 y \; .$$

By the inversion theorem for Fourier transforms, we have

$$z(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-}^{\infty} Z(\xi, y) e^{-i\xi x} d\xi$$

so that finally

$$z(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-}^{\infty} F(\xi) \cos \ \xi^2 y \ \ e^{-i\xi x} d\xi,$$

where $F(\xi)$ is the Fourier transform of $f(x)$. ■

Check Your Progress

1. The temperature θ in the semi-infinite rod $0 < x < \infty$ is determined by the differential equation

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}$$

and the conditions

(i) $\theta = 0$ when $t = 0$, x “ 0

(ii) $\theta = \theta_0 =$ const. when $x = 0$ and $t > 0$

Making use of sine transform, show that

$$\theta(x, t) = \frac{2}{\pi}\theta_0 \int_0^{\infty} \frac{\sin(\xi x)}{\xi} \ \ 1 - e^{-\kappa \xi^2 t} \ \ d\xi.$$

2. If in the last question the condition (ii) is replaced by (ii’) $\partial\theta/\partial x = -\mu$, a constant, when $x = 0$ and $t > 0$, prove that

$$\theta(x, t) = \frac{2\mu}{\pi} \int_0^{\infty} \frac{\cos(\xi x)}{\xi^2} \ \ 1 - e^{-\kappa \xi^2 t} \ \ d\xi$$

3. Show that the solution of the equation

$$\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2}$$

which tends to zero as $y \to \infty$ and which satisfies the conditions

(i) $z = f(x)$ when $y = 0$, $x > 0$

(ii) $z = 0$ when $y > 0,\ x = 0$

may be written in the form

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(\xi)e^{\xi x - y\sqrt{\xi}}\,d\xi$$

Evaluate this integral when $f(x)$ is a constant k.

4. The function $V(r,\theta)$ satisfies the differential equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

in the wedge-shaped region $r \geq 0, |\theta| \leq \alpha$ and the boundary conditions $V = f(r)$ when $\theta = \pm\alpha$. Show that it can be expressed in the form

$$V(r,\theta) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\cos(\xi\theta)}{\cos(\xi\alpha)} f(\xi) r^{-\xi}\,d\xi$$

where

$$f(\xi) = \int_0^\infty f(r) r^{\xi-1}\,dr.$$

5. The variation of the function z over the xy plane and for $t \geq 0$ is determined by the equation by the equation

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}.$$

If, when $t = 0, z = f(x,y)$ and $\partial z/\partial t = 0$, show that, at any subsequent time, $z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)\cos\left(ct\sqrt{\xi^2+\eta^2}\right) e^{-i(\xi x+\eta y)}\,d\xi\,d\eta$ where $F(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(\xi x+\eta y)}\,dx\,dy.$

4.2 Laplace Transform Technique

Definition 4.2.1. Suppose $f(t)$ is a piecewise continuous function and if it has an additional property that there exists a real number γ_0 and a finite positive number M such that

$$\underset{t\to\infty}{Lt}\,|f(t)|e^{-\gamma t} \leq M \quad \text{for} \quad \gamma \geq \gamma_0$$

and the limit does not exist when $\gamma < \gamma_0$, then such a function is said to be of exponential order γ_0, also written as

$$|f(t)| = O(e^{\gamma_0 t}).$$

Definition 4.2.2. Let $f(t)$ be a continuous and single valued function of the real variable t defined for all t, $0 < t < \infty$, and is of exponential order. Then the Laplace transform of $f(t)$ is defined as a function $F(s)$ denoted by the integral

$$L[f(t); s] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

over that range of values of s for which the integral exists. Here s is a parameter, real or complex. Obviously, $L[f(t); s]$ is a function of s. Thus

$$\begin{aligned} L[f(t); s] &= F(s) \\ f(t) &= L^{-1}[F(s); t] \end{aligned}$$

where L is the operator which transforms $f(t)$ into $F(s)$, called Laplace transform operator and

L^{-1} is the inverse Laplace transform operator.

Properties of Laplace transform

Linearity Property

If c_1 and c_2 are any two constants and if $F_1(s)$ and $F_2(s)$ are the Laplace transform, respectively of $f_1(t)$ and $f_2(t)$, then

$$L[\{c_1 f_1(t) + c_2 f_2(t)\}; s] = c_1 L[f_1(t); s] + c_2 L[f_2(t); s] = c_1 F_1(s) + c_2 F_2(s).$$

Shifting Property

If a function is multiplied by e^{at}, the transform of the resultant is obtained by replacing s by $s-a$ in the trasform of the original function. That is, if

$$L[f(t); s] = F(s)$$

then

$$L[e^{at} f(t); s] = F(s-a).$$

Multiplication by power of t

If $L[f(t); s] = F(s)$, then

$$L[t^n f(t); s] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s) \quad \text{for} \quad n = 1, 2, 3, \ldots\ldots$$

Division by t

If $L[f(t); s] = F(s)$, then

$$L\left[\frac{f(t)}{t}; s\right] = \int_s^\infty F(s)ds.$$

Differentiation Property

If $L[f(t); s] = F(s)$, then

$$L[f^{(n)}(t); s] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Initial Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s).$$

Final Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s).$$

Transform of Periodic Function

If $f(t)$ is a periodic function with period T, (i.e., $f(t+T) = f(t)$ for all t, $T > 0$), then

$$L[f(t); s] = \frac{\int_0^T e^{-st} f(t)dt}{(1 - e^{-st})}$$

Transform of Error Function

If $\mathrm{erf}(t)$ is a error function defined by $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$ then the Laplace transform is

$$L[erf(t); s] = \frac{1}{s}e^{\frac{s^2}{4}}\mathrm{erfc}\,\frac{s}{2}$$

where $\mathrm{erfc}(t)$ is the complementary error function $\mathrm{erfc}(t) = \mathrm{erf}(t)$.

Properties of Inverse Laplace Transform

Linearity Property

If $F_1(s)$ and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $f_2(t)$ and if c_1 and c_2 are any two constants, then

$$L^{-1}[\{c_1F_1(s) + c_2F_2(s)\}; t] = c_1L^{-1}[F_1(s); t] + c_2L^{-1}[F_2(s); t].$$

Shifting Property

If $L[f(t); s] = F(s)$, then

$$L^{-1}[F(s + a); t] = e^{-at}L^{-1}[F(s); t].$$

Change of Scale property

If $L^{-1}[F(s); t] = f(t)$, then

$$L^{-1}[F(as); t] = \frac{1}{a}f\left(\frac{t}{a}\right).$$

Convolution Theorem

If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then $F(s)\,G(s)$ is the Laplace transform of the convolution of $f(t)$ and $g(t)$,

$$L[(f * g)(t); s] = L\left[\int_0^t f(t-u)g(u)du; s\right] = F(s)G(s)$$

or

$$L^{-1}[F(s)G(s); t] = (f * g)(t) = \int_0^t f(t-u)g(u)du.$$

Transform of Partial Derivatives

If $u(x, t)$ is a function of two variables x and t, prove that

(i) $L\left[\frac{\partial u}{\partial t}; s\right] = sU(x, s) - u(x, 0)$

(ii) $L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2U(x, s) - su(x, 0) - u_t(x, 0)$

(iii) $L\left[\frac{\partial u}{\partial x}; s\right] = \frac{dU(x, s)}{dx}$

(iv) $L\left[\frac{\partial^2 u}{\partial x^2}; s\right] = \frac{d^2}{dx^2}U(x, s)$

(v) $L\left[\frac{\partial^2 u}{\partial x \partial t}; s\right] = s\frac{d}{dx}U(x, s) - \frac{d}{dx}u(x, 0).$

where $U(x, s) = L[u(x, t); s]$.

4.3 Fourier Transform Technique

Definition 4.3.1. Let $f(x)$ be a function defined on $(-\infty, \infty)$ and is piecewise continuous, differentiable in each finite interval and is absolutely integrable on $(-\infty, \infty)$, if

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt \tag{1}$$

then we have, for all x,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha. \tag{2}$$

Here, $F(\alpha)$ defined by equation (1) is the Fourier transform of $f(x)$, and $f(x)$ defined by equation (2) is called the Inverse Fourier transform of $F(\alpha)$ and is denoted by

$$F(\alpha) = \mathcal{F}[f(t); \alpha]$$

$$f(x) = \mathcal{F}^{-1}[F(\alpha); x]$$

which constitute the Fourier transform pair.

Definition 4.3.2. Fourier sine transform of $f(x)$ is

$$F_S(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \alpha t dt = \mathcal{F}_S[f(t); \alpha]$$

Inverse Fourier sine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S(\alpha) \sin \alpha x d\alpha = \mathcal{F}_S^{-1}[F_S(\alpha); x].$$

Definition 4.3.3. Fourier cosine transform of $f(x)$ is

$$F_C(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t dt = \mathcal{F}_C[f(t); \alpha]$$

Inverse Fourier cosine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(\alpha) \cos \alpha x d\alpha = \mathcal{F}_C^{-1}[F_C(\alpha); x].$$

Properties of Fourier Transform

Linearity Property

If $F(\alpha)$ and $G(\alpha)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$\mathcal{F}[c_1 f(t) + c_2 g(t); \alpha] = c_1 F(\alpha) + c_2 G(\alpha)$$
$$\mathcal{F}^{-1}[c_1 F(\alpha) + c_2 G(\alpha); x] = c_1 f(x) + c_2 g(x)$$

where c_1 and c_2 are constants.

Change of Scale

If $\mathcal{F}[f(t); \alpha] = F(\alpha)$, then

$$\mathcal{F}[f(at); \alpha] = \frac{1}{a} F\left(\frac{\alpha}{a}\right).$$

Shifting Property

If $\mathcal{F}[f(x); \alpha] = F(\alpha)$, then

$$\mathcal{F}[f(x - a); \alpha] = e^{ia\alpha} F(\alpha).$$

Modulation Property

If $\mathcal{F}[f(x);\alpha] = F(\alpha)$, then

$$\mathcal{F}[f(x)\cos ax;\alpha] = \frac{1}{2}[F(\alpha - a) + F(\alpha + a)].$$

Differentiation

If $f(x)$ and its first $(r-1)$ derivatives are continuous, and if its r^{th} derivative is piecewise continuous, then

$$\mathcal{F}[f^{(r)}(x);\alpha] = (-i\alpha)^r \mathcal{F}[f(x);\alpha], \quad r = 0, 1, 2, \ldots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that $f(x)$ and its first $(r-1)$ derivatives vanish as $x \to \pm\infty$.

If $\mathcal{F}[u(x,t); x \to \alpha] = U(\alpha, t)$, then

(i) $\mathcal{F}\left[\dfrac{\partial u}{\partial x}(x,t); x \to \alpha\right] = -i\alpha U(\alpha, t).$

(ii) $\mathcal{F}\left[\dfrac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = (-1)^2(i\alpha)^2 U(\alpha, t).$

(iii) $\mathcal{F}\left[\dfrac{\partial^n u}{\partial x^n}(x,t); x \to \alpha\right] = (-1)^n(i\alpha)^n U(\alpha, t).$

(iv) $\mathcal{F}\left[\dfrac{\partial u}{\partial t}(x,t); x \to \alpha\right] = U_t(\alpha, t).$

(v) $\mathcal{F}_S\left[\dfrac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = \sqrt{\dfrac{2}{\pi}}\,\alpha\, u(x,t)\big|_{x=0} - \alpha^2 \mathcal{F}_S[u(x,t); x \to \alpha].$

(vi) $\mathcal{F}_C\left[\dfrac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = -\sqrt{\dfrac{2}{\pi}}\,\dfrac{\partial u}{\partial x}(x,t)\big|_{x=0} - \alpha^2 \mathcal{F}_C[u(x,t); x \to \alpha].$

Convolution Theorem

If $\mathcal{F}[f(x);\alpha] = F(\alpha)$ and $\mathcal{F}[g(x);\alpha] = G(\alpha)$, then $\mathcal{F}[(f * g)(x);\alpha] = F(\alpha)G(\alpha)$ where

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(x-u)\,du.$$

Parseval's Relation

If $\mathcal{F}[f(x);\alpha] = F(\alpha)$, then

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

which is known as Parseval's relation.

4.4 Finite Fourier Transform

If a function $f(x)$ satisfies Dirichlet conditions in the interval $0 \le x \le \pi$, then it has Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx, \qquad n = 1, 2, \ldots. \tag{2}$$

The Fourier series in equation (1) converges pointwise to $f(x)$ at points where $f(x)$ is continuous and to the value $\frac{1}{2}[f(x+) + f(x-)]$ at other points.

If a function $f(x)$ satisfies Dirichlet conditions in the interval $0 \le x \le \pi$, then it has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{3}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx, \qquad n = 1, 2, \ldots. \tag{4}$$

4.5 Nonlinear Equations of the Second Order

Consider a second order nonlinear partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0. \tag{1}$$

4.5.1 Monge's Method

In this method, we consider one or two first integrals of the form

$$\eta = f(\xi). \tag{2}$$

where ξ and η are the functions of x, y, z, p, and q and the function f is arbitrary.

Problem 4.5.1. If the partial differential equation has the integral $\eta = f(\xi)$, where ξ and η are the functions of x, y, z, p, and q and the function f is arbitrary, then prove that the partial differential equation is of the form $Rr + Ss + Tt + U\,rt - s^2 = V$ or $Rr + Ss + Tt = V$.

Solution. Consider a second order nonlinear partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0.$$

Given the equation has the integral of the form

$$\eta = f(\xi),$$

where ξ and η are the functions of x, y, z, p and q and the function f is arbitrary.

Differentiating (2) partially with respect to x and y, we get

$$\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial\eta}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial\eta}{\partial q}\frac{\partial q}{\partial x} = f'(\xi)\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial\xi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial\xi}{\partial q}\frac{\partial q}{\partial x}\right)$$

$$\eta_x + \eta_z p + \eta_p r + \eta_q s = f'(\xi)\left(\xi_x + \xi_z p + \xi_p r + \xi_q s\right) \tag{3}$$

$$\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial\eta}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial\eta}{\partial q}\frac{\partial q}{\partial y} = f'(\xi)\left(\frac{\partial\xi}{\partial y} + \frac{\partial\xi}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial\xi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial\xi}{\partial q}\frac{\partial q}{\partial y}\right)$$

$$\eta_y + \eta_z q + \eta_p s + \eta_q t = f'(\xi)\left(\xi_y + \xi_z q + \xi_p s + \xi_q t\right) \tag{4}$$

Eliminating $f'(\xi)$ from equations (3) and (4), then

$$\frac{(3)}{(4)} \Rightarrow \frac{\eta_x + \eta_z p + \eta_p r + \eta_q s}{\eta_y + \eta_z q + \eta_p s + \eta_q t} = \frac{f'(\xi)\left(\xi_x + \xi_z p + \xi_p r + \xi_q s\right)}{f'(\xi)\left(\xi_y + \xi_z q + \xi_p s + \xi_q t\right)}$$

implies

$$\left(\eta_x + \eta_z p + \eta_p r + \eta_q s\right)\left(\xi_y + \xi_z q + \xi_p s + \xi_q t\right) = \left(\eta_y + \eta_z q + \eta_p s + \eta_q t\right)\left(\xi_x + \xi_z p + \xi_p r + \xi_q s\right)$$

On simplifying, we get

$$\left[\left(\xi_p\eta_y - \xi_y\eta_p\right) + q\left(\xi_p\eta_z - \xi_z\eta_p\right)\right] r + \left[\left(\xi_q\eta_y - \xi_y\eta_q\right) + q\left(\xi_q\eta_z - \xi_z\eta_q\right) - \left(\xi_p\eta_x - \xi_x\eta_p\right)\right.$$

$$\left.-p\left(\xi_p\eta_z - \xi_z\eta_p\right)\right] s + \left[\left(\xi_x\eta_q - \xi_q\eta_x\right) + p\left(\xi_z\eta_q - \xi_q\eta_z\right)\right] t + \left(\xi_p\eta_q - \xi_q\eta_p\right)\left(rt - s^2\right)$$

$$= \left(\xi_y\eta_x - \xi_x\eta_y\right) + p\left(\xi_y\eta_z - \xi_z\eta_y\right) + q\left(\xi_z\eta_x - \xi_x\eta_z\right)$$

$$\left[\frac{\partial(\xi,\eta)}{\partial(p,y)} + q\frac{\partial(\xi,\eta)}{\partial(p,z)}\right] r + \left[\frac{\partial(\xi,\eta)}{\partial(q,y)} + q\frac{\partial(\xi,\eta)}{\partial(q,z)} - \frac{\partial(\xi,\eta)}{\partial(p,x)} - p\frac{\partial(\xi,\eta)}{\partial(p,z)}\right] s + \left[\frac{\partial(\xi,\eta)}{\partial(x,q)} + p\frac{\partial(\xi,\eta)}{\partial(z,q)}\right] t$$

$$+ \frac{\partial(\xi,\eta)}{\partial(p,q)}\left(rt - s^2\right) = \frac{\partial(\xi,\eta)}{\partial(y,x)} + p\frac{\partial(\xi,\eta)}{\partial(y,z)} + q\frac{\partial(\xi,\eta)}{\partial(z,x)}$$

The required form of the partial differential equation

$$Rr + Ss + Tt + U\ \ rt - s^2\ \ = V \tag{5}$$

which has the first integral as $\eta = f(\xi)$.

If the Jacobian, $J = \dfrac{\partial(\xi, \eta)}{\partial(p, q)} = 0$, then equation (5) reduces to the form

$$Rr + Ss + Tt = V.$$

■

4.5.2 Solution of Second Order Nonlinear Partial Differential Equations

Let us assume a first integral of the equation

$$Rr + Ss + Tt + U\ \ rt - s^2\ \ = V \tag{1}$$

exists and that it is of the form

$$\eta = f(\xi). \tag{2}$$

For any function z of x and y, we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + sdy \quad \Rightarrow \quad r = \frac{dp - sdy}{dx} \tag{3}$$

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy \quad \Rightarrow \quad t = \frac{dq - sdx}{dy}. \tag{4}$$

Substituting (3) and (4) in (1), we get

$$(Rdpdy + Tdqdx - Udpdq - Vdxdy) - s\ \ R(dy)^2 - S\,dxdy + T(dx)^2 + Udpdx + Udqdy\ \ = 0$$

Monge's subsidiary equations are

$$L \equiv Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \tag{5}$$

and

$$M \equiv R(dy)^2 - S\,dxdy + T(dx)^2 + Udpdx + Udqdy = 0. \tag{6}$$

Let us factorise $M + \lambda L$, where λ is an undetermined multiplier. Now,

$$M + \lambda L \equiv R(dy)^2 + T(dx)^2 - (S + \lambda V)dxdy + Udpdx + Udqdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq$$
$$= 0 \tag{7}$$

and let k and m be constants such that

$$M + \lambda L \equiv (Rdy + mTdx + kUdp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0. \tag{8}$$

Comparing coefficients in (7) and (8), we get

$$\frac{R}{m} + mT = -(S + \lambda V), \tag{9}$$

and take

$$k = m \text{ and } \frac{R\lambda}{k} = U. \tag{10}$$

Using the above equations, we get

$$\lambda^2(UV + RT) + \lambda US + U^2 = 0, \tag{11}$$

which is quadratic in λ. Let λ_1 and λ_2 be its roots.
When $\lambda = \lambda_1$, (10) $\Rightarrow \frac{R\lambda_1}{k} = U \quad \Rightarrow \quad k = \frac{R\lambda_1}{U} \quad \Rightarrow \quad m = \frac{R\lambda_1}{U}$. Equation (8) gives

$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$
$$(Udy + \lambda_1 Tdx + \lambda_1 Udp)\,(Udx + \lambda_1 Rdy + \lambda_1 Udq) = 0. \tag{12}$$

Similarly for $\lambda = \lambda_2$, we have

$$(Udy + \lambda_2 Tdx + \lambda_2 Udp)\,(Udx + \lambda_2 Rdy + \lambda_2 Udq) = 0 \tag{13}$$

implies

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \tag{14}$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0, \qquad Udx + \lambda_1 Rdy + \lambda_1 Udq = 0. \tag{15}$$

Equations (14) give two integrals $u_1 = c_1$ and $v_1 = c_1$ so that one intermediate integral is

$$u_1 = f_1(v_1), \tag{16}$$

where f_1 is an arbitrary function. Similarly, the second intermediate integral

$$u_2 = f_2(v_2), \tag{17}$$

where f_2 is an arbitrary function.
On solving (16) and (17) for p and q and substitute in $dz = pdx + qdy$, which after integration gives the desired general solution.

Problem 4.5.2. Solve the equation $r + 4s + t + rt - s^2 = 2.$

Solution. Given

$$r + 4s + t + rt - s^2 = 2. \tag{1}$$

Comparing (1) with $Rr + Ss + Tt + U\,rt - s^2 = V,$ we have $R = 1, S = 4, T = 1, U = 1, V = 2.$

$$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \Rightarrow 3\lambda^2 + 4\lambda + 1 = 0$$

with roots $\lambda_1 = -\frac{1}{3}, \lambda_2 = -1.$

To find the integrals, we have

$$3dy - dx - dp = 0, \;\; dy - dx + dq = 0$$

leading to the first integral

$$3y - x - p = f(y - x + q)$$

where the function f is arbitrary. Similarly equations (23) reduce to

$$dy - dx - dp = 0, \quad dy - 3dx + dq = 0$$

and yield the first integral

$$y - 3x + q = g(y - x - p)$$

the function g being arbitrary.

Combine the general integral (24) with any particular integral of (25), we have

$$y - 3x + q = c_1$$

where c_1 is a constant. Solving equations (24) and (26), we find that

$$q = c_1 + 3x - y, \quad p = 3y - x - f(2x + c_1)$$

from which it follows that

$$dz = \{3y - x - f(2x + c_1)\}\, dx + \{c_1 + 3x - y\}\, dy$$

and hence that

$$z = 3xy - \frac{1}{2}\; x^2 + y^2 \; + F(2x + c_1) + c_1 y + c_2$$

where c_2 . is an arbitrary constant. Equation (28) gives the complete integral. To obtain the general integral we replace c_1 by c, c_2 by $G(c)$, where the function G is arbitrary, and the required integral is then obtained by eliminating c between the equations

$$z = 3xy - \frac{1}{2}\, x^2 + y^2 \;\; + F(2x + c) + cy + G(c)$$

$$0 = F'(2x + c) + y + G'(c). \qquad \blacksquare$$

In particular, $U = 0$. Monge's subsidiary equations are

$$Rdpdy + Tdqdx = Vdxdy$$

and

$$Rdy^2 - S\,dxdy + Tdx^2 = 0.$$

Problem 4.5.3. Solve the equation $q^2r - 2pqs + p^2t = 0$.

Solution. Given

$$q^2r - 2pqs + p^2t = 0. \tag{1}$$

Comparing (1) with $Rr + S\,s + Tt = V,$ we have $R = q^2, S = -2pq, T = p^2, V = 0.$

Monge's subsidiary equations becomes

$$q^2dpdy + p^2dqdx = 0 \tag{2}$$

$$(pdx + qdy)^2 = 0 \tag{3}$$

From equation (3), we have

$$dz = pdx + qdy \quad \Rightarrow \quad dz = 0 \quad \Rightarrow \quad z = c_1.$$

From equations (2) and (3), we have

$$qdp = pdq \quad \Rightarrow \quad p = c_2 q.$$

Then, the first integral becomes

$$p = qf(z)$$

where the function f is arbitrary. This is a Lagrange's equation and the auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}$$

with integrals $z = c_1$, $y + xf(c_1) = c_2$ leading to the general solution

$$y + xf(z) = g(z)$$

where the functions f and g are arbitrary. ■

Check Your Progress

1. Solve the wave equation $r = t$ by Monge's method.
2. Show that if a function z satisfies the differential equation $\frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y}\frac{\partial z}{\partial x}$ it is of the form $f\{x + g(y)\}$, where the functions f and g are arbitrary.

3. Solve the equation $z(qs - pt) = pq^2$.

4. Solve the equation $pq = x(ps - qr)$.

5. Solve the equation $rq^2 - 2pqs + tp^2 = pt - qs$.

6. Find an integral of the equation $z^2\ rt - s^2 + z\ 1 + q^2\ r - 2pqzs + z\ \ 1 + p^2\ t + 1 + p^2 + q^2 = 0$ involving three arbitrary constants.

Let us Sum up:

In this unit, the students acquired knowledge to

- solve the PDE's by using Laplace transform techniques.
- find Fourier Transform Technique.
- find Finite Fourier Transform.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.

2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-III

UNIT 5

BOUNDARY VALUE PROBLEMS

Structure

Objective

Overview

Overview

In this unit, we discuss the elementary solutions of Laplace equation, necessary conditions for a surface to be equipotential, boundary value problems for Laplace equation.

5.1 Elementary Solutions of Laplace's Equation

In this section, we investigate the elementary solution of Laplace equation.

Problem 5.1.1. Prove that $\psi = \dfrac{q}{|\mathbf{r} - \mathbf{r}'|}$ is a solution of the Laplace equation.

Solution. Consider a function ψ of the form

$$\psi = \frac{q}{|\mathbf{r} - \mathbf{r}'|} = \frac{q}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}, \tag{1}$$

where q is a constant and (x', y', z') are the coordinates of a *fixed* point, then since

$$\frac{\partial \psi}{\partial x} = -\frac{q(x - x')}{|\mathbf{r} - \mathbf{r}'|^3}, \text{ etc.}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(x - x')^2}{|\mathbf{r} - \mathbf{r}'|^5},$$

$$\frac{\partial^2 \psi}{\partial y^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(y - y')^2}{|\mathbf{r} - \mathbf{r}'|^5},$$

$$\frac{\partial^2 \psi}{\partial z^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(z - z')^2}{|\mathbf{r} - \mathbf{r}'|^5}.$$

Adding the last three equations, we get

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} &= -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(x - x')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(y - y')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(z - z')^2}{|\mathbf{r} - \mathbf{r}'|^5} \\
&= -\frac{3q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]}{|\mathbf{r} - \mathbf{r}'|^5} \\
&= -\frac{3q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q|\mathbf{r} - \mathbf{r}'|^2}{|\mathbf{r} - \mathbf{r}'|^5} \\
&= -\frac{3q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q}{|\mathbf{r} - \mathbf{r}'|^3} \\
\nabla^2 \psi &= 0
\end{aligned}$$

showing that the function (1) is a solution of Laplace's equation except possibly at the point (x', y', z'), where it is not defined. ■

If S is any sphere with center (x', y', z'), then

$$\int_S \frac{\partial \psi}{\partial n}\, dS = -4\pi q.$$

By Gauss' theorem, that equation (1) gives the solution of Laplace's equation corresponding to an electric charge $+q$.

By a superposition principle, we have

$$\psi = \sum_{i=1} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} \tag{2}$$

is the solution of Laplace's equation corresponding to n charges q_i situated at points with position vectors $\mathbf{r}_i$ $(i = 1, 2, \ldots, n)$.

In electrical problems, we encounter the situation where two charges $+q$ and $-q$ are situated very close together, say at points $\mathbf{r}'$ and $\mathbf{r}'+\delta\mathbf{r}'$, where $\delta\mathbf{r}' = (l, m, n)\, a$. The solution of Laplace's equation corresponding to this distribution of charge is

$$\psi = \frac{-q}{|\mathbf{r} - \mathbf{r}'|} + \frac{q}{|\mathbf{r} - \mathbf{r}' + \delta\mathbf{r}'|}.$$

Now

$$\frac{1}{|\mathbf{r} - \mathbf{r}' - \delta\mathbf{r}'|} = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{l(x - x') + m(y - y') + n(z - z')}{|\mathbf{r} - \mathbf{r}'|^3}\, a + O(a^2)$$

so that if $a \to 0$, $q \to \infty$ in such a way that $qa \to \mu$, i.e., an electric dipole is formed, it follows that the corresponding solution of Laplace's equation is

$$\psi = \mu\, \frac{l(x - x') + m(y - y') + n(z - z')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{1}$$

a result which may be written in other ways: If we introduce a vector $\mathbf{m} = \mu(l, m, n)$, then

$$\psi = \frac{\mathbf{m} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2}$$

Also since

$$\frac{\partial}{\partial x'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3}, \text{ etc.}$$

it follows that (1) may be written in the form

$$\psi = (\mathbf{m} \cdot \operatorname{grad}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \mu \left(l \frac{\partial}{\partial x'} + m \frac{\partial}{\partial y'} + n \frac{\partial}{\partial z'} \right) \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{3}$$

The corresponding form of the function ψ is

$$\psi = \int_V \frac{dq}{|\mathbf{r} - \mathbf{r}'|}, \tag{4}$$

where q is the Stieltjes measure of the charge at the point $\mathbf{r}'$, or if ρ denotes the charge density, by

$$\psi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\, d\tau'}{|\mathbf{r} - \mathbf{r}'|}. \tag{5}$$

By a similar argument it can be shown that the solution corresponding to a surface S carrying an electric charge of density σ is

$$\psi(\mathbf{r}) = \int_S \frac{\sigma(\mathbf{r}')dS'}{|\mathbf{r} - \mathbf{r}'|}. \tag{6}$$

Problem 5.1.2. If $\rho > 0$ and $\psi(r)$ is given by equation (5), where the volume V is bounded, prove that

$$\lim_{r=\infty} r\, \psi(\mathbf{r}) = M$$

where

$$M = \int_V \rho(\mathbf{r}')\, d\tau'.$$

Solution. Let r_1, r_2 be the maximum and minimum values of the distance $|\mathbf{r} - \mathbf{r}'|$ from the point $\mathbf{r}$ to the integration points $\mathbf{r}'$ of the bounded volume V. Then by a theorem of elementary calculus

$$\frac{M}{r_1} < \int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r} - \mathbf{r}'|} < \frac{M}{r_2}$$

an equality which may be written in the form

$$\left(\frac{r}{r_1}\right) M < r\,\phi(\mathbf{r}) < \left(\frac{r}{r_2}\right) M.$$

Now as $r \to \infty$, $\frac{r}{r_1}$ and $\frac{r}{r_2}$ both tend to unity, so that

$$\lim_{r=\infty} r\,\psi(\mathbf{r}) = M. \qquad \blacksquare$$

Check Your Progress

1. Prove that $r\cos\theta$ and $r^{-2}\cos\theta$ satisfy Laplace's equation, when r, θ, φ are spherical polar coordinates.

   An electric dipole of moment μ is placed at the center of a uniform hollow conducting sphere of radius a which is insulated and has a total charge e. Verify that V_i, the potential inside the sphere, and V_0, the potential outside the sphere, are given by

   $$V_i = \frac{e}{a} + \frac{\mu\cos\theta}{r^2} - \frac{\mu r}{a^3}\cos\theta, \qquad V_0 = \frac{e}{r},$$

   where r is measured from the center of the sphere and θ is the angle between the radius vector and the positive direction of the dipole.

2. A surface S carries an electrical charge of density σ. In the negative direction of the normal from each point P of S there is located a point P_1 at a constant distance h, thus forming a parallel surface S_1. Assuming that corresponding points P and P_1 have the same normal and that corresponding elements of area carry numerically equal charges of opposite sign, show that the potential function of the system is

   $$\psi = \int_S \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'+h\mathbf{n}|} \right) \sigma(\mathbf{r}')\,dS'.$$

   By letting $h \to 0$, $p \to \infty$ in such a way that $\sigma h \to \mu$ everywhere uniformly on S, obtain the expression

$$\psi = \int_S \frac{\mu\{\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')\}}{|\mathbf{r}-\mathbf{r}'|^3}\, dS'$$

for the potential of an electrical double layer.

3. A closed equipotential surface S contains matter which can be represented by a volume density σ. By substituting $\psi' = |\mathbf{r}-\mathbf{r}'|^{-1}$ in Green's theorem

$$\int_S \left(\psi'\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi'}{\partial n}\right) dS = \int_V (\psi'\nabla^2\psi - \psi\nabla^2\psi')d\tau$$

prove that

$$\int_S \left(\frac{\partial\psi}{\partial n}\right)\frac{dS'}{|\mathbf{r}-\mathbf{r}'|} + 4\pi\int_V \frac{\rho(r')d\tau'}{|\mathbf{r}-\mathbf{r}'|} = 0.$$

Deduce that the matter contained within any closed equipotential surface S can be thought of as spread over the surface S with surface density

$$-\frac{1}{4\pi}\frac{\partial\psi}{\partial n}$$

at any point.

4. By applying Green's theorem in the above form to the region between an equipotential surface S and the infinite sphere with $\psi' = |\mathbf{r}-\mathbf{r}'|^{-1}$ and ψ the potential of the whole distribution of matter, prove that the potential inside S due to the joint effects of Green's equivalent layer and the original matter outside S is the constant potential of S.

5. Show that

$$\int_V \frac{d\tau'}{|\mathbf{r}-\mathbf{r}'|} \le 2\pi\left(\frac{3V}{4\pi}\right)^{\frac{1}{2}}$$

irrespective of whether the point with position vector $\mathbf{r}$ is inside or outside the volume V or on the surface bounding it.

6. Prove that the potential

$$\psi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|}$$

and its first derivatives are continuous when the point P with position vector $\mathbf{r}$ lies inside or on the boundary of V.

Show further that $\nabla^2\psi = -4\pi\rho$ if $P \in V$ and that $\nabla^2\psi = 0$ if $P \notin V$.

5.2 Families of Equipotential Surfaces

If the function $\psi(x, y, z)$ is a solution of Laplace's equation, the one-parameter system of surfaces

$$\psi(x, y, z) = c$$

is called a family of *equipotential* surfaces. It is not true, however, that any one-parameter family of surfaces

$$f(x, y, z) = c \tag{1}$$

is a family of equipotential surfaces.

Theorem 5.2.1. *The necessary condition for the surface $f(x, y, z) = c$ to be equipotential if $\dfrac{\nabla^2 f}{|grad\ f|^2}$ is a function of f alone.*

Proof. The surfaces (1) will be equipotential if the potential function ψ is constant whenever $f(x, y, z)$ is constant. A functional relation must be of the form

$$\psi = F\{f(x, y, z)\} \tag{2}$$

between the functions ψ and f. Differentiating equation (2) partially with respect to x, we obtain the result

$$\frac{\partial\psi}{\partial x} = \frac{dF}{df}\frac{\partial f}{\partial x} \tag{3}$$

and hence the relation

$$\frac{\partial^2\psi}{\partial x^2} = \frac{d^2F}{df^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{dF}{df}\frac{\partial^2 f}{\partial x^2} \tag{4}$$

from which it follows that

$$\nabla^2\psi = F''(f)(\text{grad}\, f)^2 + F'(f)\nabla^2 f. \tag{5}$$

Now, in free space, $\nabla^2\psi = 0$, so that the required necessary condition is that

$$\frac{\nabla^2 f}{(\text{grad}\ f)^2} = -\frac{F''(f)}{F'(f)} = \chi(f). \tag{6}$$

Hence the condition that the surfaces (1) form a family of equipotential surfaces in free space is that the quantity

$$\chi(f) = \frac{\nabla^2 f}{|\text{grad}\, f|^2}$$

is a function of f alone. ■

Problem 5.2.1. Derive the general form of potential function.

Solution. From (6), we have

$$\frac{d^2F}{df^2} + \chi(f)\frac{dF}{df} = 0$$

from which it follows that

$$\frac{dF}{df} = Ae^{-\int \chi(f)df},$$

where A is a constant, and hence that

$$\psi = A\int e^{-\int \chi(f)df}df + B, \tag{7}$$

where A and B are constants. ■

Problem 5.2.2. Show that the surfaces

$$x^2 + y^2 + z^2 = cx^{\frac{2}{3}}$$

can form a family of equipotential surfaces, and find the general form of the corresponding potential function.

Solution. Given

$$x^2 + y^2 + z^2 = cx^{\frac{2}{3}}$$

can be written as

$$f = x^{\frac{4}{3}} + x^{-\frac{2}{3}}(y^2 + z^2)$$

so that

$$grad\, f = \frac{2}{3} x^{-\frac{2}{3}} (2x^2 - y^2 - z^2, 3xy, 3xz).$$

Hence

$$\nabla^2 f = \frac{10}{9} x^{-\frac{2}{3}} (4x^2 + y^2 + z^2)$$

and

$$|grad\, f|^2 = \frac{4}{9} x^{-\frac{10}{3}} (4x^2 + y^2 + z^2)(x^2 + y^2 + z^2)$$

Now

$$
\begin{aligned}
\chi(f) &= \frac{\nabla^2 f}{|\text{grad } f|^2} \\
&= \frac{\frac{10}{9} x^{-\frac{2}{3}}(4x^2 + y^2 + z^2)}{\frac{4}{9} x^{-\frac{10}{3}}(4x^2 + y^2 + z^2)(x^2 + y^2 + z^2)} \\
&= \frac{5}{2f}
\end{aligned}
$$

which is a function of f alone. The given set of surfaces therefore forms a family of equipotential surfaces.

General form of potential function

$$
\begin{aligned}
\psi &= A \int e^{-\int \chi(f) df} df + B \\
&= A \int e^{-\int \frac{5}{2f} df} df + B \\
&= A \int e^{-\frac{5}{2} \log f} df + B \\
&= A \int e^{\log f^{-\frac{5}{2}}} df + B \\
&= A \int f^{-\frac{5}{2}} df + B
\end{aligned}
$$

$$\psi = A f^{-\frac{3}{2}} + B$$

from which it follows that the required potential function is

$$\psi = Ax(x^2 + y^2 + z^2)^{-\frac{3}{2}} + B$$

where A and B are constants. ■

Check Your Progress

1. Show that the surfaces

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = c$$

can form a family of equipotential surfaces, and find the general form of the corresponding potential function.

2. Show that the family of right circular cones

$$x^2 + y^2 = cz^2,$$

where c is a parameter, forms a set of equipotential surfaces, and show that the corresponding potential function is of the form $A \log \tan \frac{1}{2}\theta + B$, where A and B are constants and θ is the usual polar angle.

3. Show that if the curves $f(xy) = c$ form a system of equipotential lines in free space for a two-dimensional system, the surfaces formed by their revolution about the x axis do not constitute a system of equipotential surfaces in free space unless

$$\left(\frac{1}{y}\frac{\partial f}{\partial y}\right) \div \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$$

is a constant or a function of c only.

Show that the cylinders $x^2 + y^2 = 2cx$ for a possible set of equipotential surfaces in free space but that the spheres $x^2 + y^2 = 2cx$ for a possible set of equipotential surfaces in free space but that the spheres $x^2 + y^2 + z^2 = 2cx$ do not.

4. Show that the surfaces

$$x^2 + y^2 - 2cx + a^2 = 0,$$

where a is fixed and c is a parameter specifying a particular surface of the family, form a set of equipotential surfaces.

The cylinder of parameter c_1 completely surrounds that of parameter c_2, and $c_8 > a > 0$. The first is grounded, and the second carries a charge E per unit length. Prove that its potential is

$$E \log \frac{(c_1 + a)(c_2 - a)}{(c_1 - a)(c_2 + a)}.$$

5.3 Boundary Value Problems

In addition to satisfying Laplace's equation within a certain region of space V, also satisfy certain conditions on the boundary S of this region. Any problem in which we are required to find such a function ψ is called a *boundary value problem for Laplace's equation*.

There are three main types of boundary value problem for Laplace's equation:

Interior Dirichlet Problem

If f is a continuous function prescribed on the boundary S of some finite region V, determine a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within V and $\psi = f$ on S.

Exterior Dirichlet Problem

If f is a continuous function prescribed on the boundary S of a finite simply connected region V, determine a function $\psi(x, y, z)$ which satisfies $\nabla^2\psi = 0$ outside V and is such that $\psi = f$ on S.

Interior Neumann Problem

If f is a continuous function which is defined uniquely at each point of the boundary S of a finite region V, determine a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within V and its normal derivative $\frac{\partial\psi}{\partial n}$ coincides with f at every point of S.

Exterior Neumann Problem

If f is a continuous function prescribed at each point of the (smooth) boundary S of a bounded simply connected region V, find a function $\psi(x, y, z)$ satisfying $\nabla^2\psi = 0$ outside V and $\frac{\partial\psi}{\partial n} = f$ on S.

Churchill problem

If f is a continuous function prescribed on the boundary S of a finite region V, determine a function $\psi(x, y, z)$ such that $\nabla^2 \psi = 0$ within V and

$$\frac{\partial\psi}{\partial n} + (k+1)\psi = f$$

at every point of S.

Let us Sum up:

In this unit, the students acquired knowledge to

- find the elementary solution of Laplace's Equation.
- understand the basic concepts of Families of Equipotential Surfaces.
- classify the various types of boundary value problems.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-III

UNIT 6

LAPLACE EQUATIONS

Structure

Overview

In this unit, we discuss the solutions of Laplace equation in spherical coordinates, cylindrical and rectangular Cartesian coordinates using separation of variables and also finding solution of Laplace equation using Green's function.

6.1 Separation of Variables

6.1.1 Solution of Laplace equation in spherical coordinates

The Laplace equation in spherical coordinates is given by

$$\nabla^2\psi = \frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial \varphi^2} = 0. \tag{1}$$

Let us assume the solution of the form

$$\psi(r, \theta, \varphi) = R(r)F(\theta, \varphi). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$F\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 F}{\partial \varphi^2} = 0$$

$$\Rightarrow \frac{\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}{R} = \frac{-\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\varphi^2}\right\}}{F} = -\lambda.$$

where λ is a separation constant. This leads to

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\lambda$$

$$\frac{1}{F\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\varphi^2}\right] = \lambda.$$

yields

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda R = 0 \tag{3}$$

which is a Euler's equation. Hence using the transformation $r = e^z$, and for $\lambda = -n(n+1)$, we have

$$R = c_1 r^n + \frac{c_2}{r^{n+1}}. \tag{4}$$

For $\lambda = -n(n+1)$,
$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\varphi^2} + n(n+1)F\sin\theta = 0. \tag{5}$$
Let the solution of equation (5) be
$$F = \Theta(\theta)\Phi(\varphi) \tag{6}$$
which gives
$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin\theta H\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2 \tag{7}$$
where m^2 is another separation constant. Then
$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0, \tag{8}$$
$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin\theta H\right] = m^2. \tag{9}$$

The solution of equation (8) is
$$\Phi = c_3\cos(m\varphi) + c_4\sin(m\varphi) \tag{10}$$
provided $m \neq 0$

If $m = 0$, then the solution is independent of φ which corresponds to the axisymmetric case.

Putting $\cos\theta = \mu$ in equation (9), we obtain
$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0 \tag{11}$$
which is the well-known Legendre equation. Its general solution is given by
$$\Theta(\mu) = c_5P_n^m(\mu) + c_6Q_\alpha(\mu), \qquad -1 \le \mu \le 1 \tag{12}$$
or
$$\Theta(\theta) = c_5P_n^m(\cos\theta) + c_6Q_\alpha(\cos\theta), \qquad -1 \le \cos\theta \le 1 \tag{13}$$
where P_n^m, Q_n^m are associated Legendre functions of the first and second kind respectively.

The continuity of $\Theta(\theta)$ at $\theta = 0, \pi$ implies the continuity of $\Theta(\theta)$ at $\mu = \pm 1$. Since $Q_n^m(\mu)$ has a singularity at $\mu = 1$, we choose $c_6 = 0$. Therefore, the solution of Laplace equation in spherical coordinates is given by

$$\psi(r,\theta,\varphi) = \left(c_1 r^n + \frac{c_2}{r^{n+1}}\right)\left(c_3\cos(m\varphi) + c_4\sin(m\varphi)\right) c_5 P_n^m(\cos\theta). \tag{14}$$

In the antisymmetric case $(m = 0)$, then the solution is

$$\psi(r,\theta,\varphi) = \left(c_1 r^n + \frac{c_2}{r^{n+1}}\right) c_3 c_5 P_n(\cos\theta). \tag{15}$$

By the principle of superposition, we get

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \tag{16}$$

which is the required solution.

In the general case in which $m \neq 0$ we find that when $0 \leq m \leq n$, equation (11) possesses solutions of the type

$$P_n^m(\mu) = (\mu^2 - 1)^{\frac{1}{2}m}\frac{d^m P_n(\mu)}{d\mu^m} \tag{17}$$

$$Q_n^m(\mu) = (\mu^2 - 1)^{\frac{1}{2}m}\frac{d^m Q_n(\mu)}{d\mu^m}. \tag{18}$$

When $\mu = \pm 1$, $Q_n^m(\mu)$ is infinite, so that in any physical problem in which it is known that Θ, i.e., ψ, does not become infinite on the polar axis we take $P_n^m(\mu)$ to be the solution of equation (11). In this way we obtain solutions of Laplace's equation (1) of the form

$$\psi = \sum_{n=0}^{\infty}\sum_{m<n}\left(A_{mn} r^n + B_{mn} r^{-n-1}\right) P_n^m(\cos\theta)\, e^{\pm im\varphi} \tag{19}$$

which may be written as

$$\psi = \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\left[A_n P_n(\cos\theta) + \sum_{m=1}^{n}\left(A_{nm}\cos m\varphi + B_{nm}\sin m\varphi\right) P_n^m(\cos\theta)\right]. \tag{20}$$

Problem 6.1.1. A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the disturbed stream.

Solution. Take the polar axis Oz to be in the direction of the given velocity and take polar coordinates (r, θ, φ) with origin at the center of the fixed sphere.

The velocity of the fluid is given by the vector $\mathbf{q} = -\text{grad}\psi,$ where

(i) $\nabla^2\psi = 0$

(ii) $\dfrac{\partial \psi}{\partial r} = 0$ on $r = a$

(iii) $\psi \sim -Vr\cos\theta = -VrP_1(\cos\theta)$ as $r \to \infty.$

The axially symmetrical function

$$\psi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta) \tag{1}$$

satisfies (i).

Differetiating (1) partially with respect to $r,$ we get

$$\frac{\partial \psi}{\partial r} = \sum_{n=0}^{\infty} \left(A_n n r^{n-1} - (n+1)\frac{B_n}{r^{n+2}} \right) P_n(\cos\theta). \tag{2}$$

Applying condition (ii) in (2), put $r = a$

$$0 = \sum_{n=0}^{\infty} \left(A_n n a^{n-1} - (n+1)\frac{B_n}{a^{n+2}} \right) P_n(\cos\theta)$$

$$\Rightarrow \quad A_n n a^{n-1} - (n+1)\frac{B_n}{a^{n+2}} = 0$$

$$\Rightarrow \quad B_n = \frac{na^{2n+1}A_n}{(n+1)}.$$

Equation (1) becomes

$$\psi = \sum_{n=0} \left(A_n r^n + \frac{na^{2n+1}A_n}{(n+1)r^{n+1}} \right) P_n(\cos\theta) \tag{3}$$

As $r \to \infty$, this velocity potential has the asymptotic form

$$\psi \sim \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$$-VrP_1(\cos\theta) \sim A_0P_0(\cos\theta) + A_1rP_1(\cos\theta) + +A_2r^2P_2(\cos\theta) + \cdots.$$

Comparing the coefficients of like powers of r on both sides, we obtain

$$A_0 = 0, A_1 = -V, A_2 = 0, A_3 = 0, \ldots.$$

Hence the required velocity potential is

$$\psi = -V\left(r + \frac{a^3}{2r^2}\right)\cos\theta.$$

The components of the velocity are therefore

$$q_r = -\frac{\partial\psi}{\partial r} = V\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -V\left(1 + \frac{a^3}{2r^3}\right)\sin\theta. \qquad \blacksquare$$

Problem 6.1.2. A uniform insulated sphere of dielectric constant κ and radius a carries on its surface a charge of density $\lambda P_n(\cos\theta)$. Prove that the interior of the sphere contributes an amount

$$\frac{8\pi^2\lambda^2 a^3 \kappa n}{(2n+1)(\kappa n+n+1)^2}$$

to the electrostatic energy.

Solution. The electrostatic potential ψ takes the value ψ_1 inside the sphere and ψ_2 outside, where by virtue of Sec. 1 (d) we have:

(i) $\nabla^2\psi_1 = 0,\ \nabla^2\psi_2 = 0$

(ii) ψ_1 is finite at $r = 0$; $\psi_2 \to 0$ as $r \to \infty$;

(iii) $\psi_1 = \psi_2$ and $\kappa\left(\dfrac{\partial\psi_1}{\partial r}\right) - \dfrac{\partial\psi_2}{\partial r} = 4\pi\lambda P_n(\cos\theta)$ on $r = a$.

Conditions (i), (ii) and the first of (iii) and the condition of axial symmetry are satisfied if

$$\psi_1 = A\left(\frac{r}{a}\right)^n P_n(\cos\theta), \qquad \psi_2 = A\left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta)$$

and the second of (iii) is satisfied if we choose A so that

$$\left[\frac{n\kappa}{a} + \frac{(n+1)}{a}\right] A = 4m\lambda.$$

Hence the required potential function is

$$\psi_1 = \frac{4\pi a\lambda}{\kappa n+n+1}\left(\frac{r}{a}\right)^n P_n(\cos\theta).$$

The energy due to the interior of the sphere is known from electrostatic theory to be

$$\frac{\kappa}{8\pi}\int \psi_1 \left(\frac{\partial \psi_1}{\partial n}\right) dS = \frac{\kappa}{8\pi}\frac{16\pi^2 a^2 \lambda^2}{(\kappa n + n + 1)^2}\frac{n}{a}2\pi a^2 \int_0^{\pi} \sin\theta P_n(\cos\theta)\, P_n(\cos\theta)\, d\theta.$$

Since

$$\int_{-1}^{1} \{P_n(\mu)\}^2\, d\mu = \frac{2}{2n+1}$$

Then the energy becomes

$$\begin{aligned}\frac{\kappa}{8\pi}\int \psi_1 \left(\frac{\partial \psi_1}{\partial n}\right) dS &= \frac{\kappa}{8\pi}\frac{16\pi^2 a^2 \lambda^2}{(\kappa n + n + 1)^2}\frac{n}{a}2\pi a^2 \frac{2}{2n+1} \\ &= \frac{8\pi^2 \lambda^2 a^3 \kappa n}{(2n+1)(\kappa n + n + 1)^2}.\end{aligned}$$

which is the electrostatic energy in the interior of the sphere. ■

6.1.2 Solution of Laplace equation in Cylindrical coordinates

The Laplace equation in cylindrical coordinates is given by

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \tag{1}$$

Let us assume the solution of the form

$$\psi(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$\frac{\partial^2 R}{\partial \rho^2}\Phi Z + \frac{1}{\rho}\frac{\partial R}{\partial \rho}\Phi Z + \frac{1}{\rho^2}\frac{d^2 \Phi}{d\varphi^2}RZ + \frac{d^2 Z}{dz^2}R\Theta = 0 \tag{3}$$

or

$$\frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho R}\frac{\partial R}{\partial \rho} + \frac{1}{\rho^2 \Phi}\frac{d^2\Phi}{d\varphi^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2} = k \text{ (say)} \tag{4}$$

where k is a separation constant. Therefore,

$$\frac{d^2 Z}{dz^2} + kZ = 0 \tag{5}$$

and

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} - k = 0. \tag{6}$$

We have the following three cases:

If $k > 0$, then $k = \lambda^2$ and the solution is

$$Z = c_1 \cos \lambda z + c_2 \sin \lambda z. \tag{7}$$

If $k < 0$, then $k = -\lambda^2$ and the solution of equation (5) is

$$Z = c_1 e^{\lambda z} + c_2 e^{-\lambda z}. \tag{8}$$

If $k = 0$, then the solution of equation (5) is

$$Z = c_1 z + c_2. \tag{9}$$

Under the physical suituation, the only acceptable solution is

$$Z = c_1 e^{\lambda z} + c_2 e^{-\lambda z}. \tag{10}$$

Equation (6) becomes

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + m^2 r^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = k' \text{ (say)}. \tag{11}$$

Since the solution to be periodic in φ, which can be obtained when k' is positive and we take

$k^J = n^2$. Therefore, the acceptable solution will be

$$\Phi = c_3 \cos n\varphi + c_4 \sin n\varphi. \tag{12}$$

If $k^J = n^2$, then the equation (11) becomes

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (m^2 r^2 - n^2)R = 0 \tag{13}$$

which is a Bessel's equation and its general solution is

$$R = A_{mn} J_n(m\rho) + B_{mn} Y_n(m\rho). \tag{14}$$

where $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the n th order Bessel functions of first and second kind, respectively and A_{mn} and B_{mn} are constants. The function $Y_n(mp)$ becomes infinite as $\rho \to 0$, so that if we are interested in problems in which it is obvious on physical grounds that ψ remains finite along the line $\rho = 0$, we must take $B_{mn} = 0$. In this way we obtain a solution of the type

$$\psi = \sum_m \sum_n A_{mn}\, J_n(m\rho)\, e^{\pm mz \pm in\varphi}. \tag{15}$$

For problems in which there is symmetry about the z axis we may take $n = 0$ to obtain solutions of the form

$$\psi = \sum_m A_m J_0(m\rho)\, e^{\pm mz}. \tag{16}$$

In particular if we wish a solution which is symmetrical about Oz and tends to zero as $\rho \to 0$ and as $z \to \infty$, we must take it in the form

$$\psi = \sum_m A_m J_0(m\rho)\, e^{-mz}. \tag{17}$$

Problem 6.1.3. Find the potential function $\psi(p, z)$ in the region $0 \leq p \leq 1$, $z \geq 0$ satisfying the conditions

(i) $\psi \to 0$ as $z \to \infty$

(ii) $\psi = 0$ on $\rho = 1$

(iii) $\psi = f(\rho)$ on $z = 0$ for $0 \le \rho \le 1$.

Solution. The conditions (i) and (ii) are satisfied if we take a function of the form

$$\psi(\rho, z) = \sum_s A_s J_0(\lambda_s \rho)\, e^{-\lambda_s z}, \tag{18}$$

where λ_s is a root of the equation

$$J_0(\lambda) = 0.$$

Now it is a well-known result of the theory of Bessel functions that we can write

$$f(\rho) = \sum_s A_s J_0(\lambda_s \rho)$$

where

$$A_s = \frac{2}{[J_1(\lambda_s)]^2} \int_0^1 \rho\, f(\rho)\, J_0(\lambda_s \rho)\, d\rho. \tag{19}$$

Hence the desired solution is (18), with A_s given by the formula (19). ■

6.1.3 Solution of Laplace Equation in Rectangular Cartesian Coordinates

The Laplace equation in rectangular cartesian coordinatesis given by

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{1}$$

By the variables separable method, let us assume the solution in the form

$$u(x, y, z) = X(x)Y(y)Z(z). \tag{2}$$

Substituting equation (2) into the Laplace equation (1), we get

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

which can also be written as

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \lambda_1^2$$

where λ_1 is a separation constant. Thus we have

$$X''(x) + \lambda_1^2 X(x) = 0. \tag{3}$$

After the second separation, we also have

$$\frac{Z''(z)}{Z(z)} - \lambda_1^2 = \frac{Y''(y)}{Y(y)} = \lambda_2^2$$

$$Y''(y) + \lambda_2^2 Y(y) = 0 \tag{4}$$

$$Z''(x) - \lambda_3^2 Z(z) = 0 \tag{5}$$

where $\lambda_3^2 = \lambda_1^2 + \lambda_2^2$. The general solution of equations (3), (4) and (5) are

$$X(x) = c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x \tag{6}$$

$$Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y \tag{7}$$

$$Z(z) = c_5 \cosh \lambda_5 z + c_6 \sinh \lambda_6 z. \tag{8}$$

Then the solution becomes

$$u(x, y, z) = (c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x)(c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y)(c_5 \cosh \lambda_5 z + c_6 \sinh \lambda_6 z).$$

Problem 6.1.4. Find the potential function $u(x, y, z)$ in the region $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ satisfying the conditions

(i) $u = 0$ on $x = 0, x = a, y = 0, y = b, z = 0$

(ii) $u = f(x, y)$ on $z = c$, $0 \le x \le a$, $0 \le \gamma \le b$.

Solution. The potential distribution in the rectangular box satisfies the Laplace equation.

The problem is described by

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} u(0, y, z) &= u(a, y, z) = 0 \\ u(x, 0, z) &= u(x, b, z) = 0 \\ u(x, y, 0) &= 0 \\ u(x, y, c) &= f(x, y). \end{aligned}$$

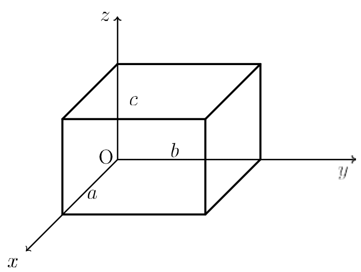


Figure 6.1.1: Boundary Conditions

The most suitable solution for the given problem is

$$u(x, y, z) = X(x)Y(y)Z(z),$$

where

$$X(x) = c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x \tag{1}$$

$$Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y \tag{2}$$

$$Z(z) = c_5 \cosh \lambda_5 z + c_6 \sinh \lambda_6 z. \tag{3}$$

From the boundary conditions, we have

$$\begin{aligned} X(0) &= X(a) = 0 \\ Y(0) &= Y(b) = 0 \\ Z(0) &= 0 \end{aligned}$$

Applying the boundary conditions $X(0) = 0$ and $X(a) = 0$ in (1), we get $c_1 = 0$ and $\lambda_1 a = m\pi$ implies $\lambda_1 = \dfrac{m\pi}{a}$, $m = 1, 2, \ldots.$

Applying the boundary conditions $Y(0) = 0$ and $Y(b) = 0$ in (2), we get $c_3 = 0$ and $\lambda_2 b = n\pi$ implies $\lambda_2 = \dfrac{n\pi}{b}$, $n = 1, 2, \ldots.$

Applying the boundary conditions $Z(0) = 0$ in (3), we get $c_5 = 0$.

Further, we note that

$$\lambda_3^2 = \lambda_1^2 + \lambda_2^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = \lambda_{mn}^2 \quad \text{say.}$$

Then

$$\lambda_3 = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \lambda_{mn}.$$

The solutions now take the form

$$\begin{aligned} X(x) &= c_{2m} \sin \frac{m\pi x}{a}, \qquad m = 1, 2, \ldots \\ Y(y) &= c_{4n} \sin \frac{n\pi y}{b}, \qquad n = 1, 2, \ldots \\ Z(z) &= c_{6mn} \sinh \lambda_{mn} z. \end{aligned}$$

Let $c_{mn} = c_{2m}c_{4n}c_{6mn}$, then, after using the principle of superposition, the required solution is

$$u(x, y, z) = X(x)Y(y)Z(z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \lambda_{mn} z. \tag{4}$$

Applying the last boundary condition $u(x, y, c) = f(x, y)$ in (4), we get

$$f(x, y) = \sum \sum c_{mn} \sinh \lambda_{mn} c \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

which is a double Fourier sine series, where

$$c_{mn} \sinh \lambda_{mn} c = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy. \tag{5}$$

Therefore, equation (4) along with the constants c_{mn} given in (5) constitute the required potential. ■

Check Your Progress

1. If ψ is a harmonic function which is zero on the cone $\theta = \alpha$ and takes the value $\sum a_n r^n$ on the cone $\theta = \beta$, show that when $\alpha < \theta < \beta$,

$$\psi = \sum_{n=0}^{\infty} a_n \left(\frac{Q_n(\cos\alpha)P_n(\cos\theta) - P_n(\cos\alpha)Q_n(\cos\theta)}{Q_n(\cos\alpha)P_n(\cos\beta) - P_n(\cos\alpha)Q_n(\cos\beta)} \right) r^n.$$

2. A small magnet of moment $\mathbf{m}$ lies at the center of a spherical hollow of radius a in medium of uniform permeability μ. Show that the magnetic field in this medium is the same as that produced by a magnet of moment $3\mathbf{m}/(1+2\mu)$ lying at the center of the hollow.

   Determine the field in the hollow.

3. A grounded nearly spherical conductor whose surface has the equation

$$r = a\left[1 + \sum_{n=z} \varepsilon_n P_n(\cos\theta)\right]$$

   is placed in a uniform electric field E which is parallel to the axis of symmetry of the conductor. Show that if the squares and products of the ε's can be neglected, the potential is given by

$$V = Ea\left[\left(1 + \frac{6}{5}\varepsilon_2\right)\left(\frac{a}{r}\right)^2 - \frac{r}{a}\right]P_1 + 3\sum_{n=2}\left(\frac{n}{2n-1}\varepsilon_{n-1} + \frac{n+1}{2n+3}\varepsilon_{n+1}\right)\left(\frac{a}{r}\right)^{n+1} P_n\, s_1 = 0.$$

4. Heat flows in a semi-infinite rectangular plate, the end $x = 0$ being kept at temperature θ_0 and the long edges $y = 0$ and $y = a$ at zero temperature. Prove that the temperature at a point (x, y) is

$$\frac{4\theta_0}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\frac{(2m+1)\pi y}{a}\, e^{-(2m+1)\pi x/a}.$$

5. V is a function of r and θ satisfying the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a, r = b, \theta = 0, \theta = \frac{1}{2}\pi$. Its value along the boundary $r = a$ is $\theta(\frac{1}{2}\pi - \theta)$, and its value along the other boundaries is zero. Prove that

$$V = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(r/b)^{4n-2} - (b/r)^{4n-2}}{(a/b)^{4n-2} - (b/a)^{4n-2}}\, \frac{\sin(4n-2)\theta}{(2n-1)^3}.$$

6.2 Problems with Axial Symmetry

The determination of a potential function ψ for a system which has axis of symmetry to be the polar axis $\theta = 0$. Suppose that we wish to determine the potential function (r, θ, φ) corresponding to a given distribution of sources (such as masses, charges, etc.) and we have to calculate its value $\psi(z, 0, 0)$ at a point on the axis of symmetry. If we expand $\psi(z, 0, 0)$ in the Laurent series

$$\psi(z, 0, 0) = \sum_{n=0}^{\infty} \left(A_n z^n + \frac{B_n}{z^{n+1}}\right), \tag{1}$$

then it is readily shown that the required potential function is

$$\psi(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \tag{2}$$

for

(i) $\nabla^2\psi = 0$;

(ii) $\psi(r, \theta, \varphi)$ takes the value (1) on the axis of symmetry, since there $P_n(\cos\theta) = 1$, $r = z$;

(iii) $\psi(r, \theta, \varphi)$ is symmetrical about Oz as required.

The determination of the potential due to a uniform circular wire of radius a charged with electricity of line density e. At a point on the axis of the wire it is readily seen that

$$\psi(z, 0, 0) = \frac{2\pi e a}{\sqrt{a^2 + z^2}}$$

so that

$$\psi(z, 0, 0) = \begin{cases} 2\pi e \displaystyle\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} \left(-\frac{z^2}{a^2}\right)^n & z < a \\ 2\pi e \displaystyle\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-1)^n \left(\frac{a}{z}\right)^{2n+1} & z > a, \end{cases}$$

where we have used the notation $(a)_n = a(a+1)\cdots(a+n-1)$. Hence at a general point we have

$$\psi(r, \theta) = \begin{cases} 2\pi e \displaystyle\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-1)^n \frac{r^{2n}}{a^{2n}} P_{2n}(\cos\theta) & r \le a \\ 2\pi e \displaystyle\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-1)^n \frac{a^{2n+1}}{r^{2n+1}} P_{2n}(\cos\theta) & r \ge a. \end{cases}$$

∞ $\overline{2}$

r

∞ $\overline{2}$

Problem 6.2.1. A uniform circular wire of radius a charged with electricity of line density e

surrounds grounded concentric spherical conductor of radius $c.$ Determine the electrical charge density at any point on the conductor.

Solution. The potential functions is of the form

$$\psi_1 = 2\pi e \sum_n (-1) \left(\tfrac{1}{}\right)_n \left(\underline{r}\right)^{2n}$$

$\underline{r}^{\;2n}$

$\underline{c}$ $^{2n+1}$,

$n=0$

$$\psi_2 = 2\pi e \sum_{n} (-1) \left(\frac{1}{\ }\right)_n \underline{a}^{\,2n+1}$$

$\underline{a}\ ^{2n+1},$

$n=0$.

$n!$ r

The boundary conditions

(i) $\psi_1 = 0$ on $r = c$

(ii) $\psi_1 = \psi_2, \ \frac{\partial \psi_1}{\partial r} = \frac{\partial \psi_2}{\partial r}$ on $r = a$

yield the equations

$$(-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{c}{a}\right)^{2n} + A_n \left(\frac{c}{a}\right)^{2n} + B_n = 0$$

$$A_n + B_n \left(\frac{c}{a}\right)^{2n+1} = C_n$$

$$2nA_n - (2n+1)B_n \left(\frac{c}{a}\right)^{2n+1} = -(2n+1)C_n$$

from which it follows that

$$A_n = 0, \quad B_n = -(-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{c}{a}\right)^{2n}.$$

Hence when $c \le r \le a$,

$$\psi_1 = 2\pi e \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} \left(\frac{r^{2n}}{a^{2n}} - \frac{c^{4n+1}}{a^{2n} r^{2n+1}} \right) P_{2n}(\cos\theta).$$

The surface density on the spherical conductor is given by the formula

$$\sigma = -\frac{1}{4\pi} \left(\frac{\partial \psi_1}{\partial r} \right)_{r=c}$$

so that

$$\sigma = -\frac{e}{2c} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} (4n+1) \left(\frac{c}{a}\right)^{2n} P_{2n}(\cos\theta). \qquad \blacksquare$$

Problems for Practice

1. Prove that the potential of a circular disk of radius a carrying a charge of surface density σ at a point $(z, 0, 0)$ on its axis $\theta = 0$ is

$$2\pi\sigma[(z^2 + a^2)^{\frac{1}{2}} - z].$$

Deduce its value at a general point in space.

2. A grounded conducting sphere of radius a has its center on the axis of a charged circular ring, any radius vector $\mathbf{c}$ from this center to the ring making an angle α with the axis. Show that the force pulling the sphere into the ring is

$$-\frac{Q^2}{c^2}\sum_{n=0}(n+1)P_{n+1}(\cos\alpha)P_n(\cos\theta)\left(\frac{a}{c}\right)^{2n+1}.$$

3. A grounded conducting sphere of radius a is placed with its center at a point on the axis of a circular coil of radius b at a distance c from the center of the coil; the coil carries a charge e uniformly distributed. Prove that if a is small, the force of attraction between the sphere and the coil is

$$\frac{e^2ac}{f^4}\left[1+\frac{a^2}{f^2}\left(\frac{3c^2}{f^2}-1\right)+O\left(\frac{a^3}{f^3}\right)\right],$$

where $f^2 = b^2 + c^2$.

4. A dielectric sphere is surrounded by a thin circular wire of large radius b carrying a charge E. Prove that the potential within the sphere is

$$\frac{E}{b}\sum_{n=0}^{\infty}(-1)^n\frac{4n+1}{1+2n(1+\kappa)}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\left(\frac{r}{b}\right)^{2n}P_{2n}(\cos\theta).$$

6.3 The Theory of Green's Function for Laplace's Equation

Suppose the values of ψ and $\dfrac{\partial\psi}{\partial n}$ are known at every point of the boundary S of a finite region V and that $\nabla^2\psi = 0$ within V. We determine ψ by Green's theorem in the form

$$\int_\Omega (\psi\nabla^2\psi' - \psi'\nabla^2\psi)d\tau = \int_\Sigma \left(\psi\frac{\partial\psi'}{\partial n} - \psi'\frac{\partial\psi}{\partial n}\right) dS, \tag{1}$$

where Σ denotes the boundary of the region Ω.

To determine the solution $\psi(\mathbf{r})$ at a point P with position vector $\mathbf{r}$, then we surround P by a sphere C which has its center at P and has radius ε and take Σ to be the region which is exterior to C and interior to S. Putting

$$\psi' = \frac{1}{|\mathbf{r}' - \mathbf{r}|},$$

we know that the above function is an elementary solution of Laplace equation, i.e.,

$$\nabla^2\psi' = \nabla^2\psi = 0$$

within Ω, we see that

$$\int_C \left(\psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial\psi}{\partial n}\right) dS' + \int_S \left(\psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial\psi}{\partial n}\right) dS' = 0, \tag{2}$$

where $\mathbf{n}$, the normals. Now, on the surface of the sphere C,

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{\varepsilon},$$
$$\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{\varepsilon^2},$$
$$dS' = \varepsilon^2 \sin\theta d\theta d\varphi$$

and

$$\psi(\mathbf{r}') = \psi(\mathbf{r}) + d\psi$$

$$= \psi(\mathbf{r}) + \left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y} + z\frac{\partial\psi}{\partial z} \right)$$
$$= \psi(\mathbf{r}) + \varepsilon \left(\sin\theta\cos\varphi\frac{\partial\psi}{\partial x} + \sin\theta\sin\varphi\frac{\partial\psi}{\partial y} + \cos\theta\frac{\partial\psi}{\partial z} \right)$$

$$\psi(\mathbf{r}') = \psi(\mathbf{r}) + O(s) \qquad \text{on } C \qquad \frac{\partial\psi(\mathbf{r}')}{\partial n} = \left(\frac{\partial\psi(\mathbf{r})}{\partial n}\right)_P + O(s)$$

so that

$$\int_C \psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|}\,dS' = 4\pi\psi(\mathbf{r}) + O(s)$$

and

$$\int_C \frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial\psi}{\partial n}\,dS' = O(s).$$

Substituting these results into equation (2) and letting ε tend to zero, we find that

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \left(\frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial\psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} \right) dS' \tag{3}$$

so that the value of ψ at an interior point of the region V can be determined in terms of the values of ψ and $\dfrac{\partial\psi}{\partial n}$ on the boundary S.

Taking the directions of the normals to be as indicated in Fig. 24 and proceeding as above, we find, in this instance, that

$$4\pi\psi(\mathbf{r}) + O(s) + \int_S \left(\frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial\psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|} \right) dS' + \int_{\Sigma'} \left(\frac{1}{R}\frac{\partial\psi}{\partial n} + \frac{\psi}{R^2} \right) dS' = 0.$$

Letting $\varepsilon \to 0$ and $R \to \infty$, the solution (3) is valid in the case of the exterior Dirichlet problem provided that $R\psi$ and $R^2\dfrac{\partial\psi}{\partial n}$ an remain finite as $R \to \infty$.

Equation (3) would seem at first sight to indicate that to obtain a solution of Dirichlet's problem we need to know not only the value of the function ψ but also the value of $\frac{\partial\psi}{\partial n}$. We define a Green's function $G(\mathbf{r}, \mathbf{r}')$ by the equation

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + \frac{1}{|\mathbf{r}' - \mathbf{r}|}, \tag{4}$$

where the function $H(\mathbf{r}, \mathbf{r}')$ satisfies the relations

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) H(\mathbf{r}, \mathbf{r}') = 0 \tag{5}$$

and

$$H(\mathbf{r}, \mathbf{r}') + \frac{1}{|\mathbf{r}' - \mathbf{r}|} = 0 \quad \text{on } S. \tag{6}$$

The derivation of equation (3) is given by

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \left(G(\mathbf{r}, \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right) dS'. \tag{7}$$

Then the solution of the Dirichlet problem is given by the relation

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\int_S \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n}\, dS'. \tag{8}$$

where $G(\mathbf{r}, \mathbf{r}')$ satisfying equations (4), (5) and (6).

The solution of the Dirichlet problem is thus reduced to the determination of the Green's function $G(\mathbf{r}, \mathbf{r}')$.

Thus the Green's function for the Dirichlet problem involving the Laplace operator is a function $G(\mathbf{r}, \mathbf{r}')$ which satisfies the following properties:

(i) The Green's function $G(\mathbf{r}, \mathbf{r}')$ has the property of symmetry, i.e.,

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1) \tag{9}$$

i.e., if P_1 and P_2 are two points within a finite region bounded by a surface S, then the value at P_2 of the Green's function for the point P_1 and the surface S is equal to the value at P_1 of the Green's function for the point P_2 and the surface S.

(ii) $\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ in the region.

6.3.1 Physical Interpretation

If S is a grounded electrical conductor and if a unit charge is situated at the point with radius vector $\mathbf{r}$, then

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r}' - \mathbf{r}|} + H(\mathbf{r}, \mathbf{r}')$$

is the value at the point $\mathbf{r}'$ of the potential due to the charge at $\mathbf{r}$ and the induced charge on S. The first term on the right of this equation is the potential of the unit charge, and the second is the potential of the induced charge. By the definition of $H(\mathbf{r}, \mathbf{r}')$ the total potential $G(\mathbf{r}, \mathbf{r}')$ vanishes on S.

6.3.2 Solution of Dirichlet Problem using Green's Function

Dirichtet's Problem for a Semi-infinite Space

If the semi-infinite space to be $x \geq 0$, then we have to determine a function ψ such that $\nabla^2\psi = 0$ in $x \geq 0$, $\psi = f(y, z)$ on $x = 0$, and $\psi \to 0$ as $r \to \infty$.

The corresponding conditions on the Green's function $G(\mathbf{r}, \mathbf{r}')$ are that equations (4) and (5) should be satisfied and that G should vanish on the plane $x = 0$.

Suppose that Π, with position vector ρ, is the image in the plane $x = 0$ of the point P with position vector $\mathbf{r}$. If

$$H(\mathbf{r}, \mathbf{r}') = -\frac{1}{|\rho - \mathbf{r}'|}, \tag{10}$$

then the equation (5) is satisfied. Since $PQ = \Pi Q$ whenever Q lies on $x = 0$, it follows that equation (6) is also satisfied.

The required Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\rho - \mathbf{r}'|}, \tag{11}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (-x, y, z)$.

Since

$$\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} = -\frac{\partial}{\partial x'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}\right]$$

on the plane $x' = 0$

$$\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} = -\frac{2x}{[x^2+(y-y')^2+(z-z')^2]^{3/2}}.$$

Substituting the above equation and $\psi(\mathbf{r}') = f(y', z')$ into equation (8), we obtain the solution of this Dirichlet problem

$$\psi(x,y,z) = \frac{x}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y',z')\,dy'dz'}{[x^2+(y-y')^2+(z-z')^2]^{3/2}}. \tag{12}$$

Dirichlet's Problem for a Sphere

To determine the function $\psi(r, \theta, \varphi)$ satisfying the conditions

$$\nabla^2\psi = 0 \qquad r < a \tag{13}$$

$$\psi = f(\theta,\varphi) \qquad \text{on } r = a. \tag{14}$$

The Green's function $G(\mathbf{r}, \mathbf{r}')$ satisfies (4) and (5) and G should vanish on the surface of the sphere $r = a$.

Suppose that Π, with position vector **æ**, is the inverse point with respect to the sphere $r = a$ of the point P with position vector $\mathbf{r}$. Then from

$$H(\mathbf{r},\mathbf{r}') = -\frac{a}{r|\rho - \mathbf{r}'|} = -\frac{a}{r\left|\frac{a^2}{r^2}\mathbf{r} - \mathbf{r}'\right|} \tag{15}$$

the equation (5) is satisfied, and if Q lies on the surface of the sphere, $PQ = \frac{r}{a}\,\Pi Q$, so that equation (6) is also satisfied. The Green's function is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a/r}{\left|\frac{a^2}{r^2}\mathbf{r} - \mathbf{r}'\right|}. \tag{16}$$

Now

$$\frac{\partial G}{\partial r'} = -\frac{1}{R^3}\left(R\frac{\partial R}{\partial r'} - \frac{r^2}{a^2}R'\frac{\partial R'}{\partial r'}\right),$$

where

$$R^2 = r^2 + r'^2 - 2rr'\cos\Theta,$$

$$R'^2 = \frac{a^4}{r^2} + r'^2 - \frac{2a^2}{r}r'\cos\Theta \tag{17}$$

and

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{18}$$

Thus

$$\frac{\partial G}{\partial r'} = -\frac{r'(a^2-r^2)}{a^2R^3}$$

and when $r' = a$,

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'} = -\frac{a^2-r^2}{a(r^2+a^2-2ar\cos\Theta)^{\frac{3}{2}}}. \tag{19}$$

Hence if $\psi = f(\theta,\varphi)$ on $r = a$, it follows from equations (8) and (19) that the solution of the interior Dirichlet problem for a sphere is given by the equation

$$\psi(r,\theta,\varphi) = \frac{a(a^2-r^2)}{4\pi}\int_0^{2\pi}d\varphi'\int_0^{\pi}\frac{f(\theta',\varphi')\sin\theta' d\theta'}{(a^2+r^2-2ar\cos\Theta)^{\frac{3}{2}}}, \tag{20}$$

where $\cos\Theta$ is defined by equation (18).

The solution of the exterior Dirichlet problem is

$$\psi(r,\theta,\varphi) = \frac{a(r^2-a^2)}{4\pi}\int_0^{2\pi}d\varphi'\int_0^{\pi}\frac{f(\theta',\varphi')\sin\theta' d\theta'}{(a^2+r^2-2ar\cos\Theta)^{\frac{3}{2}}}. \tag{21}$$

The integral on the right-hand side of the solution (20) of the interior Dirichlet problem is called Poisson's integral. The function

$$\psi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\sum_{m=0}^{\infty}(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)P_n^m(\cos\theta) \tag{22}$$

is a solution of Laplace's equation which is finite at the origin. If this function is to provide a solution of our interior Dirichlet problem, then the constants A_{mn}, B_{mn} must be chosen so that

$$f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)P_n^m(\cos\theta),$$

where

$$A_{0n} = \frac{2n+1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta',\varphi')P_n(\cos\theta')\sin\theta'\,d\theta'\,d\varphi'$$

$$A_{mn} = \frac{(2n+1)}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta',\varphi')P_n^m(\cos\theta')\sin\theta'\cos(m\varphi')\,d\theta'\,d\varphi'$$

$$B_{mn} = \frac{(2n+1)}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta',\varphi')P_n^m(\cos\theta')\sin\theta'\sin(m\varphi')\,d\theta'\,d\varphi'.$$

Then the solution becomes

$$\psi(r,\theta,\varphi) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta',\varphi')\,g\sin\theta'\,d\theta'\,d\varphi', \tag{23}$$

where

$$g = \sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n\left[P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi')\right].$$

and

$$\frac{1-h^2}{(1-2h\cos\Theta+h^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}(2n+1)h^nP_n(\cos\Theta)$$

$$P_n(\cos\Theta) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{n=1}^{\infty}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi'),$$

where Θ is defined by equation (18), we have

$$g = \frac{a(a^2-r^2)}{(a^2-2ar\cos\Theta+r^2)^{\frac{3}{2}}}. \tag{24}$$

Check Your Progress

1. Suppose that P_1 and P_2 are two points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, which lie in the interior of a finite region V bounded by a surface S. By applying Green's theorem in the form (1) to the region bounded by S and two spheres of small radii surrounding P_1 and P_2 and taking $\psi(\mathbf{r}') = G(\mathbf{r}_1, \mathbf{r}')$, $\psi'(\mathbf{r}') = G(\mathbf{r}_2, \mathbf{r}')$, prove that

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1)$$

2. If the function $\psi(x, y, z)$ is harmonic in the half space $x \geq 0$, and if on $x = 0$, $\psi = 1$ inside a closed curve C and $\psi = 0$ outside C, prove that $2\pi\psi(x, y, z)$ is equal to the solid angle subtended by C at the point with coordinates (x, y, z).

3. If $\psi(x, y, z)$ is such that $\nabla^2\psi = 0$ for $x \geq 0$, $\psi = f(y)$ on $x = 0$, and $\psi \to 0$ as $r \to \infty$, prove that

$$\psi(x, y, z) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(y')dy'}{x^2 + (y - y')^2}.$$

4. The function $\psi(\mathbf{r})$ is harmonic within a sphere S and is continuous on the boundary. Prove that the value of ψ at the center of the sphere is equal to the arithmetic mean of its values on the surface of the sphere.

5. Use Green's theorem to show that, in a usual notation, if at all points of space

$$\nabla^2\varphi = -4\pi\rho,$$

where ρ is a function of position, and if φ and $r\,\mathrm{grad}\varphi$ tend to zero at infinity, then

$$\varphi = \int \frac{\rho dV}{r}.$$

Let us Sum up:

In this unit, the students acquired knowledge to

- find the solution of Laplace Equation in Rectangular Cartesian Coordinates.
- understand the concept of problems with axial symmetry.
- analysis Dirchlet Problem using green's function.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-IV

UNIT 7

THE WAVE EQUATION-I

Structure

Objective

Overview

7. 1 The Occurence of the Wave Equation in Physics.

7. 2 Elementary Solutions of the One-dimensional Wave Equation.

7. 3 Vibrating Membrances: Application of the Calculus of Variations.

Let us Sum Up

Check Your Progress

Suggested Readings

Overview

In this unit, we consider the elementary solutions of the wave equation and Vibrating Membrances.

Now. we consider the wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

which is a typical hyperbolic equation. This equation can be written in the form

$$\square^2 \psi = 0,$$

where $\square^2$ denotes the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Assume a solution of the wave equation of the form

$$\psi = \Psi(x, y, z)\, e^{\pm ikct}$$

then the function Ψ must satisfy the equation

$$(\nabla^2 + k^2)\, \Psi = 0$$

which is called the *space form of the wave equation* or *Helmholtz's equation*.

7.1 The Occurrence of the Wave Equation in Physics

In this section, we present the list of situations in physics where the wave equation arise.

Transverse Vibrations of a String

If a string of uniform linear density ρ is stretched to a uniform tension T, and if in the equilibrium position, the string coincides with the x axis, then when the string is disturbed slightly from its equilibrium position, the transverse displacement $y(x, t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{1}$$

where $c^2 = \frac{T}{\rho}$. At any point $x = a$ of the string which is fixed $y(a, t) = 0$ for all values of t.

Longitudinal Vibrations in a Bar

If a uniform bar of elastic material of uniform cross section whose axis coincides with Ox is stressed in such a way that each point of a typical cross section of the bar takes the same displacement $\xi(x, t)$, then

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}, \tag{2}$$

where $c^2 = \frac{E}{\rho}$, E being the Young's modulus and ρ the density of the material of the bar.

Longitudinal Sound Waves

If plane waves of sound are being propagated in a horn whose cross section for the section with abscissa x is $A(x)$ in such a way that every point of that section has the same longitudinal displacement $\xi(x, t)$, then ξ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial}{\partial x}(A\xi)\right) = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \tag{3}$$

which reduces to the one-dimensional wave equation (2) in the case in which the cross section is uniform.

Electric Signals in Cables

If the resistance per unit length R, and the leakage parameter G are both zero, the voltage $V(x, t)$ and the current $z(x, t)$ both satisfy the one-dimensional wave equation, with wave velocity c defined by the equation

$$c^2 = \frac{1}{LC}, \tag{4}$$

where L is the inductance and C the capacity per unit length.

Transverse Vibrations of a Membrane

If a thin elastic membrane of uniform areal density σ is stretched to a uniform tension T, and if in the equilibrium position, the membrane coincides with the xy plane, then the small transverse vibrations of the membrane are governed by the wave equation

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \tag{5}$$

where $z(x, y, t)$ is the transverse displacement (assumed small) at time t of the point (x, y) of the membrane. The wave velocity c is defined by the equation

$$c^2 = \frac{T}{\sigma}. \tag{6}$$

Sound Waves in Space

Consider a sound wave at the point (x, y, z) at time t has velocity $\mathbf{v} = (u, v, w)$ and that the pressure and density there and then are p, ρ, respectively; then if p_0, ρ_0 are the corresponding values in the equilibrium state, we write

$$\rho = \rho_0(1 + s), \qquad p = p_0 + c^2 \rho_0 s, \tag{7}$$

where s is called the condensation of the gas and c^2 is given by $c^2 = \left(\frac{dp}{d\rho}\right)_0$. Then the equations of motion governed by the wave equation

$$\nabla^2 \varphi = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2}, \tag{8}$$

where the motion of the gas is irrotational.

Electromagnetic Waves

If we write

$$\mathbf{H} = \operatorname{curl} \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \varphi,$$

then Maxwell's equations

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad \operatorname{curl}\mathbf{H} = \frac{4\pi\mathbf{i}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

are satisfied identically provided that $\mathbf{A}$ and φ satisfy the equations

$$\nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{4\pi\mathbf{i}}{c}, \qquad \nabla^2\varphi = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - 4\pi\rho.$$

Therefore in the absence of charges or currents φ and the components of $\mathbf{A}$ satisfy the wave equation.

Elastic Waves in Solids

If (u, v, w) denote the components of the displacement vector $\mathbf{v}$ at the point (x, y, z), then the components of the stress tensor are given by the equations

$$(\sigma_x, \sigma_y, \sigma_z) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right)$$

$$(\tau_{yz}, \tau_{zx}, \tau_{xy}) = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

where λ and μ are Lame's constants. The equations of motion are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho X = \rho\frac{\partial^2 u}{\partial t^2}, \text{ etc.}$$

where $\mathbf{F} = (X, Y, Z)$ is the body force at (x, y, z). If we write

$$\dot{\mathbf{F}} = \text{grad}\varphi + \text{curl}\Psi,$$

then it is easily shown that the displacement vector can be taken in the form

$$\mathbf{v} = \text{grad}\varphi + \text{curl}\Psi$$

provided that φ and Ψ satisfy the equations

$$\frac{\partial^2\varphi}{\partial t^2} - c_1^2\nabla^2\varphi = \Phi, \qquad \frac{\partial^2\Psi}{\partial t^2} - c_2^2\nabla^2\Psi = \Psi,$$

where the wave velocities c_1, c_2 are given by

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}.$$

Hence, in the absence of body forces, φ and the components of Ψ each satisfies a wave equation.

7.2 Elementary Solutions of the One-dimensional Wave Equation

Consider the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{1}$$

The canonical form of the one-dimensional wave equation (1) is

$$y_{\xi\eta} = 0, \tag{2}$$

with the transformation variables are $\xi = x + ct$ and $\eta = x - ct$. Then the solution of (1) is

$$\frac{\partial^2 y}{\partial\xi\partial\eta} = 0$$

$$\frac{\partial y}{\partial\xi} = A(\xi)$$

$$y = \int A(\xi)d\xi + B(\eta)$$

$$y = f(\xi) + g(\eta)$$

$$\boxed{y = f(x + ct) + g(x - ct)} \qquad (f \text{ and } g \text{ are arbitrary functions}) \qquad (3)$$

which is called the elementary solution of one dimensional wave equation.

Problem 7.2.1. Derive the **D'Alembert solution** of one-dimensional wave equation.

or

The displacement and the velocity of an infinite string is given by $\eta(x)$ and $v(x)$ respectively, at the initial time (i.e., $t = 0$). Determine the motion of the string.

Solution. Consider the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \quad -\infty < x < \infty, \quad t \geq 0 \qquad (1)$$

subject to the initial conditions

(i) $y(x, 0) = \eta(x)$, (2)

(ii) $\dfrac{\partial y}{\partial t}(x, 0) = v(x)$, (3) where $\eta(x)$ and $v(x)$ are twice continuously differentiable.

The general solution of the one-dimensional wave equation is

$$y(x, t) = f(x + ct) + g(x - ct) \tag{4}$$

where f and g are arbitrary functions.

Applying the condition (i) $y = \eta$ on $t = 0$ in (4), we get

$$\eta = f(x) + g(x). \tag{5}$$

Differentiating (4) with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = cf'(x + ct) - cg'(x - ct) \tag{6}$$

Applying the condition (ii) $\frac{\partial y}{\partial t}(x, 0) = v(x)$ in (6), we get

$$v(x) = cf'(x) - cg'(x) \tag{7}$$

Integrating equation (7), we have

$$f(x) - g(x) = \frac{1}{c}\int_b^x v(\xi)d\xi, \tag{8}$$

where b is arbitrary.

Solving (5) and (8) yields

$$f(x) = \frac{1}{2}\eta(x) + \frac{1}{2c}\int_b^x v(\xi)d\xi$$
$$g(x) = \frac{1}{2}\eta(x) - \frac{1}{2c}\int_b^x v(\xi)d\xi$$

Substituting these expressions in equation (4), we obtain

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi \tag{9}$$

which is the required **D'Alembert's solution** of the one-dimensional wave equation.

If $v \equiv 0$, i.e., the string is released from rest, then the solution (9) becomes

$$y = \frac{1}{2}\left[\eta(x+ct) + \eta(x-ct)\right]. \qquad \blacksquare$$

Note 7.2.1. The above solution shows the subsequent displacement of the string is produced by two pulses of "shape" $y = \frac{1}{2}\eta(x)$, each moving with velocity c, one to the right and the other to the left. By taking the initial displacement is

$$\eta(x) = \begin{cases} 0 & x < -a \\ 1 & |x| < a \\ 0 & x > a. \end{cases}$$

Problem 7.2.2. Consider the motion of a semi-infinite string $x \geq 0$ fixed at the point $x = 0$ with the conditions

$$y = \eta(x), \quad \frac{\partial y}{\partial t} = v(x) \quad x \geq 0 \quad \text{at} \quad t = 0$$
$$y = 0, \quad \frac{\partial y}{\partial t} = 0 \quad t \geq 0 \quad \text{at} \quad x = 0.$$

Solution. The motion of the string is governed by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \quad 0 \leq x < \infty, \quad t \geq 0.$$

Then the D'Alembert's solution

$$y = \frac{1}{2}\{\eta(x + ct) + \eta(x - ct)\} + \frac{1}{2c}\int_{x-}^{x+ct} v(\xi)d\xi$$

is no longer applicable, since $\eta(x - ct)$ would not have a meaning if $t > x/c$. ■

Problem 7.2.3. Consider the motion of an infinite string subject to the initial conditions

$$y = Y(x), \quad \frac{\partial y}{\partial t} = V(x) \quad \text{at } t = 0,$$

where

$$Y(x) = \begin{cases} \eta(x) & \text{if } x \geq 0 \\ -\eta(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad V(x) = \begin{cases} v(x) & \text{if } x \geq 0 \\ -v(-x) & \text{if } x < 0. \end{cases}$$

Solution. The motion of the string is governed by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \quad -\infty < x < \infty, \quad t \geq 0 \tag{1}$$

Then its displacement is given by

$$y = \frac{1}{2}\{Y(x+ct) + Y(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\xi)d\xi \tag{2}$$

when $x = 0$

$$y = \frac{1}{2}\{Y(ct) + Y(-ct)\} + \frac{1}{2c}\int_{-ct}^{ct} V(\xi)d\xi \tag{3}$$

and

$$\frac{\partial y}{\partial t} = \frac{1}{2}c\{Y'(ct) - Y'(-ct)\} + \frac{1}{2}\{V(ct) + V(-ct)\}. \tag{4}$$

From the definitions of Y and V and from the equations (3) and (4) we get y and $\frac{\partial y}{\partial t}$ at $x = 0$ are identically zero for all values of t.

The function (3) also satisfies the condition $y = 0, \frac{\partial y}{\partial t} = 0, t \geq 0$ at $x = 0$.

In particular, if the string is released from rest so that v, and consequently V, is identically zero, then the appropriate solution is

$$y = \begin{cases} \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] & x \geq ct \\ \frac{1}{2}[\eta(x+ct) - \eta(x-ct)] & x \leq ct. \end{cases}$$

■

Using the above problems, we construct the wave problem in finite string with the initial conditions.

Problem 7.2.4. Consider the motion of a finite string of length l with the initial conditions

$$y = Y(x), \quad \frac{\partial y}{\partial t} = V(x) \quad 0 \leq x \leq l \text{ at } t = 0$$
$$y = 0, \quad \frac{\partial y}{\partial t} = 0 \quad t \geq 0 \text{ at } x = 0 \text{ and } x = l,$$

where $Y(x)$ is defined by

$$Y(x) = \begin{cases} \eta(x) & \text{if } 0 \leq x \leq l \\ -\eta(-x) & \text{if } -l \leq x \leq 0 \end{cases} \quad \text{and} \quad V(x) = \begin{cases} v(x) & \text{if } 0 \leq x \leq l \\ -v(-x) & \text{if } -l \leq x \leq 0 \end{cases}$$

and

$$Y(x + 2rl) = Y(x); \quad V(x + 2rl) = V(x) \text{ if } -l \leq x \leq l, \; r = \pm 1, \pm 2, \ldots.$$

In other words, $Y(x)$ and $V(x)$ are odd periodic function of period $2l$.

Solution. The motion of the string is governed by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad 0 < x < l, \quad t \geq 0. \tag{1}$$

Then its displacement is given by

$$y = \frac{1}{2}\{Y(x + ct) + Y(x - ct)\} + \frac{1}{2c}\int_{x-}^{x+ct} V(\xi)d\xi. \tag{2}$$

Given that $Y(x)$ and $V(x)$ are odd periodic function and has a Fourier sine expansion of the form

$$Y(x) = \sum_{m=0}^{\infty} \eta_m \sin\frac{m\pi x}{l}, \tag{3}$$

where the coefficients η_m are given by the formula

$$\eta_m = \frac{2}{l}\int_0^l \eta(\xi)\sin\frac{m\pi\xi}{l}\,d\xi. \tag{4}$$

Similarly

$$V(x) = \sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l}, \tag{5}$$

where

$$v_m = \frac{2}{l}\int_0^l v(\xi)\sin\frac{m\pi\xi}{l}\,d\xi. \tag{6}$$

Using the above equations, we get

$$\frac{1}{2}\{Y(x+ct)+Y(x-ct)\} = \sum_{m=0}^{\infty}\eta_m \sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l}$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} V(\xi)d\xi = \frac{l}{\pi c}\sum_{m=0}^{\infty}\frac{v_m}{m}\sin\frac{m\pi x}{t}\sin\frac{m\pi ct}{l}$$

Substituting the above equations in (2), we obtain the solution of the present problem is

$$y = \sum_{m=1}^{\infty}\eta_m \sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l} + \frac{l}{\pi c}\sum_{m=1}^{\infty}\frac{v_m}{m}\sin\frac{m\pi x}{l}\sin\frac{m\pi ct}{l}, \tag{7}$$

where η_m and v_m are defined by equations (4) and (6), respectively. ■

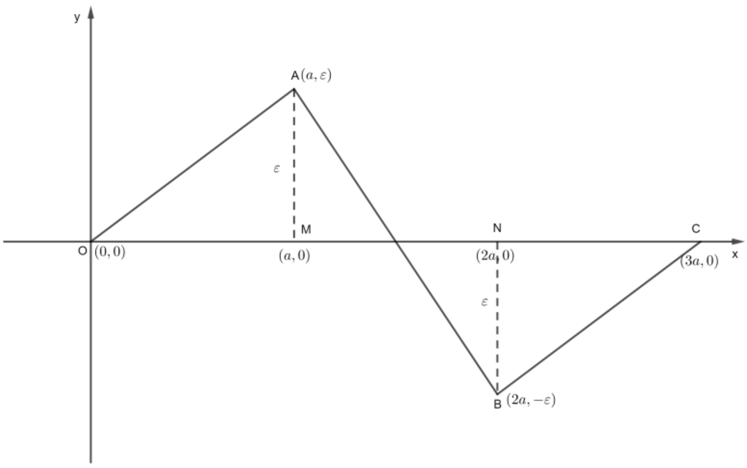
Problem 7.2.5. The points of trisection of a string are pulled aside through a distance s on opposite sides of the position or equilibrium, and the string is released from rest. Derive an expression for the displacement of the string at any subsequent time and show that the mid-point of the string always remains at rest.

Solution. The motion of the string is governed by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad 0 < x < l, \quad t \geq 0. \tag{1}$$

Take the length of the string $l = 3a$.

From the given conditions, the string OC ($3a$) is trisected at A and C. We have to find the three line equation OA, AB and BC for the initial position of the string. Equation of straight line with



two given points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Equation of OA is

$$y - 0 = \frac{(\varepsilon - 0)}{(a - 0)}(x - 0) \Rightarrow y = \frac{\varepsilon x}{a}.$$

Equation of AB is

$$y - \varepsilon = \frac{(-\varepsilon - \varepsilon)}{(2a - a)}(x - a) \;\Rightarrow\; y - \varepsilon = -\frac{2\varepsilon}{a}(x - a) \;\Rightarrow\; y = \frac{\varepsilon(3a - 2x)}{a}.$$

Equation of BC is

$$y - 0 = \frac{(-\varepsilon - 0)}{(2a - 3a)}(x - 3a) \;\Rightarrow\; y = \frac{\varepsilon(x - 3a)}{a}.$$

Therefore the initial position of the string is

$$y(x, 0) = \eta(x) = \begin{cases} \dfrac{\varepsilon x}{a} & 0 \le x \le a \\[2mm] \dfrac{\varepsilon(3a - 2x)}{a} & a \le x \le 2a \\[2mm] \dfrac{\varepsilon(x - 3a)}{a} & 2a \le x \le 3a \end{cases}$$

and the string is released from rest so $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = v(x) \equiv 0.$

Then its displacement is given by

$$y = \sum_{m=1} \eta_m \sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l} \tag{2}$$

where

$$\eta_m = \frac{2}{l}\int_0^l \eta(\xi)\sin\frac{m\pi\xi}{l}\,d\xi$$

$$\eta_m = \frac{2\varepsilon}{3a^2}\left[\int_0^a x\sin\frac{m\pi x}{3a}dx + \int_a^{2a}(3a - 2x)\sin\frac{m\pi x}{3a}dx + \int_{2a}^{3a}(x - 3a)\sin\frac{m\pi x}{3a}dx\right]$$

$$= \frac{18\varepsilon}{\pi^2 m^2}\{1 + (-1)^m\}\sin\left(\frac{1}{3}m\pi\right) \qquad \text{(by Bernoulli's integration formula)}$$

$$= \begin{cases} \frac{36\varepsilon}{\pi^2 m^2} \sin \frac{m\pi}{3}, & m - \text{even} \\ 0, & m - \text{odd} \end{cases}$$

and

$$v_m = 0$$

so that the displacement (2) is

$$y = \frac{36\varepsilon}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{m\pi}{3} \sin \frac{m\pi x}{3a} \cos \frac{m\pi ct}{3a} \tag{3}$$

Replace m by $2n$, we get

$$\boxed{y = \frac{9s}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{2n\pi}{3} \sin \frac{2n\pi x}{3a} \cos \frac{2n\pi ct}{3a}.} \tag{4}$$

To find the displacement at the midpoint of the string:

Put $x = \frac{3a}{2}$ in (4), we get

$$y = \frac{9s}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{2n\pi}{3} \sin(n\pi) \cos \frac{2n\pi ct}{3a}.$$

$$\boxed{y = 0} \qquad (\because \sin n\pi = 0 \quad \forall \ n \in \mathbb{Z})$$

Therefore, the displacement of the mid-point of the string is always zero. ■

Check Your Progress

1. A uniform string of line density ρ is stretched to tension ρc^2 and executes a small transverse vibration in a plane through the undisturbed line of the string. The ends $x = 0, l$ of the string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance ε and released at time $t = 0$. Show that at any subsequent time t the transverse displacement y is given by the Fourier expansion $y = \frac{2\varepsilon l^2}{\pi^2 b(l-b)} \sum_{s=1}^{\infty} \frac{1}{s^2} \sin\left(\frac{s\pi b}{l}\right) \sin\frac{s\pi x}{l} \cos\frac{s\pi ct}{l}$.
2. If the string is released from rest in the position $y = \frac{4\varepsilon}{l^2} x(l - x)$, show that its motion is described by the equation $y = \frac{32\varepsilon}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\frac{(2n+1)\pi x}{l} \cos\frac{(2n+1)\pi ct}{l}$.
3. If the string is released from rest in the position $y = f(x)$, show that the total energy of the string is $\frac{\pi^2 T}{4l} \sum_{s=1}^{\infty} s^2 k_s^2$, where $k_s = \frac{2}{l} \int_0^l f(x) \sin\frac{s\pi x}{l}\, dx$. The mid-point of a string is pulled aside through a small distance and then released. Show that in the subsequent motion the fundamental mode contributes $8/\pi^2$ of the total energy.

7.3 Vibrating Membranes: Application of the Calculus of Variations

The transverse vibrations of a thin membrane S bounded by the curve Γ in the xy plane is described by a function $z(x, y, t)$ satisfies the wave equation

$$\nabla_1^2 z = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} \tag{1}$$

the boundary condition

$$z = 0 \text{ on } \Gamma \text{ for all } t \tag{1}$$

and the initial conditions

$$z = f(x, y), \quad \frac{\partial z}{\partial t} = g(x, y) \quad t = 0, \quad (x, y) \in S. \tag{2}$$

7.3.1 Solution of the Equation of the Vibrating Membrane (Rectangular Membrane) - Integral Transforms Method

In this subsection, we discuss the integral methods to solve the vibrating membrane problem.

Problem 7.3.1. A thin membrane of great extent is released from rest in the position $z = f(x, y)$. Determine the displacement at any subsequent time.

Solution. The motion of the vibrating membrane is governed by the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}. \tag{1}$$

From the given data, the boundary condition

$$z = 0 \text{ on } \Gamma \text{ for all } t \tag{2}$$

and the initial conditions

$$z = f(x, y), \quad \frac{\partial z}{\partial t} = 0 \quad t = 0, \text{ for all } (x, y) \text{ of the plane.} \tag{3}$$

The two dimensional Fourier transform of $z(x, y, t)$ is

$$Z(\xi, \eta, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(x, y, t)e^{i(\xi x+\eta y)}dxdy.$$

Taking the Fourier transform of (1) on both sides, we get

$$\frac{d^2 Z}{dt^2} + c^2(\xi^2 + \eta^2)Z = 0 \tag{4}$$

and the conditions (3) becomes

$$Z = F(\xi, \eta), \qquad \frac{dZ}{dt} = 0 \quad t = 0. \tag{5}$$

The solution of (4) is

$$Z = A\cos\left(c\sqrt{\xi^2+\eta^2}\,t\right) + B\sin\left(c\sqrt{\xi^2+\eta^2}\,t\right). \tag{6}$$

Using the condition (5), we obtain

$$A = F(\xi, \eta) \text{ and } B = 0.$$

Substituting the values of A and B in (6), we have

$$Z = F(\xi, \eta)\cos[c(\xi^2+\eta^2)^{\frac{1}{2}}t]. \tag{7}$$

Applying the inverse Fourier transform to find the solution of the given problem

$$z = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta)\cos[c(\xi^2+\eta^2)^{\frac{1}{2}}t]e^{-i(\xi x+\eta y)}d\xi d\eta. \qquad \blacksquare$$

7.3.2 Solution of the Equation of the Vibrating Membrane (Rectangular Membrane) - Separation of Variables

The motion of the vibrating membrane is governed by the two-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}. \tag{1}$$

Let us assume the solution of the form

$$z(x, y, t) = X(x)Y(y)T(t). \tag{2}$$

Substituting equation (2) in equation (1), we have

$$\frac{1}{c^2} XYT'' - [X''YT + XY''T] = 0$$

Dividing throught by XYT, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{T''}{T} = -k^2 \quad \text{(a separation constant)}$$

implies

$$\frac{X''}{X} = -k_1^2 \qquad \frac{Y''}{Y} = -k_2^2 \qquad \frac{1}{c^2}\frac{T''}{T} = -k^2$$

$$X'' = -k_1^2 X \qquad Y'' = -k_2^2 Y \qquad T'' = -k^2c^2T$$

$$X'' + k_1^2 X = 0 \qquad Y'' + k_2^2 Y = 0 \qquad T'' + k^2c^2T = 0$$

$$X = c_1 \cos k_1 x + c_2 \sin k_1 x \qquad Y = c_3 \cos k_2 y + c_4 \sin k_2 y \qquad T = c_5 \cos kct + c_6 \sin kct.$$

Then the solution becomes

$$z(x, y, t) = (c_1 \cos k_1 x + c_2 \sin k_1 x)(c_3 \cos k_2 y + c_4 \sin k_2 y)(c_5 \cos kct + c_6 \sin kct), \tag{3}$$

where $k^2 = k_1^2 + k_2^2$.

Problem 7.3.2. A rectangular membrane with fastened edges makes free transverse vibrations. Find the displacement of the vibrating membrane.

Solution. The transverse vibration of a rectangular membrane is described by

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad 0 \le x \le a, \quad 0 \le y \le b \tag{1}$$

subject to the boundary conditions

(i) $z(0, y, t) = 0$

(ii) $z(a, y, t) = 0$

(iii) $z(x, 0, t) = 0$

(iv) $z(x, b, t) = 0$

and initial conditions

$$z(x, y, 0) = f(x, y), \qquad \frac{\partial z}{\partial t}(x, y, 0) = 0.$$

The suitable solution is

$$z(x, y, t) = (c_1 \cos k_1 x + c_2 \sin k_1 x)(c_3 \cos k_2 y + c_4 \sin k_2 y)(c_5 \cos kct + c_6 \sin kct), \tag{2}$$

where $k^2 = k_1^2 + k_2^2$.

Applying the boundary condition $z(0, y, t) = 0$, we get

$$0 = c_1(c_3 \cos k_2 y + c_4 \sin k_2 y)(c_5 \cos kct + c_6 \sin kct)$$

which gives $c_1 = 0$, then equation (2) becomes

$$z(x, y, t) = c_2 \sin k_1 x\, (c_3 \cos k_2 y + c_4 \sin k_2 y)(c_5 \cos kct + c_6 \sin kct), \tag{3}$$

Applying the boundary condition $z(x, 0, t) = 0$, we get

$$0 = c_2 \sin k_1 x \, c_3 \, (c_5 \cos kct + c_6 \sin kct)$$

which gives $c_3 = 0$, then equation (3) becomes

$$z(x, y, t) = c_2 \, c_4 \, \sin k_1 x \, \sin k_2 y \, (c_5 \cos kct + c_6 \sin kct), \tag{4}$$

Applying the boundary condition $z(a, y, t) = 0$, we get

$$0 = c_2 \, c_4 \sin k_1 a \, \sin k_2 y \, (c_5 \cos kct + c_6 \sin kct)$$

implies that $k_1 = m\pi/a$, $m = 1, 2, \ldots,$ applying the boundary condition $u(x, b, t) = 0$, we get

$$0 = c_2 \, c_4 \sin k_1 x \, \sin k_2 b \, (c_5 \cos kct + c_6 \sin kct)$$

implies that $k_2 = n\pi/b$, $n = 1, 2, \ldots\ldots$ Then, the equation (4) becomes

$$z(x, y, t) = c_2 \, c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (c_5 \cos kct + c_6 \sin kct).$$

After adjusting the constants and by the principle of superposition, we get

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos k_{mn} ct + B_{mn} \sin k_{mn} ct] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5}$$

where

$$k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

Applying the initial condition $\frac{\partial z}{\partial t}(x, y, 0) = 0$ yields $B_{mn} = 0$, then the solution becomes

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos k_{mn}ct \ \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{6}$$

Applying the initial condition $z(x, y, 0) = f(x, y)$, we have

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy. \tag{7}$$

Hence, the required solution is given by equation (6) along with the coefficients A_{mn} in the above equation (7). ■

7.3.3 Solution of the Equation of the Vibrating Membrane (circular Membrane) - Separation of Variables

The motion of the vibrating membrane takes the form

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} \tag{1}$$

and the curve Γ can be taken as $r = a$. Let us assume the solution of the form

$$z = R(r)\Theta(\theta)e^{\pm ikct}, \tag{2}$$

then the functions $R,\ \Theta$ must satisfies

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+k^2R\right]+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=0$$

using the separation constants we obtain the ordinary differential equations for $R,\ \Theta$ are

$$\frac{d^2\Theta}{d\theta^2}+m^2\Theta=0 \tag{3}$$

and

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(k^2-\frac{m^2}{r^2}\right)R=0. \tag{4}$$

The solutions of (3) is of the form

$$\Theta=e^{-\pm im\theta}.$$

If the displacement $z(r,\theta,t)$ is periodic, i.e., $z(r,\ \theta+2\pi,\ t)=z(r,\theta,t)$, then we must choose m to be an *integer*. Furthermore, at $r=0$ the solution of (4) is

$$R=J_m(kr),$$

where $J_m(x)$ denotes the Bessel function of the first kind of order m and argument x. Thus the solution of the equation (1) of the form

$$z=\sum_{m,k}A_{mk}J_m(kr)e^{\pm im\theta\pm ikct}. \tag{5}$$

If z vanishes on the circle $r=a$, then the numbers k must be chosen so that

$$J_m(ka)=0 \tag{6}$$

and we obtain the solution

$$z=\sum_{m,n}A_{mn}J_m(k_{mn}r)\exp\{\pm im\theta\pm ik_{mn}ct\}, \tag{7}$$

where A_{mn} are constants and $k_{m1}, k_{m2}, \ldots$ are the positive roots of the transcendental equation (6). In the symmetrical case in which z is a function of r and t alone the corresponding solution is

$$z(r,t) = \sum_n A_n J_0(k_n r) e^{\pm i c k_n t}, \tag{3}$$

where $k_1, k_2, \ldots$ are the positive zeros of the function $J_0(ka)$.

Problem 7.3.3. Find the displacement of the vibrating circular membrane.

Solution. The initial condition is

$$z = f(r), \qquad \frac{\partial z}{\partial t} = 0 \text{ at } t = 0,$$

then the solution of the problem is

$$z = \sum_n A_n J_0(k_n r) \cos(k_n c t), \tag{1}$$

where the constants A_n are chosen so that

$$f(r) = \sum_n A_n J_0(k_n r).$$

Using the Bessel functions we obtain

$$A_n = \frac{2}{a^2} J_1^2(k_n a) \int_0^a r f(r) J_0(k_n r) dr. \tag{2}$$

The complete solution of our problem is therefore given by the equations (1) and (2). ■

7.3.4 Solution of the Equation of the Vibrating Membrane whose boundary Γ

The transverse vibrations of a thin membrane S bounded by the curve Γ in the xy plane is described by a function $z(x, y, t)$ satisfies the wave equation

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \tag{1}$$

the boundary condition

$$z = 0 \text{ on } \Gamma \text{ for all } t. \tag{2}$$

When the boundary curve Γ is fixed is of the form $f(x, y)e^{it\sqrt{\lambda}}$, then the nth eigenvalue λ_n is the minimum of the integral

$$I = T\iint_S \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dxdy \tag{3}$$

with respect to those sufficiently regular functions φ which vanish on Γ and satisfy the normalization condition

$$\sigma\iint_S \varphi^2 dxdy = 1 \tag{4}$$

and the $n - 1$ orthogonality relations

$$\iint_S \varphi\,\varphi_m dxdy = 0, \tag{5}$$

where φ_m is the minimizing function which makes I equal to λ_m. If

$$z = \psi_m(x, y)e^{ik_m ct} \tag{6}$$

is an approximate solution of the problem stated in equations (1) and (2), then if $\Phi_1, \ldots, \Phi_n$ are n functions which are continuously differentiable in S and which vanish on Γ, an approximate solution is

$$\psi_m(x, y) = \sum_{i=1}^{n} C_i^m \Phi_i(x, y), \tag{7}$$

where the coefficients $C_i^{(m)}$ are the solutions of the linear algebraic equations

$$\sum_{j=1}^{n} (\sigma_{ij} k_m^2 - \Gamma_{ij}) C_j^{m} = 0 \qquad i = 1, 2, \dots, n \tag{8}$$

with

$$\sigma_{ij} = \sigma_{ji} = \iint_S \Phi_i \Phi_j \, dxdy \tag{9}$$

$$\Gamma_{ij} = \Gamma_{ji} = \iint_S \left(\frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} + \frac{\partial \Phi_i}{\partial y} \frac{\partial \Phi_j}{\partial y} \right) dxdy \tag{10}$$

and the first n approximate eigenvalues $k_1, k_2, \dots, k_n$ are given by the n positive roots of the determinantal equation

$$\begin{vmatrix} \sigma_{11}k^2 - \Gamma_{11} & \sigma_{12}k^2 - \Gamma_{12} & \cdots & \sigma_{1n}k^2 - \Gamma_{1n} \\ \sigma_{21}k^2 - \Gamma_{21} & \sigma_{22}k^2 - \Gamma_{22} & \cdots & \sigma_{2n}k^2 - \Gamma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1}k^2 - \Gamma_{n1} & \sigma_{n2}k^2 - \Gamma_{n2} & \cdots & \sigma_{nn}k^2 - \Gamma_{nn} \end{vmatrix} = 0. \tag{11}$$

In addition the coefficients must be chosen to satisfy the normalization condition

$$\sigma \sum_{i,j=1} C_i^{(m)} C_j^{(m)} \sigma_{ij} = 1. \tag{12}$$

If the boundary curve Γ of the membrane S has equation $u(xy) = 0$, a simple choice of the approximate functions Φ_i $(i = 1, 2, \dots, n)$ is to take

$\Phi_1 = u(x, y)$, $\quad \Phi_2 = xu(x, y)$, $\quad \Phi_3 = yu(x, y)$

$\Phi_4 = x^2u(x, y)$, $\quad \Phi_5 = xyu(x, y)$, $\quad \Phi_6 = y^2u(x, y)$, etc.

The variational approach to eigenvalue problems is useful not only in the derivation of approximate solutions but also in the establishing of quite general theorems about the eigenvalues of a system.

Problem 7.3.4. Find approximate values for the first three eigenvalues of a square membrane of side 2.

Solution. Here the membrane is bounded by the lines $x = \pm 1$, $y = \pm 1$.

Assume

$$\Phi_1 = (1 - x^2)(1 - y^2),$$

$$\Phi_2 = x(1 - x^2)(1 - y^2),$$

$$\Phi_3 = y(1 - x^2)(1 - y^2)$$

and we obtain

$$\sigma_{11} = \frac{256}{225}, \qquad \sigma_{22} = \sigma_{33} = \frac{256}{1575}, \qquad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$$

$$\Gamma_{11} = \frac{256}{45}, \qquad \Gamma_{22} = \Gamma_{33} = \frac{3328}{1575}, \qquad \Gamma_{12} = \Gamma_{23} = \Gamma_{31} = 0.$$

The determinantal equation is

$$(k^2 - 5)(k^2 - 13)^2 = 0$$

Then the first three approximate eigenvalues of the square are

$$k_1 = \sqrt{5} = 2.236, \quad k_2 = k_3 = \sqrt{13} = 3.606,$$

whereas the exact results are

$$k_1 = \frac{\pi\sqrt{2}}{2} = 2.221, \quad k_2 = k_3 = \frac{\pi\sqrt{5}}{2} = 3.942.$$

■

Check Your Progress

1. A very large membrane which is in its equilibrium position lies in the shape $z = f(r)$ $(r^2 = x^2 + y^2)$. Show that its subsequent displacement is given by the equation $z(r, t) = \int_0^\infty \xi \bar{f}(\xi) \cos(c\xi t) J_0(\xi r) d\xi$, where $\xi = \int_0^\infty r f(r) J_0(\xi r) dr$.

2. A square membrane whose edges are fixed receives a blow in such a way that a concentric and similarly situated square area one-sixteenth of the area of the membrane acquires a transverse velocity v without sensible displacement, the remainder being undisturbed. Find a series for the displacement of the membrane at any subsequent time.

3. A membrane of uniform density σ per unit area is stretched on a circular frame of radius a to uniform stress σc^2. When $t = 0$, the membrane is released from rest in the position $x = \varepsilon(a^2 - r^2)$, where s is small, and r is the distance from the center. Show that the displacement of the center at time t is $8\varepsilon a^2 \sum_{n=1}^{\infty} \frac{\cos(\xi_n ct/a)}{\xi_n J_1(\xi_n)}$, where ξ_n is the nth positive zero of the Bessel function J_0.

4. Using the approximations $\Phi_1 = 1 - \sqrt{x^2 + y^2}$, $\Phi_2 = x - x\sqrt{x^2 + y^2}$, $\Phi_3 = y - y\sqrt{x^2 + y^2}$ show that the first three approximate values of the constant k in the solution $f(r)e^{ikct}$, describing the transverse vibrations of a circular membrane of unit radius, are $K_1 = \sqrt{6}$, $K_2 = K_3 = \sqrt{15}$.

Let us Sum up:

In this unit, the students acquired knowledge to

- find D'Alembert solution of the one-dimensional wave equation.
- find the motion of the string is governed by one-dimensional wave equation.
- the application of the Calculus of Variations in Vibraring Membranes.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.

2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-IV

UNIT 8

THE WAVE EQUATION-II

Structure

Overview

In this unit, we will illustrate the method to find the solution of three-dimensional wave equation in Rectangular Cartesian Coordinates.

8.1 Three-dimensional Problems

In this section, we consider the three-dimensional wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}. \tag{1}$$

8.1.1 Solution of the Three-Dimensional Wave Equation in Cartesian Coordinates

Consider a three-dimensional wave equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

Let us assume the solution of the form

$$\psi(x,y,z,t)=X(x)Y(y)Z(z)T(t). \tag{2}$$

Substituting equation (2) in equation (1), we have

$$X''YZT+XY''ZT+XYZ''T=\frac{1}{c^2}T''$$

Dividing throught by $XYZT$, we get

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=\frac{1}{c^2}\frac{T''}{T}=-k^2 \quad \text{(a separation constant)}$$

implies

$$\frac{X''}{X}=-l^2 \qquad\qquad \frac{Y''}{Y}=-m^2$$

$$X''=-l^2X \qquad\qquad Y''=-m^2Y$$

$$X''+l^2X=0 \qquad\qquad Y''+m^2Y=0$$

$$X=c_1\cos lx+c_2\sin lx \qquad\qquad Y=c_3\cos my+c_4\sin my$$

$$\frac{Z''}{Z}=-n^2 \qquad\qquad \frac{1}{c^2}\frac{T''}{T}=-k^2$$

$$Z''=-n^2Z \qquad\qquad T''=-k^2c^2T$$

$$Z''+n^2Z=0 \qquad\qquad T''+k^2c^2T=0$$

$$Z=c_5\cos nz+c_6\sin nz \qquad\qquad T=c_7\cos kct+c_8\sin kct$$

Then the solution becomes

$$\psi(x, y, z, t) = (c_1 \cos lx + c_2 \sin lx)(c_3 \cos my + c_4 \sin my)(c_5 \cos nz + c_6 \sin nz)(c_7 \cos kct + c_8 \sin kct), \quad (3)$$

where $k^2 = l^2 + m^2 + n^2$.

Problem 8.1.1. A gas is contained in a cubical box of side a. Show that if c is the velocity of sound in the gas, the periods of free oscillations are

$$\frac{2a}{c\sqrt{n_1^2 + n_2^2 + n_3^2}},$$

where n_1, n_2, n_3 are integers.

Solution. In this problem the governing equation is three-dimensional wave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \quad (1)$$

The solution of (1) is

$$\psi(x, y, z, t) = (c_1 \cos lx + c_2 \sin lx)(c_3 \cos my + c_4 \sin my)(c_5 \cos nz + c_6 \sin nz)(c_7 \cos kct + c_8 \sin kct), \quad (2)$$

where $k^2 = l^2 + m^2 + n^2$.

From the given data, solution (2) is valid in the space $0 \leq (x, y, z) \leq a$ and such that $\frac{\partial\psi}{\partial n} = 0$ on the boundaries of the cube. Therefore, the solution can be obtaine das follows

$$\psi(x, y, z, t) = \sum_{n_1, n_2, n_3} A_{n_1,n_2,n_3} \cos\frac{n_1\pi x}{a} \cos\frac{n_2\pi y}{a} \cos\frac{n_3\pi z}{a} \cos\left(n_1^2 + n_2^2 + n_3^2\right)^{\frac{1}{2}}\frac{\pi ct}{a},$$

where n_1, n_2, n_3 are integers. The periods of the free oscillations of the gas are

$$\frac{2a}{c\sqrt{n_1^2 + n_2^2 + n_3^2}}. \qquad \blacksquare$$

8.1.2 Solution of the Three-Dimensional Wave Equation in Spherical Polar Coordinates

Consider a three-dimensional wave equation in spherical polar coordinates (r, θ, φ)

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

Let us assume the solution of the form

$$\psi(r, \theta, \varphi, t) = \Psi(r)\, P_n^m(\cos\theta)\, e^{\pm im\varphi \pm ikct}, \tag{2}$$

where $\Psi(r)$ is a function of r and $P_n^m(\cos\theta)$ is the associated Legendre function. Substituting equation (2) into equation (1), we find that $\Psi(r)$ satisfies the ordinary differential equation

$$\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Psi}{dr} - \frac{n(n+1)}{r^2}\Psi + k^2\Psi = 0. \tag{3}$$

Put

$$\Psi = r^{-\frac{1}{2}}R(r),$$

Equation (3) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right)R = 0. \tag{4}$$

If $n + \frac{1}{2}$ is neither zero nor an integer, then the solution of (4) is

$$R(r) = AJ_{n+\frac{1}{2}}(kr) + BJ_{-n-\frac{1}{2}}(kr), \tag{5}$$

where A and B are constants and $J_\nu(z)$ denotes the Bessel function of the first kind of order ν and argument z. By symmetry property,

$$\psi(r,\ \theta+\pi,\ \varphi) = \psi(r,\theta,\varphi), \qquad \psi(r,\ \theta,\ \varphi+2\pi) = \psi(r,\theta,\varphi),$$

we take m and n to be integers.

Hence the function

$$\psi(r,\theta,\varphi,t) = r^{-\frac{1}{2}}\ J_{\pm(n+\frac{1}{2})}(kr)\ P_n^m(\cos\theta)\ e^{\pm im\varphi\pm ikct} \tag{6}$$

is a solution of the wave equation (1). The functions $J_{\pm(n+\frac{1}{2})}(kr)$, which occur in the solution (6), are called *spherical Bessel functions*.

If n is half of an odd integer, then

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} [f_n(x)\sin x - g_n(x)\cos x]$$

$$J_{-n}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} (-1)^{n-\frac{1}{2}}[g_n(x)\sin x + f_n(x)\cos x],$$

where $f_n(x)$ and $g_n(x)$ are polynomials in x^{-1}.

- When $n = \frac{1}{2}$, $f_{\frac{1}{2}}(x) = 1$, $g_{\frac{1}{2}}(x) = 0$.
- When $n = \frac{3}{2}$, $f_{\frac{3}{2}}(x) = 1/x$ and $g_{\frac{3}{2}}(x) = 1$.

Then, we have

$$\psi(r) = \frac{1}{r}\, e^{\pm ikr\pm ikct} \tag{7}$$

$$\psi(r,\theta) = \frac{1}{r}\left[\frac{\sin(kr)}{kr} - \cos(kr)\right]\cos\theta e^{\pm ikct} \tag{8}$$

are particular solutions of the wave equation (1).

In the case of spherical symmetry, i.e., if ψ is a function of r and t alone, then it must satisfy the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \tag{9}$$

Put $\psi = \frac{\varphi}{r}$, we get

$$\frac{\partial^2\varphi}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$$

then

$$\varphi = f(r - ct) + g(r + ct),$$

where the functions f and g are arbitrary. Therefore, the general solution of the equation (9) is

$$\psi = \frac{1}{r}[f(r - ct) + g(r + ct)], \tag{10}$$

where the functions f and g are arbitrary.

The function $r^{-1}f(r - ct)$ represents a diverging wave. Taking

$$\varphi = \frac{1}{4\pi r} f\left(t - \frac{r}{c}\right) \tag{11}$$

to be the velocity potential of a gas, then the velocity of the gas is

$$u = -\frac{\partial\varphi}{\partial r} = \frac{1}{4\pi r^2} f\left(t - \frac{r}{c}\right) + \frac{1}{4\pi rc} f'\left(t - \frac{r}{c}\right)$$

so that the total flux through a sphere of center the origin and small radius ε is

$$4\pi\varepsilon^2 u = f(t) + O(\varepsilon).$$

The difference between the pressure at an instant t and the equilibrium value is given by

$$p - p_0 = \rho\frac{\partial\varphi}{\partial t} = \frac{\rho}{4\pi r} f'\left(t - \frac{r}{c}\right). \tag{12}$$

Problem 8.1.2. A gas is contained in a rigid sphere of radius a. Show that if c is the velocity of sound in the gas, the frequencies of purely radial oscillations are $c\xi_i/a$, where $\xi_1, \xi_2, \ldots$ are the positive roots of the equation $\tan \xi = \xi$.

Solution. The given problem is governed by the given equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \tag{1}$$

subject to the condition that ψ remains finite at the origin, and given that $u = \partial\psi/\partial r = 0$ at $r = a$.

Using the conditions, we find the solution of (1) is given by

$$\psi = A\,\frac{\sin(kr)}{r}\,e^{ikct}, \tag{2}$$

where A is a constant. Now,

$$u = -\frac{\partial \psi}{\partial r} = A\left[\frac{k\cos(kr)}{r} - \frac{\sin(kr)}{r^2}\right]e^{ikct}.$$

Applying the second condition, we obtain

$$\tan(ka) = ka.$$

The possible frequencies are $c\xi_i/a$ ($i = 1, 2, \ldots$), where $\xi_1, \xi_2, \ldots$ are the positive roots of the transcendental equation

$$\tan \xi = \xi. \qquad \blacksquare$$

8.1.3 Solution of the Three-Dimensional Wave Equation in Cylindrical Coordinates

Consider a three-dimensional wave equation in cylindrical coordinates (ρ, φ, z)

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

Let us assume the solution of the form

$$\psi(\rho, \varphi, z, t) = R(r)\Phi(\varphi)Z(z)T(t) \tag{2}$$

Substitute equation (2) in equation (1), we get the following equations for each variables

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\omega^2 - \frac{m^2}{\rho^2}\right)R = 0,$$

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0,$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0,$$

$$\frac{d^2T}{dt^2} + k^2c^2T = 0,$$

where

$$\gamma^2 = k^2 - \omega^2.$$

The solutions of the form

$$\psi(\rho, \varphi, z, t) = J_m(\omega\rho)e^{ikct - i\gamma z \pm im\varphi},$$

where γ is related to k and ω. The phase velocity is

$$V = \frac{kc}{\gamma}$$

and the group velocity is

$$W = \frac{d}{d\gamma}(kc) = \frac{c\gamma}{k}.$$

Solution in terms of Hankel Functions

The general solution of (1) is

$$\psi(p, \varphi, z, t) = [A_m J_m(\omega\rho) + B_m Y_m(\omega\rho)]e^{ikct - i\gamma z \pm im\varphi}, \tag{3}$$

where $Y_m(\omega\rho)$ denotes Bessel's function of the second kind and A_m, B_m denote complex constants.

Relation between Bessel's functions and Hankel functions are given by

$$H_m^{(1)}(\omega\rho) = J_m(\omega\rho) + i\mathrm{Y}_m(\omega\rho),$$
$$H_m^{(2)}(\omega\rho) = J_m(\omega p) - i\mathrm{Y}_m(\omega\rho).$$

The the solution (3) can be written as

$$\psi(\rho, \varphi, z, t) = [A_m H_m^{(1)}(\omega\rho) + B_m H_m^{(2)}(\omega\rho)]e^{ikct - i\gamma z \pm im\varphi}. \tag{4}$$

In case of axial symmetry $(m = 0)$, we obtain solutions of the form

$$\psi(\rho, z, t) = [AH_0^{(1)}(\omega\rho) + BH_0^{(2)}(\omega\rho)]e^{ikct - i\gamma z}. \tag{5}$$

Now, for large values of ρ

$$\begin{aligned} H_0^{(1)}(\omega\rho) &\sim \left(\frac{2}{\pi\omega\rho}\right)^{\frac{1}{2}} e^{i(\omega\rho - \frac{1}{4}\pi)}, \\ H_0^{(2)}(\omega\rho) &\sim \left(\frac{2}{\pi\omega\rho}\right)^{\frac{1}{2}} e^{-i(\omega\rho - \frac{1}{4}\pi)} \end{aligned} \tag{6}$$

so as $\rho \to \infty$,

$$\psi(\rho, z, t) \sim \left(\frac{2}{\pi\omega\rho}\right)^{\frac{1}{2}} \left[Ae^{i(kct + \omega\rho - i\gamma z - \frac{1}{4}\pi)} + Be^{i(kct - \omega\rho - i\gamma z + \frac{1}{4}\pi)}\right].$$

Thus the solution

$$\psi_0(\rho, z, t) = H_0^{(1)}(\omega\rho)e^{ikct - i\gamma z} \tag{7}$$

represents waves diverging from the axis $\rho = 0$, while the solution

$$\psi_i(\rho, z, t) = H_0^{(2)}(\omega\rho)e^{ikct-i\gamma z} \tag{8}$$

represents waves converging to this axis.

In the two-dimensional case, the solution becomes

$$\psi(\rho, \varphi, t) = [A_m H_m^{(1)}(k\rho) + B_m H_m^{(2)}(k\rho)]e^{ikct \pm im\varphi} \tag{9}$$

with

$$\psi_0(\rho, t) = H_0^{(1)}(k\rho)e^{ikct}$$

and

$$\psi_i(\rho, t) = H_0^{(2)}(k\rho)e^{ikct}$$

respectively.

Problem 8.1.3. Harmonic sound waves of period $2\pi/kc$ and small amplitude are propagated along a circular wave guide bounded by the cylinder $\rho = a$. Prove that solutions independent of the angle variable φ are of the form

$$\psi = e^{i(kct-\gamma z)} J_0\left(\frac{\xi_n p}{a}\right),$$

where ξ_n is a zero of $J_1(\xi)$ and $\gamma^2 = k^2 - (\xi_n^2/a^2)$.

Show that this mode is propagated only if $k > \xi_n/a$.

Solution. Since ψ is independent of φ, then by taking $m = 0$ in equation, we obtain the solution of the form

$$\psi = J_0(\omega\rho)e^{i(kct-\gamma z)}, \tag{1}$$

where $\gamma^2 = k^2 - \omega^2$. The boundary condition is that the velocity of the gas vanishes on the cylinder; i.e.,

$$\frac{\partial\psi}{\partial\rho} = 0 \text{ on } \rho = a. \tag{2}$$

Since $J_0'(x) = -J_1(x)$, and by condition $J_1(\omega a) = 0$; $\omega = \xi_n/a$, where $\xi_1, \xi_2, \ldots$ are the zeros of $J_1(\xi)$. Then

$$\psi = e^{i(kct-\gamma z)}\, J_0\left(\frac{\xi_n\rho}{a}\right), \tag{3}$$

where $\gamma^2 = k^2 - (\xi_n^2/a^2)$.

For the mode given by equation (3) to be propagated we must have γ real; i.e., $k > \xi_n/a$. ■

Check Your Progress

1. A wave of frequency ν is propagated inside an endless uniform tube whose cross section is rectangular.

   (a) Calculate the phase velocity and the wavelength along the direction of propagation.

   (b) Show that if a wave is to be propagated along the tube, its fre quency cannot be lower than $\nu_{\min} = \frac{c}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{\frac{1}{2}}$, where a and b are the lengths of the sides of the cross section.

   (c) Verify that the group velocity is always less than c. Show that the group velocity tends to zero as the frequency decreases to $\nu_{\min}$.

2. Show that the flux of energy through unit area of a fixed surface produced by sound waves of velocity potential ψ in a medium of average density ρ is $-\rho\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial n}$. A source of strength

$A\cos(\sigma t)$ is situated at the origin. Show that the average rate at which the source loses energy to the air is $\dfrac{\rho A^2\sigma^2}{8\pi c}$, where c is the velocity of sound in air.

3. A symmetrical pressure disturbance $\rho_0 A \sin kct$ is maintained over the surface of a sphere of radius a which contains a gas of mean density ρ_0. Find the velocity potential of the forced oscillation of the gas, and show that the radial velocity at any point of the surface of the sphere varies harmonically with amplitude $\dfrac{A}{c}\left(\dfrac{1}{ka} - \cot ka\right)$.

8.2 General Solutions of the Wave Equation

In this section, we state the theorem to discuss conditions on general solutions of the wave equation with $\psi(r,t)$ and its normal derivative $\partial\psi/\partial n$ are prescribed on a surface S.

Suppose that Ψ is a solution of the space form of the wave equation

$$\nabla^2\Psi + k^2\Psi = 0 \tag{1}$$

and that the singularities of Ψ all lie outside a closed surface S bounding the volume V. Put

$$\Psi' = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{2}$$

we observe that if the point with position vector $\mathbf{r}$ lies outside S, then

$$\int_S \left(\Psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial\Psi(\mathbf{r}')}{\partial n} \right) dS' = 0. \tag{3}$$

By Green's theorem, we have

$$\int_S \left(\Psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial\Psi(\mathbf{r}')}{\partial n} \right) dS' = \lim_{\varepsilon\to 0} \int_C \left(\left(ik - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\Psi(\mathbf{r}') - \frac{\partial\Psi(\mathbf{r}')}{\partial r'} \right) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, dS'$$

and the value of the limit on the right-hand side of this equation is $-4\pi\Psi(\mathbf{r})$.

We now state the following standard theorems:

Helmholtz's First Theorem.

If $\Psi(\mathbf{r})$ is a solution of the space form of the wave equation $\nabla^2\Psi + k^2\Psi = 0$ whose partial derivatives of the first and second orders are continuous within the volume V on the closed surface S bounding V, then

$$\frac{1}{4\pi}\int_S \left(\Psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Psi(\mathbf{r}')}{\partial n}\right)dS' = \begin{cases}\Psi(\mathbf{r}) & \text{if } \mathbf{r}\in V\\ 0 & \text{if } \mathbf{r}\notin V,\end{cases} \tag{1}$$

where $\mathbf{n}$ is the outward normal to S.

Helmholtz's Second Theorem.

If $\Psi(\mathbf{r})$ is a solution of the space form of the wave equation whose partial derivatives of the first and second orders are continuous outside the volume V and on the closed surface S bounding V, if $r\Psi(\mathbf{r})$ is bounded, and if

$$r\left(\frac{\partial\Psi}{\partial r} - ik\Psi\right) \to 0$$

uniformly with respect to the angle variables as $r \to \infty$, then

$$\frac{1}{4\pi}\int_S \left(\Psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Psi(\mathbf{r}')}{\partial n}\right)dS' = \begin{cases}\Psi(\mathbf{r}) & \text{if } \mathbf{r}\notin V\\ 0 & \text{if } \mathbf{r}\in V,\end{cases} \tag{2}$$

where $\mathbf{n}$ is the outward normal to S.

Weber's Theorem.

If $\Psi(\mathbf{p})$ is a solution of the space form of the two-dimensional wave equation $\nabla_1^2\Psi + k^2\Psi = 0$ whose partial derivatives of the first and second orders are continuous within the area S and on the closed curve Γ bounding S, then

$$\frac{1}{4i}\int_\Gamma \left(\Psi(\textbf{æ}')\frac{\partial}{\partial n}H_0^{(1)}(k|\textbf{æ}-\textbf{æ}'|) - H_0^{(1)}(k|\textbf{æ}-\textbf{æ}'|)\frac{\partial\Psi(\textbf{æ}')}{\partial n}\right)ds' = \begin{cases}\Psi(\textbf{æ}) & \text{if } \textbf{æ}\in S\\ 0 & \text{if } \textbf{æ}\notin S,\end{cases}$$

where $\mathbf{n}$ is the outward normal to Γ.

Kirchhoff's First Theorem.

If $\psi(\mathbf{r}, t)$ is a solution of the wave equation whose partial derivatives of the first and second orders are continuous within the volume V and on the surface S bounding V, then

$$\frac{1}{4\pi}\int_S \left(-[\psi]\frac{\partial}{\partial n}\left(\frac{1}{\lambda}\right) + \frac{1}{c\lambda}\frac{\partial \lambda}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] + \frac{1}{\lambda}\left[\frac{\partial \psi}{\partial n}\right]\right) dS' = \begin{cases} \psi(\mathbf{r}, t), & \text{if } P(\mathbf{r}) \in V, \\ 0, & \text{if } P(\mathbf{r}) \notin V, \end{cases} \tag{3}$$

where $\lambda = |\mathbf{r} - \mathbf{r}'|$ and $\mathbf{n}$ is the outward normal to S.

Kirchhoff's Second Theorem.

If $\psi(\mathbf{r}, t)$ is a solution of the wave equation which has no singularities outside the region V bounded by the surface S for all values of the time from $-\infty$ to t, and if as $r \to \infty$,

$$\psi(\mathbf{r}, t) \sim \frac{f(ct - r)}{r},$$

where $f(u)$, $f'(u)$ are bounded near $u = -\infty$, then

$$\frac{1}{4\pi}\int_S \left([\psi]\frac{\partial}{\partial n}\left(\frac{1}{\lambda}\right) - \frac{1}{c\lambda}\frac{\partial \lambda}{\partial n}\left[\frac{\partial \psi}{\partial t}\right] - \frac{1}{\lambda}\left[\frac{\partial \psi}{\partial n}\right]\right) dS' = \begin{cases} \psi(\mathbf{r}), & \text{if } P(\mathbf{r}) \in V, \\ 0, & \text{if } P(\mathbf{r}) \notin V, \end{cases} \tag{4}$$

where $\mathbf{n}$ is the outward normal to S.

Let us Sum up:

In this unit, the students acquired knowledge to

- find the solution interms of Hankel functions.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.

2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-V

UNIT 9

THE DIFFUSION EQUATION

Overview

In this unit, we will illustrate to find the one-dimensional Diffusion Equation and two-dimensional Diffusion Equation.

In this chapter, we consider the one-dimensional diffusion equation

$$k\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$$

which is parabolic type equation. The generalized form of diffusion equation is given by

$$k\nabla^2\theta = \frac{\partial\theta}{\partial t},$$

where k is a constant.

- If $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then $k\nabla^2\theta = \frac{\partial\theta}{\partial t}$ is called two-dimensional diffusion equation.
- If $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, then $k\nabla^2\theta = \frac{\partial\theta}{\partial t}$ is called three-dimensional diffusion equation.

9.1 Elementary Solutions of the Diffusion Equation

In this section, we consider elementary solutions of the one-dimensional diffusion equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}. \tag{1}$$

Consider the expression

$$\theta = \frac{1}{\sqrt{t}}\, e^{-x^2/4\kappa t}, \tag{2}$$

then

$$\frac{\partial^2\theta}{\partial x^2} = \frac{x^2}{4\kappa^2 t^{\frac{5}{2}}}\, e^{-x^2/4\kappa t} - \frac{1}{2\kappa t^{\frac{3}{2}}}\, e^{-x^2/4\kappa t}$$

and

$$\frac{\partial\theta}{\partial t} = \frac{x^2}{4\kappa t^{\frac{5}{2}}}\, e^{-x^2/4\kappa t} - \frac{1}{2t^{\frac{3}{2}}}\, e^{-x^2/4\kappa t}$$

substituting the above expressions of $\frac{\partial^2\theta}{\partial x^2}$ and $\frac{\partial\theta}{\partial t}$ in (1), we observe that it satisfies (1) and hence the expression (2) is a solution of the equation (1).

Again, we consider the another expression

$$\theta = \frac{1}{2\sqrt{\pi\kappa t}}e^{-(x-\xi)^2/4\kappa t}, \tag{3}$$

where ξ is a real constant. It is easy to verify that expression (3) is also a solution of equation (1).

If $\varphi(x)$ is a bounded function on real numbers, then the Poisson integral

$$\theta(x, t) = \frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty} \varphi(\xi)\, e^{-(x-\xi)^2/4\kappa t} d\xi \tag{4}$$

is also, in some sense, a solution of the equation (1).

Problem 9.1.1. Prove that the Poisson integral

$$\theta(x, t) = \frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty} \varphi(\xi)\, e^{-(x-\xi)^2/4\kappa t} d\xi$$

is the solution of the one-dimensional diffusion equation with initial condition

$$\begin{aligned} \frac{\partial^2\theta}{\partial x^2} &= \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad -\infty < x < \infty \\ \theta(x, 0) &= \varphi(x) \text{ is bounded.} \end{aligned} \tag{5}$$

Solution. It is easy to observe that the integral (4) is convergent if $t > 0$ and that the integrals obtained from it by differentiating under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x, t).

The function $\theta(x, t)$ and its derivatives of all orders therefore exist for $t > 0$, and since the integrand satisfies the one-dimensional diffusion equation, it follows that $\theta(x, t)$ itself satisfies that equation for $t > 0$.

Now

$$\left|\frac{1}{2(\pi\kappa t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-(x-\xi)^2/4\kappa t}d\xi - \varphi(x)\right| = |I_1 + I_2 + I_3 - I_4|,$$

where

$$I_1 = \frac{1}{\sqrt{\pi}}\int_{-N}^{N}\{\varphi(x + 2u\sqrt{\kappa t}) - \varphi(x)\}\, e^{-u^2}\, du$$

$$I_2 = \frac{1}{\sqrt{\pi}}\int_{N}^{\infty}\varphi(x + 2u\sqrt{\kappa t})\, e^{-u^2}\, du$$

$$I_3 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-N}\varphi(x + 2u\sqrt{\kappa t})\, e^{-u^2}\, du$$

$$I_4 = \frac{2\varphi(x)}{\sqrt{\pi}}\int_{N}^{\infty} e^{-u^2}\, du.$$

Taking N sufficiently large and if the function $\varphi(x)$ is bounded, the integrals I_2, I_3, I_4 are small and by the continuity of the function φ, and for sufficiently small values of t, the integral I_1 becomes small. Thus as $t \to 0$, $\theta(x, t) \to \varphi(x)$, i.e., $\theta(x, 0) = \varphi(x)$. Hence, it is proved that the Poisson integral (4) is a solution of the initial value problem (5). ■

Remark 9.1.1. Put $u = \dfrac{(x - \xi)}{2\sqrt{\kappa t}}$ in the solution (4) of the IVP (5), can be written as

$$\theta(x, t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(x + 2u\sqrt{\kappa t})\, e^{-u^2}\, du. \tag{6}$$

Problem 9.1.2. Solve the boundary value problem (BVP)

$$
\begin{aligned}
\frac{\partial^2\theta}{\partial x^2} &= \frac{1}{\kappa}\frac{\partial\theta}{\partial t} && 0 \le x < \infty \\
\theta(x, 0) &= f(x) && x > 0 \\
\theta(0, t) &= 0 && t > 0.
\end{aligned}
\tag{7}
$$

Solution. If we write

$$
\varphi(x) = \begin{cases} f(x) & \text{for } x > 0 \\ -f(-x) & \text{for } x < 0, \end{cases}
$$

then we rewrite the Poisson integral (4) as

$$
\theta(x, t) = \frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty f(\xi)\left[e^{-(x-\xi)^2/4\kappa t} - e^{-(x+\xi)^2/4\kappa t}\right] d\xi. \tag{8}
$$

It is easy to verify that (8) is the solution of the boundary value problem (14). ■

Problem 9.1.3. Solve the boundary value problem (BVP)

$$
\begin{aligned}
\frac{\partial^2\theta}{\partial x^2} &= \frac{1}{\kappa}\frac{\partial\theta}{\partial t} && 0 \le x < \infty \\
\theta(x, 0) &= 0 && x > 0 \\
\theta(0, t) &= \theta_0 && t > 0.
\end{aligned}
\tag{9}
$$

Solution. The solution (8) can be express in the form

$$
\theta(x, t) = \frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{\kappa t}}}^\infty f(x + 2u\sqrt{\kappa t})e^{-u^2} du - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa t}}}^\infty f(-x + 2u\sqrt{\kappa t})e^{-u^2} du. \tag{10}
$$

Thus if the initial temperature is a constant, θ_0 say, then

$$\theta(x, t) = \theta_0 \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right) \tag{11}$$

where

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du \tag{12}$$

The function

$$\theta(x, t) = \theta_0 \left[1 - \operatorname{erf}\frac{x}{2\sqrt{\kappa t}}\right] \tag{13}$$

will therefore have the property that $\theta(x, 0) = 0$. ■

Problem 9.1.4. Solve the boundary value problem (BVP)

$$\begin{aligned} \frac{\partial^2\theta}{\partial x^2} &= \frac{1}{\kappa}\frac{\partial\theta}{\partial t} && 0 \le x < \infty \\ \theta(x, 0) &= 0 && x > 0 \\ \theta(0, t) &= g(t) && t > 0. \end{aligned} \tag{14}$$

Solution. Thus the function

$$\theta(x, t, t') = g(t') \left[1 - \operatorname{erf}\frac{x}{2\sqrt{\kappa t}}\right]$$

is the function which satisfies the one-dimensional diffusion equation and the conditions $\theta(x, 0, t') = 0$, $\theta(0, t, t') = g(t')$. By applying Duhamel's theorem it follows that the solution of the boundary value problem

$$\theta(x, 0) = 0, \quad \theta(0, t) = g(t) \tag{15}$$

is

$$\begin{aligned}\theta(x, t) &= \frac{2}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t g(t')dt' \int_{x/2(\kappa t-\kappa t')^{\frac{1}{2}}}^{\infty} e^{-u^2} du \\ &= \frac{x}{2\sqrt{\pi\kappa}} \int_0^t g(t') \frac{e^{-x^2/4\kappa(t-t')}}{(t-t')^{\frac{3}{2}}} dt'.\end{aligned}$$

Changing the variable of integration from t' to u where

$$t' = t - \frac{x^2}{4\kappa u^2}$$

we see that the solution may be written in the form

$$\theta(x, t) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty g\left(t - \frac{x^2}{4\kappa u^2}\right) e^{-u^2} du, \quad \eta = \frac{x}{2\sqrt{\kappa t}}.$$

■

9.2 Separation of Variables

In this section, we derive the solution of the diffusion equation

$$\nabla^2\theta = \frac{1}{\kappa} \frac{\partial \theta}{\partial t} \tag{1}$$

by using the method of the separation of variables.

Let us assume the solution of the form

$$\theta = \varphi(\mathbf{r})\, T(t), \tag{2}$$

Substituting equation (2) in equation (1), we get

$$\frac{1}{\varphi}\nabla^2\varphi = \frac{1}{\kappa T}\frac{dT}{dt}$$

then the governing equations of the functions T and φ is of the form

$$\frac{dT}{dt} + \kappa\lambda^2 T = 0 \tag{3}$$

$$(\nabla^2 + \lambda^2)\varphi = 0, \tag{4}$$

where λ is a constant which may be complex.

The solution of linear first order equation (3) is

$$T(t) = e^{-\kappa\lambda^2 t}.$$

Thus the general solution (2) of equation (1) takes the form

$$\theta(\mathbf{r}, t) = \varphi(\mathbf{r})e^{-\kappa\lambda^2 t}, \tag{5}$$

where the function φ is a solution of the Helmholtz equation (4).

9.2.1 One-dimensional Diffusion Equation

Consider the one-dimensional diffusion equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \tag{1}$$

Let us assume the solution of the form

$$\theta(x, t) = X(x)T(t) \tag{2}$$

Substituting equation (2) in equation (1), we get

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T} = c, \quad \text{(a separation constant)}$$

Then we have

$$\begin{aligned} \frac{d^2X}{dx^2} - cX &= 0 \\ \frac{dT}{dt} - \alpha cT &= 0. \end{aligned}$$

The following three cases arises:

Case I Let $c > 0$, then $c = \lambda^2$, we get

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} - \alpha\lambda^2 T = 0.$$

which gives

$$X = c_1e^{\lambda x} + c_2e^{-\lambda x}, \qquad T = c_3e^{\alpha\lambda^2 t}.$$

Case II Let $c < 0$, then $c = -\lambda^2$, we get

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + \alpha\lambda^2 T = 0.$$

which gives

$$X = c_1\cos\lambda x + c_2\sin\lambda x, \qquad T = c_3e^{-\alpha\lambda^2 t}.$$

Case III Let $c = 0$. Then

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0.$$

which gives

$$X = c_1x + c_2, \qquad T = c_3.$$

Thus, various possible solutions of the diffusion equation (1) are

$$\theta(x,\,t) \;=\; (c_\lambda e^{\lambda x} + d_\lambda e^{-\lambda x})e^{\alpha\lambda^2 t}$$
$$\theta(x,\,t) = (c_\lambda \cos\lambda x + d_\lambda \sin\lambda x)e^{-\alpha\lambda^2 t}$$
$$\theta(x,\,t) \;=\; c_\lambda x + d_\lambda$$

where

$$c_\lambda = c_1c_3, \qquad d_\lambda = c_2c_3.$$

Problem 9.2.1. The faces $x = 0$, $x = a$ of an infinite slab are maintained at zero temperature. The initial distribution of temperature in the slab is described by the equation $\theta = f(x)$ $(0 \le x \le a)$. Determine the temperature at a subsequent time t.

Solution. The temperature function $\theta(x, t)$ which satisfies the one-dimensional diffusion equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}. \tag{1}$$

From the given problem, we get the following boundary conditions

(i) $\theta(0, t) = 0$ for all $t \ge 0$.

(ii) $\theta(a, t) = 0$ for all $t \ge 0$.

(iii) $\theta(x, 0) = f(x), \quad 0 \le x \le a$

The possible solutions are

$$\theta(x,\,t) \;=\; (c_\lambda e^{\lambda x} + d_\lambda e^{-\lambda x})e^{\kappa\lambda^2 t}$$

$$\theta(x, t) = (c_\lambda \cos \lambda x + d_\lambda \sin \lambda x)e^{-\kappa\lambda^2 t}$$

$$\theta(x, t) = c_\lambda x + d_\lambda$$

The most suitable solution satisfying the boundary conditions of the given problem is

$$\theta(x, t) = (c_\lambda \cos \lambda x + d_\lambda \sin \lambda x)e^{-\kappa\lambda^2 t}. \tag{2}$$

Applying the boundary condition (i) in equation (2), we get

$$0 = c_\lambda e^{-\kappa\lambda^2 t}.$$

Here $e^{-\kappa\lambda^2 t} \neq 0$, since it is defined for all t.

$$\therefore c_\lambda = 0.$$

Substituting $c_\lambda = 0$ in (2), we get

$$\theta(x, t) = d_\lambda \sin \lambda x \, e^{-\kappa\lambda^2 t}. \tag{3}$$

Applying the boundary condition (ii) in equation (3), we get

$$\theta(a, t) = d_\lambda \sin \lambda a \, e^{-\kappa\lambda^2 t} = 0$$

$$\sin \lambda \pi = 0 \qquad (\because d_\lambda \neq 0 \ \& \ e^{-\kappa\lambda^2 t} \neq 0)$$

$$\lambda a = n\pi$$

$$\lambda = \frac{n\pi}{a},$$

where n is an integer.

Hence the solution is

$$\theta(x, t) = d_\lambda \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{\kappa n^2\pi^2 t}{a}}.$$

By the principle of superposition, the most general solution is

$$\theta(x, t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{\kappa n^2\pi^2 t}{a}}.$$

Applying the condition (iii), we get

$$\begin{aligned} \theta(x, 0) &= \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{a} \\ f(x) &= \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{a} \end{aligned}$$

which is a half-range Fourier Sine series in the interval $(0, a)$ and therefore the d_n can be obtained using the Fourier coefficient formula,

$$d_n = \frac{2}{a}\int_0^a f(u) \sin\left(\frac{n\pi u}{a}\right) du.$$

Thus, the general solution is

$$\theta(x, t) = \frac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a f(u) \sin\left(\frac{n\pi u}{a}\right) du\right] \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{\kappa n^2\pi^2 t}{a}}. \qquad \blacksquare$$

Problem 9.2.2. Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi$, $t \ge 0$, subject to the conditions

(i) T remains finite as $t \to \infty$

(ii) $T = 0$, if $x = 0$ and π for all t

(iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

Solution. The one-dimensional diffusion equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{1}$$

The possible solutions are:

$$\begin{aligned} T(x,t) &= (c_1 e^{\lambda x} + c_2 e^{-\lambda x}) e^{\alpha \lambda^2 t} \\ T(x,t) &= (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha \lambda^2 t} \\ T(x,t) &= (c_1 x + c_2). \end{aligned}$$

As $t \to \infty$, $e^{\alpha\lambda^2 t} \to \infty$, this violates the first boundary condition hence we reject the first solution.

Applying the boundary condition (ii), the third solution gives

$$0 = c_1 \cdot 0 + c_2, \qquad 0 = c_1 \cdot \pi + c_2$$

implies that $c_1 = 0$ and $c_2 = 0$ and hence $T = 0$ for all t which is a trivial solution.

The suitable solution satisfying the conditions is

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)e^{-\alpha\lambda^2 t}. \tag{2}$$

Applying the boundary condition $T = 0$ when $x = 0$ in (2), we have

$$0 = (c_1 \cos \lambda x + c_2 \sin \lambda x)|_{x=0}$$

implying $c_1 = 0$ since $e^{-\alpha\lambda^2 t} \neq 0$ defined for all t. Then the solution (2) becomes

$$T(x, t) = c_2 \sin \lambda x \, e^{-\alpha\lambda^2 t}. \tag{3}$$

Applying the boundary condition $T = 0$ when $x = \pi$ in (3), we get

$$0 = c_2 \sin \lambda\pi \, e^{-\alpha\lambda^2 t}$$

$$\Rightarrow \quad \sin \lambda\pi = 0$$

$$\Rightarrow \quad \lambda\pi = n\pi$$

$$\lambda = n$$

where n is an integer.

Equation (3) becomes

$$T(x, t) = c \sin nx \;\; e^{-\alpha n^2 t}.$$

By the principle of superposition, the most general solution is

$$T(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \; e^{-\alpha n^2 t}. \tag{4}$$

Applying the condition (iii), we get

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

which is a half-range Fourier-sine series and, therefore,

$$\begin{aligned}
c_n &= \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx\,dx \\
&= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right] \\
&= \frac{2}{\pi}\left[\left(-x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left(-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right] \\
c_n &= \frac{4\sin(n\pi/2)}{n^2\pi}.
\end{aligned}$$

Thus, the required solution is

$$\boxed{T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(nx)\,e^{-\alpha n^2 t}.}$$

■

9.2.2 Two-dimensional Diffusion Equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \tag{6}$$

Let us assume the solution of the form

$$\theta(x, y, t) = X(x)Y(y)T(t). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2.$$

Then $\beta' + \alpha\lambda^2\beta = 0$ whose solution is

$$\beta = e^{-\alpha\lambda^2 t}$$

and

$$\frac{X''}{X} = -\left(\lambda^2 + \frac{Y''}{Y}\right) = -p^2 \text{ (say)}.$$

Hence,

$$\begin{aligned} X'' + p^2X &= 0 \\ \frac{Y''}{Y} = -\lambda^2 + p^2 &= -q^2\text{(say)} \Rightarrow Y'' + q^2Y = 0. \end{aligned}$$

which gives

$$X = A\cos px + B\sin px$$

and

$$Y = C\cos qy + D\sin qy$$

Thus, the general solution of the given PDE is

$$T(x, y, t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2.$$

The solution

$$\theta(x, y, t) = \sum_{\lambda}\sum_{\mu} c_{\lambda\mu}\cos(\lambda x + \varepsilon_\lambda)\cos(\mu y + \varepsilon_\mu)e^{-(\lambda^2+\mu^2)\kappa t} \tag{7}$$

of the two-dimensional equation which we derived in Sec. 9 of Chap. 3 may be treated in a precisely similar way (cf. Prob. 3 below).

9.3 Solution of Diffusion Equation in Cylindrical Coordinates

Consider a three-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T.$$

In cylindrical coordinates (r, θ, z), it becomes

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{1}$$

where $T = T(r, \theta, z, t)$.

Let us assume solution of the form

$$T(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)\Phi(t). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$R''\Theta Z\Phi + \frac{1}{r}R'\Theta Z\Phi + \frac{1}{r^2}R\Theta''Z\Phi + R\Theta Z''\Phi = \frac{\Phi'}{\alpha}R\Theta Z$$

Dividing by $R\Theta Z\Phi$

$$\Rightarrow \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \frac{1}{\alpha}\frac{\Phi'}{\Phi} = -\lambda^2$$

where $-\lambda^2$ is a separation constant. Then

$$\frac{1}{\alpha}\frac{\Phi'}{\Phi} = -\lambda^2 \quad \Rightarrow \quad \Phi' + \alpha\lambda^2\Phi = 0 \tag{3}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = -\lambda^2$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2 \quad \text{(say)}.$$

$$-\frac{Z''}{Z} = -\mu^2 \quad \Rightarrow \quad Z'' - \mu^2 Z = 0 \tag{4}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = -\mu^2$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2 + \mu^2)r^2 = -\frac{\Theta''}{\Theta} = \nu^2 \quad \text{(say)}.$$

$$-\frac{\Theta''}{\Theta} = \nu^2 \quad \Rightarrow \quad H'' + \nu^2 H = 0 \tag{5}$$

$$R'' + \frac{1}{r}R' + \left[(\lambda^2 + \mu^2) - \frac{\nu^2}{r^2}\right]R = 0. \tag{6}$$

Equations (3), (4) and (5) have particular solutions of the form

$$\Phi = e^{-\alpha\lambda^2 t}$$
$$H = c\cos\nu\theta + D\sin\nu\theta$$
$$Z = Ae^{\mu z} + Be^{-\mu z}.$$

The differential equation (6) is called Bessel's equation of order ν and its general solution is known as

$$R(r) = c_1 J_\nu\left(\sqrt{\lambda^2 + \mu^2}\,r\right) + c_2 Y_\nu\left(\sqrt{\lambda^2 + \mu^2}\,r\right) \tag{7}$$

where $J_\nu(r)$ and $Y_\nu(r)$ are Bessel functions of order ν of the first and second kind, respectively. Equation (7) is singular when $r = 0$. The physically meaningful solutions must be twice continuously differentiable in $0 \le r \le a$. Hence, equation (7) has only one bounded solution, i.e.

$$R(r) = J_\nu\left(\sqrt{\lambda^2 + \mu^2}\,r\right).$$

Finally, the general solution of equation (1) is given by

$$T(r, \theta, z, t) = e^{-\alpha\lambda^2 t}[Ae^{\mu z} + Be^{-\mu z}][c\cos\nu\theta + D\sin\nu\theta]J_\nu\left(\sqrt{\lambda^2 + \mu^2}\,r\right).$$

Problem 9.3.1. Determine the temperature $T(r, t)$ in the infinite cylinder $0 \le r \le a$ when the initial temperature is $T(r, 0) = f(r)$, and the surface $r = a$ is maintained at 0° temperature.

Solution. According to the problem, the governing equation is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where T is a function of r and t only. Therefore,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

The corresponding boundary and initial conditions are given by

BC: $T(a, t) = 0$

IC : $T(r, 0) = f(r)$.

The general solution of Eq. (3.63) is

$$T(r, t) = A \exp(-\alpha\lambda^2 t) J_0(\lambda r).$$

Applying the boundary condition $T(a, t) = 0$, we get

$$0 = A \exp(-\alpha\lambda^2 t) J_0(\lambda a)$$

implying that $J_0(\lambda a) = 0$ which has an infinite number of roots, $\xi_n a (n = l, 2, \ldots, \infty)$.

By superposition principle,

$$T(r, t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha\xi_n^2 t) J_0(\xi_n r).$$

Now applying the initial condition $T(r, 0) = f(r)$, we get

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r).$$

We employ the orthogonal properties of Bessel functions to determine A_n

Multiply both sides of the above equation by $rJ_0(\xi_m r)$ and integrate with respect to r from 0 to a, we get

$$\int_0^a r f(r) J_0(\xi_m r) dr = \sum_{n=1}^{\infty} A_n \int_0^a J_0(\xi_m r) J_0(\xi_n r) dr$$

$$= \begin{cases} 0, & \text{for } n \neq m \\ A_m \left(\dfrac{a^2}{2}\right) J_1^2(\xi_m a), & \text{for } n = m \end{cases}$$

which gives

$$A_m = \frac{2}{a^2 J_1^2(\xi_m a)} \int_0^a u f(u) J_0(\xi_m u) du.$$

Hence, the solution is

$$T(r, t) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_0(\xi_m r)}{J_1^2(\xi_m a)} \exp(-\alpha \xi_m^2 t) \left[\int_0^a u f(u) J_0(\xi_m u) du \right]. \quad \blacksquare$$

9.4 Solution of Diffusion Equation in Spherical Coordinates

Consider the three-dimensional diffusion equation in spherical coordinates

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

The above equation can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial T}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{1}$$

Let us assume the solution of the form

$$T = R(r)\Theta(\theta)\Phi(\varphi)\beta(t). \tag{2}$$

Substituting equation (2) into equation (1), we get

$$R''\Theta\Phi\beta + \frac{2}{r}R'\Theta\Phi\beta + \frac{1}{r^2}R\Theta''\Phi\beta + \frac{\cos\theta}{r^2\sin\theta}R\Theta'\Phi\beta + \frac{1}{r^2\sin^2\theta}R\Theta\Phi''\beta = \frac{1}{\alpha}R\Theta\Phi\beta''.$$

Dividing by $R\Theta\Phi\beta$,

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi'}{\Phi} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \quad \text{(say)}$$

where λ^2 is a separation constant. Then,

$$\begin{aligned}
\frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \quad \Rightarrow \quad \beta' + \alpha\lambda^2\beta &= 0\\
\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi'}{\Phi} &= -\lambda^2\\
r^2\sin^2\theta\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta}\right] + \lambda^2 r^2\sin^2\theta = -\frac{\Phi'}{\Phi} &= m^2 \quad \text{(say)}\\
-\frac{\Phi'}{\Phi} = m^2 \quad \Rightarrow \quad \Phi' + m^2\Phi &= 0\\
r^2\sin^2\theta\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta}\right] + \lambda^2 r^2\sin^2\theta &= m^2\\
r^2\left[\frac{R''}{R} + \frac{2}{r}\frac{R'}{R}\right] + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \left[\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}\right] &= n(n+1) \quad \text{(say)}\\
\frac{m^2}{\sin^2\theta} - \left[\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}\right] &= n(n+1)\\
\Theta'' + \cot\theta\,\Theta' + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta &= 0\\
\frac{\partial^2\Theta}{\partial\theta^2} + \cot\theta\frac{\partial\Theta}{\partial\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta &= 0
\end{aligned}$$

$$r^2\left[\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\lambda^2 r^2\right] = n(n+1)$$

$$R''+\frac{2}{r}R'+\left[\lambda^2+\frac{n(n+1)}{r^2}\right]R = 0$$

$$\frac{\partial^2 R}{\partial r^2}+\frac{2}{r}\frac{\partial R}{\partial r}+\left[\lambda^2+\frac{n(n+1)}{r^2}\right]R = 0.$$

Then, the particular solution of the form

$$\beta = ce^{-\alpha\lambda^2 t}$$

$$\Phi(\varphi) = c_1e^{im\varphi}+c_2e^{-im\varphi}.$$

Let $R(r)=(\lambda r)^{-\frac{1}{2}}H(r)$, then

$$R' = (\lambda r)^{-\frac{1}{2}}H'(r)-\frac{(\lambda r)^{-\frac{1}{2}}H(r)}{2r}$$

$$R'' = (\lambda r)^{-\frac{1}{2}}H''(r)-\frac{(\lambda r)^{-\frac{1}{2}}H'(r)}{r}+\frac{3(\lambda r)^{-\frac{1}{2}}H(r)}{4r^2}$$

On simplifying, we get

$$H''(r)+\frac{1}{r}H'(r)+\left[\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]H(r)=0$$

which is Bessel's equation of order $n+\frac{1}{2}$ whose solution is

$$H(r)=AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r).$$

Therefore,

$$R(r)=(\lambda r)^{-\frac{1}{2}}\left[AJ_{n+\frac{1}{2}}(\lambda r)+BY_{n+\frac{1}{2}}(\lambda r)\right]$$

where J_n and Y_n are Bessel functions of first and second kind, respectively.

Now, by introducing a new independent variable $\mu=\cos\theta$ so that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}=\frac{\cos\theta}{\sqrt{1-\sin^2\theta}}=\frac{\mu}{\sqrt{1-\mu^2}}$$

$$\frac{dH}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dH}{d\mu}$$

$$\frac{d^2H}{d\theta^2} = (1-\mu^2)\frac{d^2H}{d\mu^2} - \mu\frac{dH}{d\mu}.$$

Then

$$(1-\mu^2)\frac{d^2H}{d\mu^2} - 2\mu\frac{dH}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]H = 0$$

which is an associated Legendre differential equation whose solution is

$$H(\theta) = CP_n^m(\mu) + DQ_n^m(\mu)$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are associated Legendre functions of degree n and of order m, of first and second kind, respectively. Since the solution is singular for $r = 0$, the solution must be twice continuously differentiable implying that $B = 0$, therefore

$$R(r) = (\lambda r)^{-\frac{1}{2}} A J_{n+\frac{1}{2}}(\lambda r).$$

The continuity of $H(\theta)$ at $\theta = 0, \pi$ implies that the continuity of $H(\theta)$ at $\mu = \pm 1$. Since $Q_n^m(\mu)$ has a singularity at $\mu = 1$, we choose $D = 0$, then

$$H(\theta) = CP_n^m(\mu).$$

After adjusting the constants, we have

$$T(r,\theta,\varphi,t) = A(\lambda r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda r) P_n^m(\cos\theta) e^{\pm im\varphi - \alpha\lambda^2 t}.$$

By the principle of superposition, we have

$$T(r,\theta,\varphi,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda r) P_n^m(\cos\theta) e^{\pm im\varphi - \alpha\lambda^2 t}$$

which is the required solution.

Problem 9.4.1. Find the temperature in a sphere of radius a, when its surface is kept at zero temperature and its initial temperature is $f(r, \theta)$.

Solution. The governing equation is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) \tag{1}$$

subject to

$$\text{BC:} \quad T(a, \theta, t) = 0$$

$$\text{IC:} \quad T(r, \theta, 0) = f(r, \theta).$$

The general solution of (1) is

$$T(r, \theta, t) = \sum_{\lambda, n} A_{\lambda n} (\lambda r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda r) P_n(\cos\theta) e^{-\alpha\lambda^2 t}. \tag{2}$$

Applying the boundary condition $T(a, \theta, t) = 0$, we get

$$0 = \sum_{\lambda, n} A_{\lambda n} (\lambda a)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda a) P_n(\cos\theta) e^{-\alpha\lambda^2 t}.$$

implying that $J_{n+\frac{1}{2}}(\lambda a) = 0$ which has infinitely many positive roots. Denoting them by ξ_i, we have

$$T(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\xi_i r) P_n(\cos\theta) \exp(-\alpha \xi_i^2 t). \tag{3}$$

Now, applying the initial condition $T(r, \theta, 0) = f(r, \theta)$, we have

$$f(r, \theta) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\xi_i r) P_n(\cos \theta).$$

Denoting $cos\theta$ by μ, we get

$$f(r, \cos^{-1}(\mu)) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\xi_i r) P_n(\mu).$$

We employ the orthogonality properties of both Legendre's and Bessel's functions to determine A_{ni}

Multiplying both sides by $P_m(\mu)$ and integrating between the limits, -1 to 1, we obtain

$$\begin{aligned} \int_{-1}^{1} f(r, \cos^{-1}(\mu)) P_m(\mu) d\mu &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\xi_i r) \int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu \\ &= \sum_{i=1}^{\infty} A_{mi} (\xi_i r)^{-\frac{1}{2}} J_{m+\frac{1}{2}} \left(\frac{2}{2m+1} \right) \end{aligned}$$

or

$$\left(\frac{2m+1}{2} \right) \int_{-1}^{1} f(r, \cos^{-1}(\mu)) P_m(\mu) d\mu = \sum_{i=1}^{\infty} A_{mi} (\xi_i r)^{-\frac{1}{2}} J_{m+\frac{1}{2}}(\xi_i r).$$

for $m = 0, 1, 2, 3, \ldots.$

Since

$$\int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu = \begin{cases} 0, & \text{for } n = m \\ \dfrac{2}{2m+1}, & \text{for } n = m. \end{cases}$$

Now, multiply both sides of the above equation by $r^{\frac{3}{2}} J_{m+\frac{1}{2}}(\xi_j r)$ and integrate with respect to r

between the limits 0 to a and use the orthogonality property of Bessel functions, we get

$$\xi_j^{\frac{1}{2}}\left(\frac{2m+1}{2}\right)\int_0^a r^{\frac{3}{2}} J_{m+\frac{1}{2}}(\xi_j r)dr\left[\int_{-1}^{1} P_m(\mu) f(r,\cos^{-1}(\mu))d\mu\right]$$

$$= \sum_{i=1}^{\infty} A_{mi}\int_0^a rJ_{m+\frac{1}{2}}(\xi_i r)J_{m+\frac{1}{2}}(\xi_j r)dr$$

$$= A_{mj}\frac{a^2}{2}\left[J'_{m+\frac{1}{2}}(\xi_j r)\right]^2$$

Since

$$\int_0^a rJ_{m+\frac{1}{2}}(\xi_i r)J_{m+\frac{1}{2}}(\xi_j r)dr = \begin{cases} 0, & \text{for } i \neq j \\ \frac{a^2}{2}\left[J'_{m+\frac{1}{2}}(\xi_j r)\right]^2, & \text{for } i = j. \end{cases}$$

$$A_{mj} = \frac{(2m+1)\xi_j^{\frac{1}{2}}}{a^2\left[J'_{m+\frac{1}{2}}(\xi_j r)\right]^2}\int_0^a r^{\frac{3}{2}} J_{m+\frac{1}{2}}(\xi_j r)dr\int_{-1}^{1} P_m(\mu) f(r,\cos^{-1}(\mu))d\mu \tag{4}$$

The required solution is

$$T(r,\theta,t) = \sum_{m,j} A_{mj}(\xi_j r)^{-\frac{1}{2}} J_{m+\frac{1}{2}}(\xi_j r) P_m(\cos\theta)\exp(-\alpha\xi_j^2 t)$$

where A_{mj}'s are given in (4). ■

Check Your Progress

1. Solve the one-dimensional diffusion equation in the range $0 \leq x \leq 2\pi$, $t > 0$ subject to the boundary conditions

$$\theta(x,0) = \sin^3 x \qquad \text{for } 0 \leq x \leq 2\pi$$

$$\theta(0,t) = \theta(2\pi,t) = 0 \qquad \text{for } t \geq 0.$$

2. The edges $x = 0, a$ and $y = b$ of the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ are maintained at zero

temperature while the temperature along the edge $y = 0$ is made to vary according to the rule $\theta(x, 0, t) = f(x)$, $0 \le x \le a$, $t > 0$. If the initial temperature in the rectangle is zero, find the temperature at any subsequent time t, and deduce that the steady-state temperature is

$$\frac{2}{a}\sum_{m=1}^{\infty}\frac{\sinh[m\pi(b-y)/a]}{\sinh(m\pi b/a)}\sin\frac{m\pi x}{a}\int_0^a f(u)\sin\frac{m\pi u}{a}\,du$$

3. A circular cylinder of radius a has its surface kept at a constant temperature θ_0. If the initial temperature is zero throughout the cylinder, prove that for $t > 0$

$$\theta(r, t) = \theta_0\left[1 - \frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\xi_n a)}{\xi_n J_1(\xi_n a)}e^{-\xi_n^2 \kappa t}\right],$$

where $\pm\xi_1, \pm\xi_2, \ldots, \pm\xi_n, \ldots$ are the roots of $J_0(\xi a) = 0$.

Let us Sum up:

In this unit, the students acquired knowledge to

- solution of Diffusion Equation in Cylindrical Coordinates.
- solution of Diffusion Equation in Spherical Coordinates.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.

2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.

BLOCK-V

UNIT 10

INTEGRAL TRANSFORMS

Overview

In this unit, we will we discuss the use of the Laplace and Fourier transform to obtain the solution of the diffusion equation.

10.1 The Use of Integral Transforms

10.1.1 Solution of Diffusion Equation by Laplace Transform

Consider the diffusion equation

$$\nabla^2\theta = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \tag{1}$$

in the region bounded by the two surfaces S_1 and S_2, the initial condition

$$\theta = f(\mathbf{r}) \quad \text{when} \quad t = 0 \tag{2}$$

and the boundary conditions

$$a_1\theta + b_1\frac{\partial\theta}{\partial n} = g_1(\mathbf{r}, t) \qquad \text{on} \quad S_1 \tag{3}$$

$$a_2\theta + b_2\frac{\partial\theta}{\partial n} = g_2(\mathbf{r}, t) \qquad \text{on} \quad S_2, \tag{4}$$

where the functions f, g_1 and g_2 are prescribed. The quantities a_1, a_2, b_1, b_2 may be functions of x, y and $z,$ but we shall assume that they do not depend on $t.$

Using Laplace transform technique, we solve the equation (1) with the initial and boundary conditions.

The Laplace transform of the function $\theta(\mathbf{r}, t)$ is given by

$$\bar{\theta}(\mathbf{r}, s) = \int_0^\infty \theta(\mathbf{r}, t)e^{-st}dt. \tag{5}$$

Taking the Laplace transform with respect to t on both sides of equation (1), we have

$$L\left[\nabla^2\theta(\mathbf{r}, t); s\right] = L\left[\frac{1}{\kappa}\frac{\partial\theta}{\partial t}(\mathbf{r}, t); s\right]$$

$$\nabla^2\bar{\theta}(\mathbf{r}, s) = \frac{1}{\kappa}\left\{s\bar{\theta}(\mathbf{r}, s) - \theta(\mathbf{r}, 0)\right\} \quad \left(\because \ \bar{\theta}(\mathbf{r}, s) = L\left[\theta(\mathbf{r}, t); s\right]\right)$$

$$\nabla^2\bar{\theta}(\mathbf{r}, s) - \frac{s}{\kappa}\bar{\theta}(\mathbf{r}, s) = -\frac{1}{\kappa}f(\mathbf{r})$$

$$(\nabla^2 - k^2)\bar{\theta}(\mathbf{r}, s) = -\frac{1}{\kappa}f(\mathbf{r}), \tag{6}$$

where $k^2 = \frac{s}{\kappa}$ and $\bar{\theta}(\mathbf{r}, s)$ satisfies the nonhomogeneous Helmholtz equation. Also the Laplace transform of the boundary conditions (3) and (4) becomes

$$a_1\bar{\theta} + b_1\frac{\partial\bar{\theta}}{\partial n} = \bar{g}_1(\mathbf{r}, s) \qquad \text{on } S_1 \tag{7}$$

$$a_2\bar{\theta} + b_2\frac{\partial\bar{\theta}}{\partial n} = \bar{g}_2(\mathbf{r}, s) \qquad \text{on } S_2. \tag{8}$$

We can determine the function $\bar{\theta}(\mathbf{r}, s)$ satisfies the equation (6) with the boundary conditions (7) and (8). Then the temperature function $\theta(\mathbf{r}, t)$ is determined by the Laplace's inversion transform and is given by

$$\theta(\mathbf{r}, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{\theta}(\mathbf{r}, s)e^{st}\,ds.$$

Problem 10.1.1. Determine the function $\theta(r, t)$ satisfying $\dfrac{\partial^2\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\theta}{\partial r} = \dfrac{1}{\kappa}\dfrac{\partial\theta}{\partial t}$, $t > 0$, $0 < r < a$ and the conditions $\theta(r, 0) = 0$, $\theta(a, t) = f(t)$.

Solution. Given

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}, \qquad t > 0,\ 0 < r < a \tag{1}$$

subject to the condition

$$\theta(r, 0) = 0 \tag{2}$$

$$\theta(a, t) = f(t). \tag{3}$$

The Laplace transform of the function $\theta(r, t)$ is

$$\bar{\theta}(r, s) = L\,[\theta(r, t); s] = \int_0^\infty \theta(r, t)e^{-st}dt. \tag{4}$$

The inverse Laplace transform of $\bar{\theta}(r, s)$ is

$$\theta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{\theta}(r, s)e^{st}\,ds. \tag{5}$$

Taking the Laplace transform with respect to t on both sides of equation (1), we have

$$L\left[\frac{\partial^2\theta}{\partial r^2}; s\right] + \frac{1}{r}L\left[\frac{\partial\theta}{\partial r}; s\right] = \frac{1}{\kappa}L\left[\frac{\partial\theta}{\partial t}; s\right]$$

$$\frac{\partial^2\bar{\theta}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\theta}(r,s)}{\partial r} = \frac{1}{\kappa}\left[s\,\bar{\theta}(r,s) - \theta(r,0)\right].$$

Using the initial condition $\theta(r, 0) = 0$, we get

$$\frac{\partial^2\bar{\theta}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\theta}(r,s)}{\partial r} - \frac{s}{\kappa}\frac{\partial\bar{\theta}(r,s)}{\partial t} = 0 \qquad (6)$$

which is a Bessel's equation of order zero.

The Laplace transform of the boundary condition $\theta(a, t) = f(t)$ is

$$\bar{\theta}(a, s) = \bar{f}(s), \qquad (7)$$

where $\bar{f}(s)$ is the Laplace transform of the function $f(t)$. The solution of equation (6) is

$$\bar{\theta}(r, s) = A\, I_0(kr) + B\, K_0(kr). \qquad (8)$$

Using the physical condition of $\theta(r, t)$ and hence $\bar{\theta}(r, s)$, cannot be infinite along the axis $r = 0$ of the cylinder, then the solution (8) becomes

$$\bar{\theta}(r, s) = A\, I_0(kr).$$

Using the boundary condition (7), we have

$$\bar{\theta}(r, s) = \bar{f}(s)\,\frac{I_0(kr)}{I_0(ka)},$$

where $k^2 = \frac{s}{\kappa}$. The solution of (1) becomes

$$\theta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\frac{I_0(kr)}{I_0(ka)}e^{st}\, ds.$$

If $\frac{I_0(kr)}{I_0(ka)}$ is the Laplace transform of the function $g(t)$, i.e., if

$$g(t) = \frac{1}{2\pi i}\int_{c-i}^{c+i\infty} \frac{I_0(kr)}{I_0(ka)}e^{st}\, ds, \tag{8}$$

then by convolution theeorem, we have

$$\theta(r, t) = \int_0^t f(t') \, g(t - t')dt' . \tag{9}$$

To evaluate the contour integral (8), we note that the integrand is a single-valued function of s, so that we may make use of the contour. The poles of the integrand are at the points

$$s = s_n = -\kappa\xi_n^2 \qquad n = 1, 2, \ldots,$$

where the quantities $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are the roots of the transcendental equation

$$J_0(a\xi) = 0. \tag{10}$$

Taking the radius of the circle MNL to be $\kappa(n+\frac{1}{2})^2\pi^2/a^2$, there will be no poles of the integrand on the circumference of the circle, and from the asymptotic expansions of the modified Bessel functions $I_0(kr)$, $I_0(ka)$ it is readily shown that the integral round the circular arc MNL tends to the value 0 as $n \to \infty$. We may therefore replace the line integral for $g(t)$ by the integral of the same function. The residue of this function at the pole $s = s_n$ is

$$\frac{I_0(ir\xi_n)e^{-\kappa\xi_n^2 t}}{a/(2i\kappa\xi_n)I_1(ia\xi_n)} = \frac{2\kappa\xi_n J_0(r\xi_n)e^{-\kappa\xi_n^2 t}}{aJ_1(a\xi_n)} \tag{11}$$

since $I_1(x) = I_0'(x)$. Thus

$$g(t) = \sum_{n=1}^{\infty} \frac{2\kappa\xi_n J_0(r\xi_n)}{aJ_1(a\xi_n)} e^{-\kappa\xi_n^2 t}. \tag{12}$$

Substituting (12) into (9), we obtain the required solution of (1) is given by

$$\theta(r,t) = \frac{2\kappa}{a} \sum_{n=1}^{\infty} \frac{\xi_n J_0(r\xi_n)}{J_1(a\xi_n)} \int_0^t f(t')e^{-\kappa\xi_n^2(t-t')}dt', \tag{1}$$

where the sum is taken over the positive roots of the transcendental equation (10). ■

10.1.2 Solution of Diffusion Equation by Fourier Transform

In this subsection, we discuss the solution of diffusion equation using Fourier transform technique. Recall the basic definitions of Fourier transform:

- Fourier transform of $f(t)$ is
  $$F[f(x); s] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx.$$
- Two-dimensional Fourier transform of $f(x, y)$ is
  $$F[f(x, y); \xi, \eta] = F(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{i(\xi x+\eta y)}dxdy.$$
- Three-dimensional Fourier transform of $f(x, y, z)$ is
  $$F[f(x, y, z); \xi, \eta, \zeta] = F(\xi, \eta, \zeta) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)e^{i(\xi x+\eta y+\zeta z)}dxdydz.$$
- Generalized Fourier transform of $f(\mathbf{r})$ is

$$F[f(\mathbf{r});\boldsymbol{\rho}] = F(\boldsymbol{\rho}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n \text{ times}} f(\mathbf{r})e^{i(\boldsymbol{\rho}\,\cdot\,\mathbf{r})}d\boldsymbol{\tau},$$

where $\mathbf{r}$ is n-dimensional vector, i.e., $\mathbf{r} = (x_1, x_2, \ldots, x_n)$ and $d\boldsymbol{\tau} = dx_1dx_2\ldots dx_n$.

Problem 10.1.2. Find the solution of the equation $\kappa\nabla^2\theta = \dfrac{\partial\theta}{\partial t}$ for an infinite solid whose initial distribution of temperature is given by $\theta(\mathbf{r}, 0) = f(\mathbf{r})$, where the function f is prescribed.

Solution. Given

$$\kappa\nabla^2\theta = \frac{\partial\theta}{\partial t} \tag{1}$$

subject to the initial condition

$$\theta(\mathbf{r}, 0) = f(\mathbf{r}). \tag{2}$$

Here we consider the three-dimensional diffusion equation, i.e., $\mathbf{r} = (x, y, z)$.

The three-dimensional Fourier transform of $\theta(\mathbf{r}, t)$ is

$$\Theta(\boldsymbol{\rho}, t) = (2\pi)^{-\frac{3}{2}} \int \theta(\mathbf{r}, t)e^{i(\boldsymbol{\rho}\,\cdot\,\mathbf{r})}d\boldsymbol{\tau}, \tag{3}$$

where $\boldsymbol{\rho} = (\xi, \eta, \zeta)$, $d\boldsymbol{\tau} = dxdydz$ and the integration extends throughout the entire xyz space.

Taking the Laplace transform of the given equation (1) and initial condition (2), we get the ordinary differential equation with initial condition

$$\frac{d\Theta}{dt} + \kappa p^2\Theta = 0 \tag{4}$$

$$\Theta(\boldsymbol{\rho}, 0) = F(\boldsymbol{\rho}), \tag{5}$$

where $F(\boldsymbol{\rho})$ is the Fourier transform of the function $f(\mathbf{r})$. The solution of equation (4) subject to the initial condition (5) is

$$\Theta(\boldsymbol{\rho}, t) = F(\boldsymbol{\rho})\, e^{-\kappa\rho^2 t}. \tag{6}$$

Let

$$G(\boldsymbol{\rho}) = e^{-\kappa\rho^2 t} \tag{7}$$

is the Fourier transform of the function

$$g(\mathbf{r}) = (2\kappa t)^{-\frac{3}{2}}\, e^{-r^2/4\kappa t}. \tag{8}$$

Equation (6) becomes

$$\Theta(\boldsymbol{\rho}, t) = F(\boldsymbol{\rho})\, G(\boldsymbol{\rho}). \tag{9}$$

By the convolution theorem of Fourier transform, the solution of (1) is

$$\begin{aligned} \theta(\mathbf{r}, t) &= (f * g)(\mathbf{r}) \\ &= (2\pi)^{-\frac{3}{2}} \int f(\mathbf{r}')g(\mathbf{r} - \mathbf{r}')d\tau' \\ \theta(\mathbf{r}, t) &= (2\kappa t)^{-\frac{3}{2}} \int f(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4\kappa t}d\tau', \end{aligned} \tag{10}$$

where the integration extends over the whole $x'y'z'$ space. If

$$\mathbf{u} = (u, v, w) = (4\kappa t)^{-\frac{1}{2}}\,(\mathbf{r}' - \mathbf{r})$$

The solution (10) reduces to

$$\theta(\mathbf{r}, t) = \pi^{-\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r} + 2\mathbf{u}\sqrt{\kappa t})e^{-(u^2+v^2+w^2)}dudvdw$$

which is known as **Fourier's solution**. ■

Check Your Progress

1. Use the theory of the Laplace transform to derive the solution of the boundary value problem:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad 0 \le x \le a, \quad t > 0$$

$$\theta(0, t) = f(t), \ \ \theta(a, t) = 0, \ \ \theta(x, 0) = 0.$$

2. If $\theta(r, t)$ satisfies the equations

   (i) $\dfrac{\partial^2\theta}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\theta}{\partial r} = \dfrac{1}{\kappa}\dfrac{\partial\theta}{\partial t} \quad 0 \le r \le a, \ t > 0$

   (ii) $\theta(r, 0) = f(r) \quad 0 \le r \le a$

   (iii) $\left(\dfrac{\partial\theta}{\partial r} + h\theta\right)_{r=a} = 0 \quad t > 0$

   show that

$$\theta(r, t) = \frac{2}{a^2}\sum_i \frac{\xi_i^2 e^{-k\xi_i^2 t}J_0(\xi_i r)}{(h^2 + \xi_i^2)[J_0(\xi_i a)]^2}\int_0^a u f(u)J_0(\xi_i u)du,$$

   where the sum is taken over the positive roots $\xi_1, \xi_2, \ldots, \xi_i, \ldots$ of the equation

$$hJ_0(a\xi_i) = \xi_i J_1(a\xi_0).$$

3. Using the Fourier sine transform $\Theta_s(\xi, t) = \left(\dfrac{2}{\pi}\right)^{\frac{1}{2}}\displaystyle\int_0^\infty \theta(x, t)\sin(\xi x)dx$, show that the Poisson integral $\theta(x, t) = \dfrac{1}{2\sqrt{\pi\kappa t}}\displaystyle\int_0^\infty f(\xi)\left[e^{-(x-\xi)^2/4\kappa t} - e^{-(x+\xi)^2/4\kappa t}\right]d\xi$ and $\theta(x, t) = \theta_0\left[1 - \operatorname{erf}\left(\dfrac{x}{2\sqrt{\kappa t}}\right)\right]$ are the solution of one-dimensional diffusion equation.

10.2 The Use of Green's Functions

In this section, we explain the procedure to determine the solution of diffusion equation using Green's function.

Consider the diffusion equation

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta \tag{1}$$

in the volume V, which is bounded by the simple surface S, subject to the boundary condition

$$\theta(\mathbf{r}, t) = \varphi(\mathbf{r}, t) \qquad \text{if } \mathbf{r} \in S \tag{2}$$

and the initial condition

$$\theta(\mathbf{r}, 0) = f(\mathbf{r}) \qquad \text{if } \mathbf{r} \in V. \tag{3}$$

Define the Green's function $G(\mathbf{r}, \mathbf{r}', t - t')$ $(t > t')$ satisfies the equation

$$\frac{\partial G}{\partial t} = \kappa \nabla^2 G \tag{4}$$

the boundary condition

$$G(\mathbf{r}, \mathbf{r}', t - t') = 0 \qquad \text{if } \mathbf{r}' \in S \tag{5}$$

and the initial condition that $\lim\limits_{t \to t'} G$ is zero at all points of V except at the point $\mathbf{r}$ where G takes the form

$$\frac{1}{8[\pi\kappa(t - t')]^{\frac{3}{2}}} \exp\left[-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4\kappa(t - t')}\right] \tag{6}$$

since G depends on t only in that it is a function of $t - t'$, then equation (4) becomes

$$\frac{\partial G}{\partial t'} + \kappa \nabla^2 G = 0. \tag{7}$$

The time t' lies within the interval of t for which equations (1) and (2) are valid, we have

$$\frac{\partial \theta}{\partial t'} = \kappa \nabla^2 \theta \qquad t' < t \tag{8}$$

$$\theta(\mathbf{r}', t') = \varphi(\mathbf{r}', t') \qquad \text{if } \mathbf{r}' \in S. \tag{9}$$

From (7) and (8), we obtain

$$\frac{\partial}{\partial t'}(\theta G) = \theta\frac{\partial G}{\partial t'} + G\frac{\partial \theta}{\partial t} = \kappa[G\nabla^2\theta - \theta\nabla^2 G]$$

$$\Rightarrow \int_0^{t-\varepsilon}\left(\int_V \frac{\partial}{\partial t'}(\theta G)d\tau'\right)dt' = \kappa\int_0^{t-\varepsilon}\left(\int_V [G\nabla^2\theta - \theta\nabla^2 G]d\tau'\right)dt', \tag{10}$$

where ε is an arbitrarily small positive constant.

By changing the order of integrations on the left-hand side, we get

$$\int_V (\theta G)_{t'=t-\varepsilon}d\tau' - \int_V (\theta G)_{t'=0}d\tau' = \theta(\mathbf{r}, t)\int_V [G(\mathbf{r}, \mathbf{r}', t-t')]_{t'=t-\varepsilon}d\tau' - \int_V G(\mathbf{r}, \mathbf{r}', t)f(\mathbf{r}')d\tau'.$$

From equation (6), we have

$$\int_V [G(\mathbf{r}, \mathbf{r}'\ t-t')]_{t'=t-0}d\tau' = 1$$

and letting $\varepsilon \to 0$, the left-hand side of equation (10) reduces

$$\theta(\mathbf{r}, t) - \int_V G(\mathbf{r}, \mathbf{r}'\ t-)d\tau'.$$

Applying Green's theorem to the right-hand side of equation (10) and the boundary conditions (2) and (5) gives

$$-\kappa\int_0^t dt'\int_S \varphi(\mathbf{r}', t')\frac{\partial G}{\partial n}dS'.$$

Again, let $\varepsilon \to 0$, we obtain

$$\theta(\mathbf{r}, t) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}', t)d\tau' - \kappa\int_0^t dt'\int_S \varphi(\mathbf{r}', t')\frac{\partial G}{\partial n}dS' \tag{11}$$

which is the required solution of the boundary value problem.

Problem 10.2.1. If the surface $z = 0$ of the semi-infinite solid $z > 0$ is maintained at temperature $\varphi(x, y, t)$ for $t > 0$, and if the initial temperature of the solid is $f(x, y, z)$, determine the distribution of temperature in the solid.

Solution. The Green's function for the given problem is

$$G(\mathbf{r}, \mathbf{r}', t-t') = \frac{1}{8[\pi\kappa(t-t')]^{\frac{3}{2}}}\left\{\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\kappa(t-t')}\right] - \exp\left[-\frac{|\mathbf{r}-\boldsymbol{\rho}'|^2}{4\kappa(t-t')}\right]\right\},$$

where $\boldsymbol{\rho}' = (x', y', -z')$ is the position vector of the image of the point $\mathbf{r}'$ in the plane $z = 0$.

Then

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial z'}\right)_{z'=0} = -\frac{z}{8\pi^{\frac{3}{2}}\kappa^{\frac{5}{2}}(t-t')^{\frac{5}{2}}}\exp\left[-\frac{(x-x')^2+(y-y')^2+z^2}{4\kappa(t-t')}\right].$$

We know that the solution of the diffusion equation using Green's function is given by

$$\theta(\mathbf{r}, t) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}', t)d\tau' - \kappa\int_0^t dt'\int_S \varphi(\mathbf{r}', t')\frac{\partial G}{\partial n}dS'$$

Then the above equation becomes

$$\theta(\mathbf{r}, t) = \frac{1}{8(\pi\kappa t)^{\frac{3}{2}}}\int_V f(\mathbf{r}')[e^{-|\mathbf{r}-\mathbf{r}'|^2/4\kappa t} - e^{-|\mathbf{r}-\boldsymbol{\rho}'|^2/4\kappa t}]d\tau'$$
$$+ \frac{z}{8(\pi\kappa)^{\frac{3}{2}}}\int_0^t\int_\Pi \frac{\varphi(x', y', t')}{(t-t')^{\frac{3}{2}}}\exp\left[-\frac{(x-x')^2+(y-y')^2+z^2}{4\kappa(t-t')}\right]dx'\,dy'\,dt',$$

where V denotes the half space $z > 0$ and Π the entire xy plane. ■

Problem 10.2.2. Determine the Green's function for the thick plate of infinite radius bounded by the parallel planes $z = 0$ and $z = a$.

Solution. We know that the Green's function $G(\mathbf{r}, \mathbf{r}', t-t')$ $(t > t')$ satisfies the equation

$$\frac{\partial G}{\partial t} = \kappa\nabla^2 G \tag{1}$$

the boundary condition

$$G(\mathbf{r},\ \mathbf{r}',\ t-t') = 0 \qquad \text{if } \mathbf{r}' \in S \tag{2}$$

and the initial condition that $\lim_{t\to t'} G$ is zero at all points of V except at the point $\mathbf{r}$ where G takes the form

$$\frac{1}{8[\pi\kappa(t-t')]^{\frac{3}{2}}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\kappa(t-t')}\right] \tag{3}$$

To determine a function G which vanishes on the planes $z=0, z=a$ and has a singularity of the type (3). We write

$$G(\mathbf{r},\mathbf{r}',t) = \frac{1}{8[\pi\kappa t]^{\frac{3}{2}}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\kappa t}\right] + G_1(\mathbf{r},\mathbf{r}',t), \tag{4}$$

and from (1), we get

$$\kappa\nabla^2 G_1 = \frac{\partial G_1}{\partial t}$$

$$\text{i.e., } \frac{\partial^2 G_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G_1}{\partial \rho} + \frac{\partial^2 G_1}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 G_1}{\partial \varphi^2} = \frac{\partial G_1}{\partial t}. \tag{5}$$

Taking the Laplace transform of (4), we get

$$L\{G(\mathbf{r},\mathbf{r}',t)\} = L\left[\frac{1}{8[\pi\kappa t]^{\frac{3}{2}}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\kappa t}\right]\right] + L\{G_1(\mathbf{r},\mathbf{r}',t)\}$$

$$\overline{G}(\mathbf{r},\mathbf{r}',s) = \frac{1}{4\pi\kappa}\int_0^\infty e^{-\mu|z-z'|}\frac{J_0(\lambda R)}{\mu}\lambda\, d\lambda + \overline{G}_1(\mathbf{r},\mathbf{r}',s) \tag{6}$$

where $R^2 = (x-x')^2+(y-y')^2$ and $\mu^2 = \lambda^2 + s/\kappa$ and $\overline{G}, \overline{G_1}$ are the Laplace transforms of G, G_1.

Again taking the Laplace transform of (5) on both sides, we get

$$\frac{\partial^2 \bar{G}_1}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \bar{G}_1}{\partial \rho} + \frac{\partial^2 \bar{G}_1}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 \bar{G}_1}{\partial \varphi^2} = \frac{s}{K}\bar{G}_1, \tag{7}$$

where ρ, z, φ denote cylindrical Coordinates.

Then the solution of equation (7) is of the form

$$\bar{G}_1(\mathbf{r}, \mathbf{r}', s) = \frac{1}{4\pi\kappa}\int_0^\infty \frac{\lambda}{\mu} J_0(\lambda R)\{F \sinh(\mu z) + H \sinh[\mu(a - z)]\}d\lambda,$$

where the functions $F(\lambda)$ and $H(\lambda)$ must be chosen so that $\bar{G}$ vanishes on the planes $z = 0, z = a$.

Therefore

$$F = -e^{-\mu(a-z')} \text{cosech } (\mu a),$$

$$H = -e^{-\mu z'} \text{cosech } (\mu a).$$

If $0 < z < z'$, we obtain

$$\bar{G} = \frac{1}{2\pi\kappa}\int_0^\infty \frac{\lambda J_0(\lambda R) \sinh[\mu(a - z')] \sinh(\mu z)}{\mu \sinh(\mu a)} d\lambda.$$

Substitution $\lambda = i\xi$, we get

$$\bar{G} = \frac{1}{4\pi\kappa}\int_{-i}^{i\infty} \frac{\xi I_0(\xi R) \sinh[\eta(a - z')] \sinh(\eta z)}{\eta \sinh(\eta a)} d\xi,$$

where $\eta^2 = s/\kappa - \xi^2$. By using the residue technique, we have

$$\overline{G} = \frac{1}{2\kappa a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right) K_0(\xi_n R),$$

where $\xi_n = \sqrt{n^2\pi^2/a^2 + s/\kappa}$. Using the fact that $K_0[x\sqrt{s/\kappa}]$ is the Laplace transform of $(2t)^{-1}e^{-x^2/4\kappa t}$ and that the Laplace transform of $e^{-at}f(t)$ is $f(s+a)$, we obtain

$$G = \frac{e^{-R^2/4\kappa t}}{2\pi\kappa t a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right) e^{-n^2\pi^2\kappa t/a^2}$$

which is the required Green's function. ■

Check Your Progress

1. Solve the boundary value problem

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial x^2} \quad x \geq 0,\ t > 0$$
$$\theta(0,t) = \varphi(t), \quad t > 0;$$
$$\theta(x,0) = f(x), \quad x \geq 0.$$

2. By using the theory of Laplace transforms derive the Green's function for the segment $0 \leq x \leq a$.

3. Show that the Green's function for problems with radial symmetry, in which the temperature vanishes on $r = a$, can be expressed in the form

$$G(r, r', t) = \frac{1}{2\pi a r r'}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi r}{a}\right)\sin\left(\frac{n\pi r'}{a}\right) e^{-n^2\pi^2\kappa t/a^2}.$$

Let us Sum up:

In this unit, the students acquired knowledge to

- find the solution of diffusion equation by Fourier transform technique.
- understand the concepts of the uses of Green's funciton.

Suggested Readings:

1. M.D. Raisinghania, *Advanced Differential Equations*, S. Chand & Company Ltd., New Delhi, 2001.
2. K. Sanakara Rao, *Introduction to Partial Differential Equations*, Second Edition, Prentice-Hall of India, New Delhi, 2006.